Information-Theoretic PIR: Constructions and Applications

Seyoon Ragavan

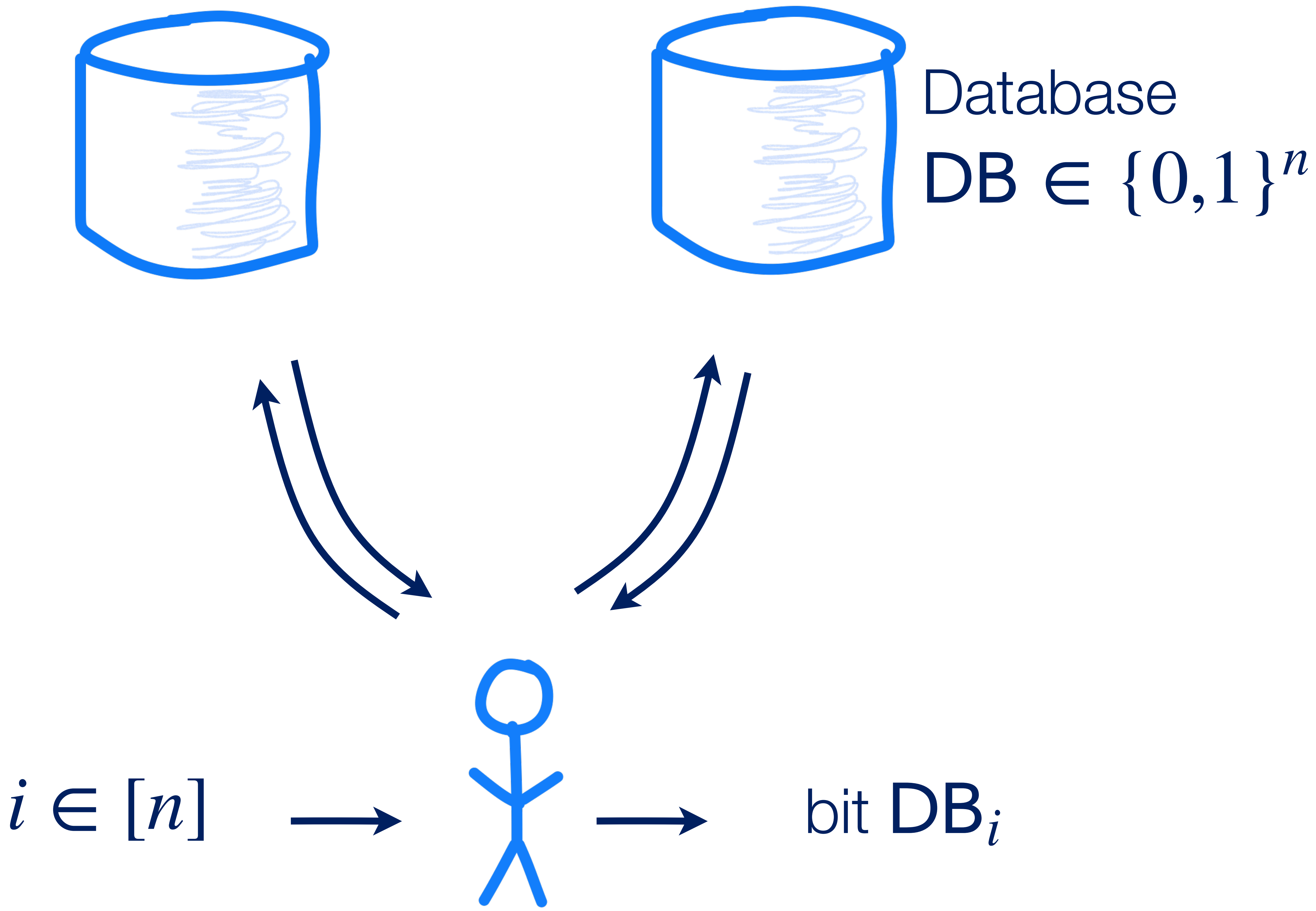
Based on joint works with Alexandra Henzinger and (if time permits) Ted Pyne

Thanks Alexandra for some of these slides!

ePrint: 2025/2008, 2026/265

Private Information Retrieval [CGKS95,KO97]

Goal: privately read an entry from a remote database



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$i \in [n]$ ⟶ ⟶ bit DB_i

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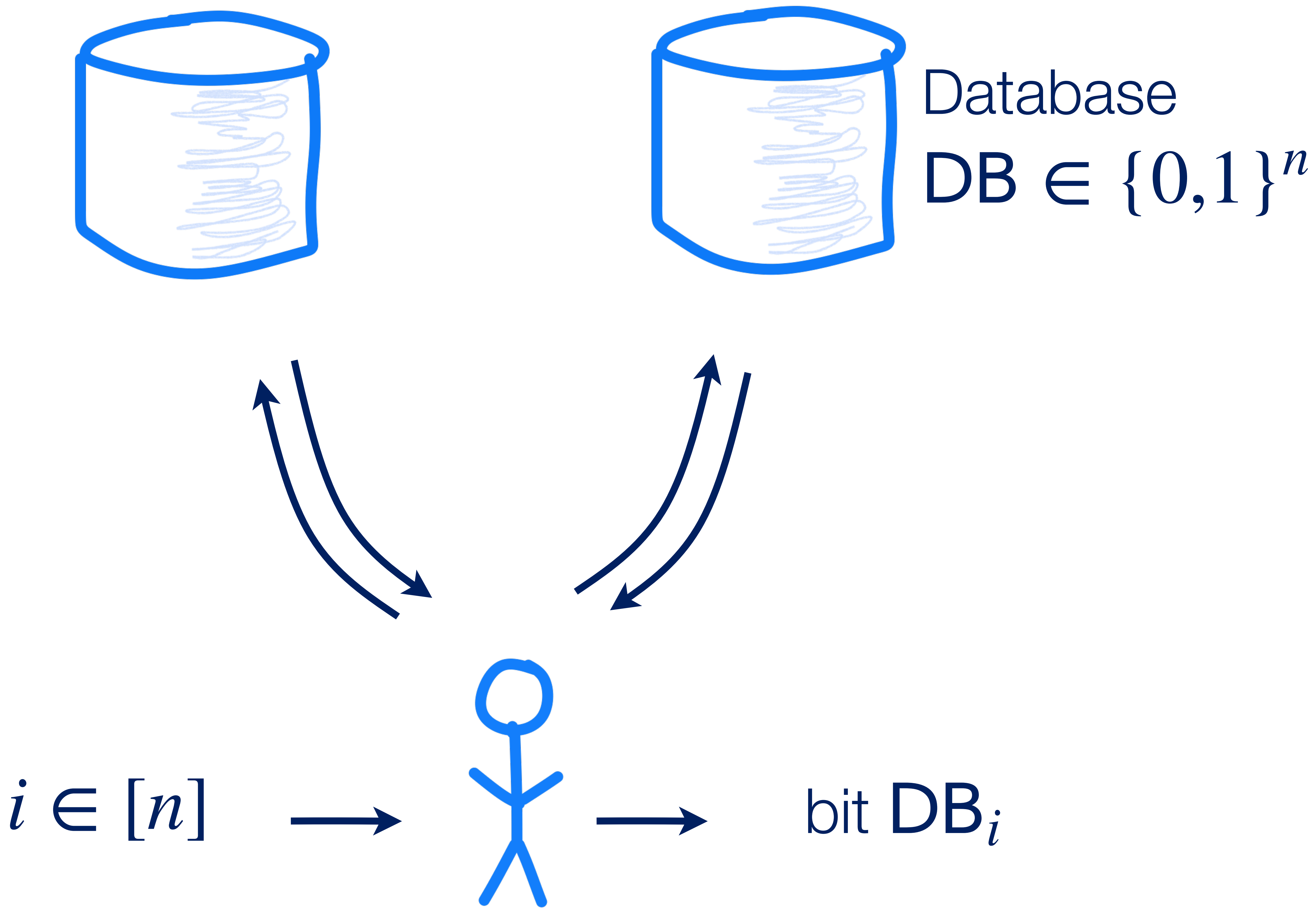

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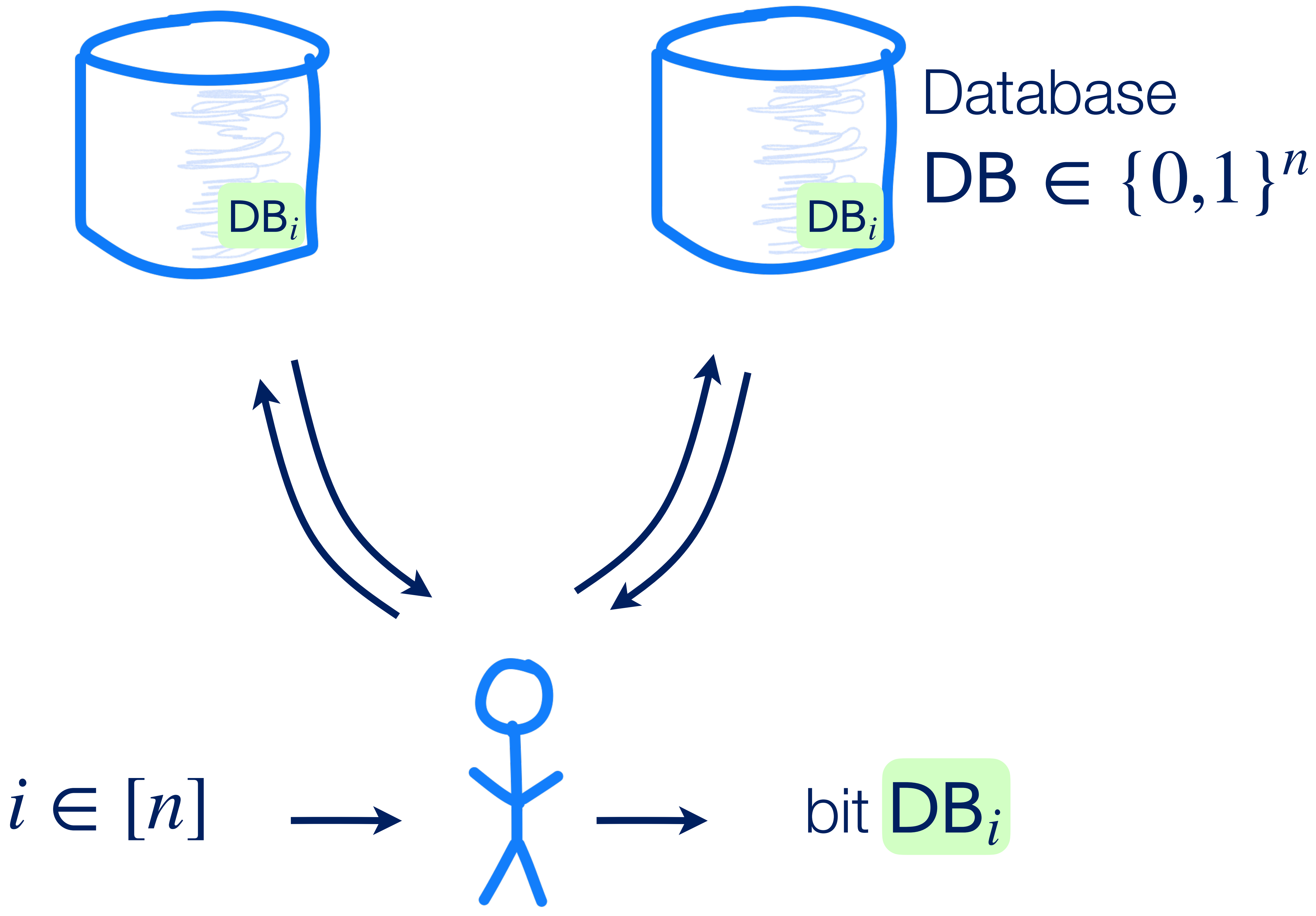

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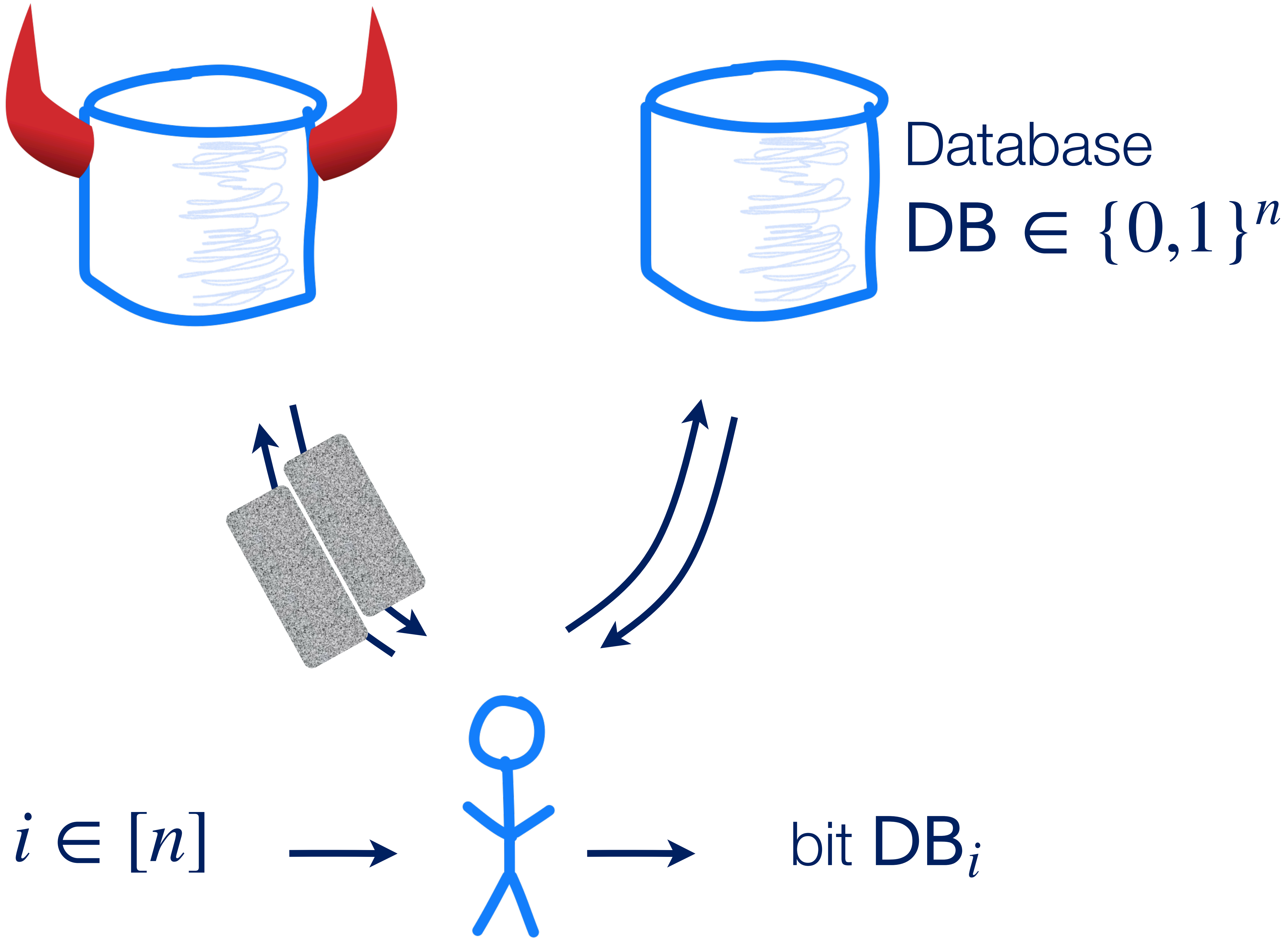

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Formally: information-theoretic/computational indistinguishability of the server's view, on reads to any two indices i or i'.

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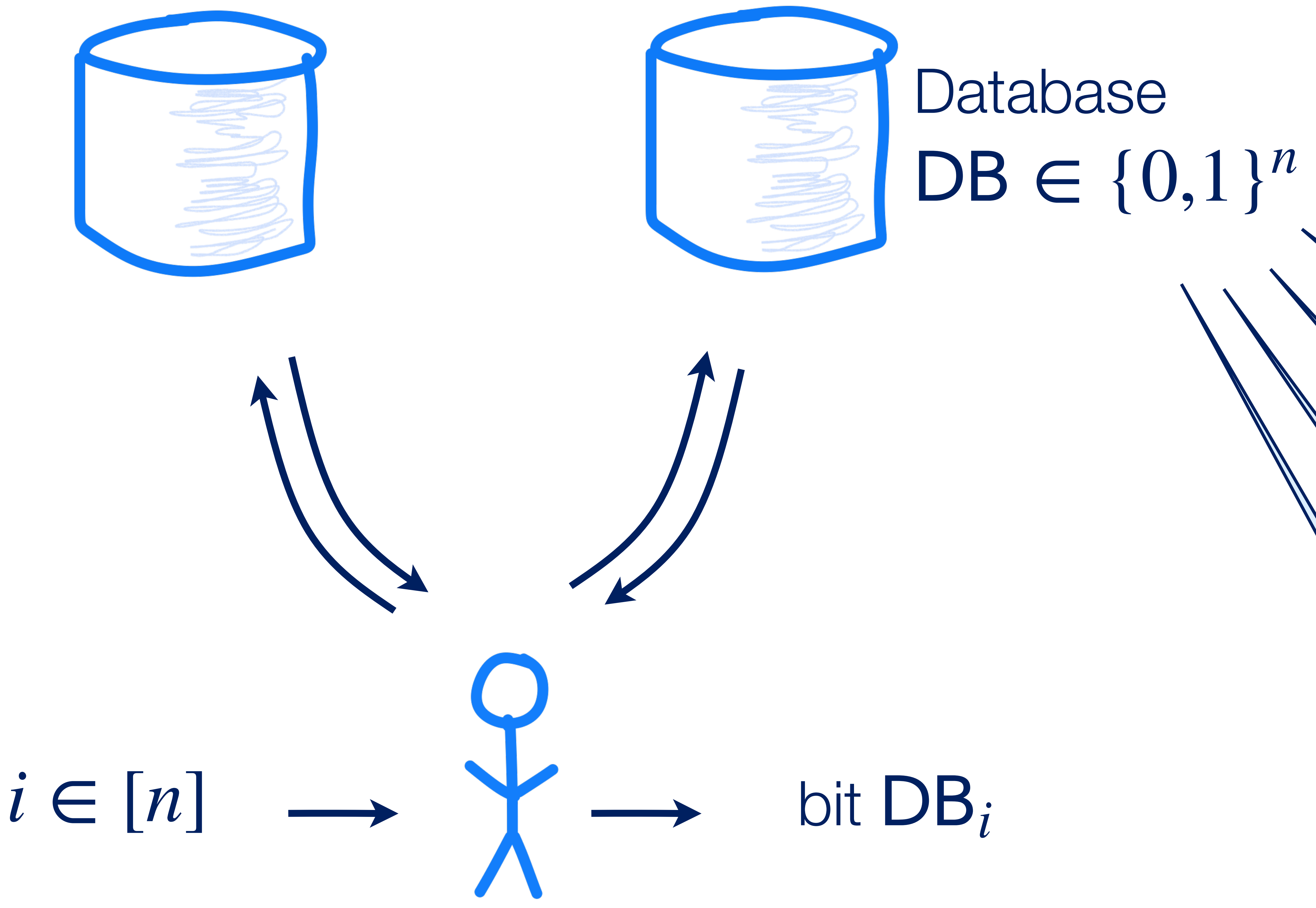

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Private web search
[Coeus'21, Tiptoe'23, Wally'24]

Private advertising
[J'01,ObliviAd'12,AdVeil'20…]

Private media delivery
[Popcorn'16,PIRSONA'21,CrowdSurf'25,…]

Metadata-hiding messaging
[PIRTor'11,Pung'16,Talek'20,Addra'21,…]

Private auditing and blocklists
[Checklist'20,SimplePIR'23,HPW'25,…]

Private file storage
[Dory'20,Peer2PIR'25,…]

On your iPhone: Private caller-ID

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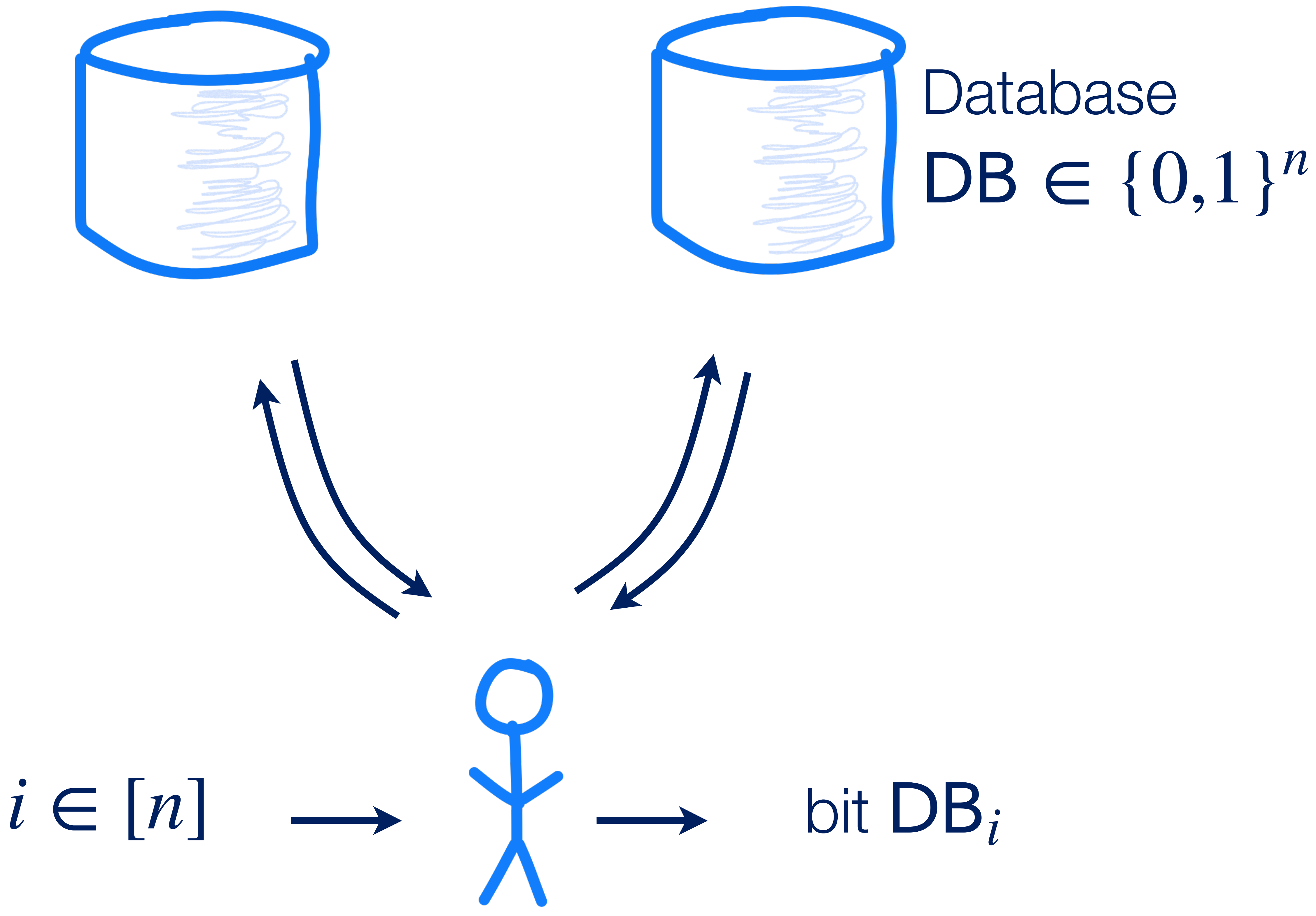

Modern PIR needs very little communication:

- No privacy: $\log n + 1$
- Info-theoretic privacy: $n^{o(1)}$
- Comp. privacy: $O(\lambda \cdot \log n)$
  [KO97,CMS99,DG16,BGI16]

…but lots of server work:

- No privacy: $O(1)$ time
- With privacy: $\Omega(n)$ time
  [BIM00, PY22]

Solution: Change the PIR model to get sublinear time

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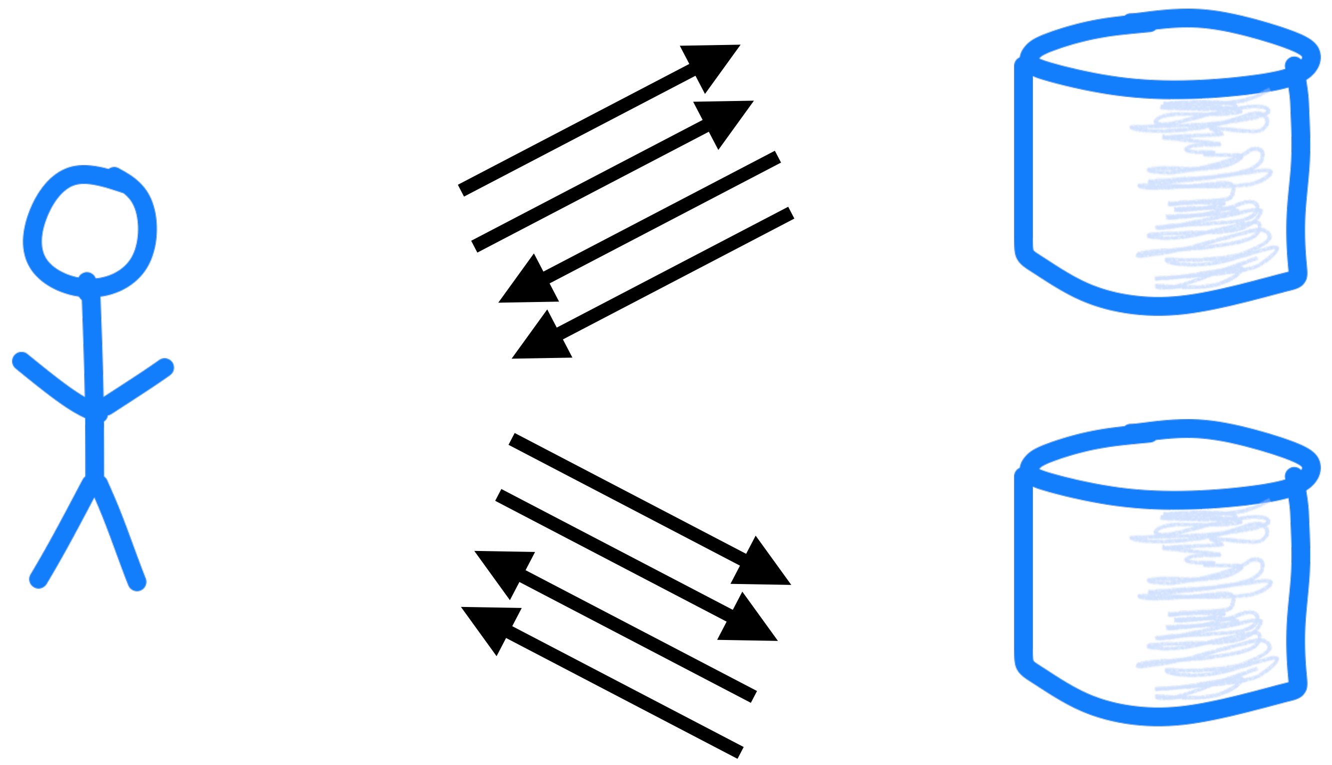
Batch PIR with many, non-adaptive queries



[IKOS’04,HHG’13,GKL’10,LG’15,AS’16,H’16,ACLS’18,CHLR’18]

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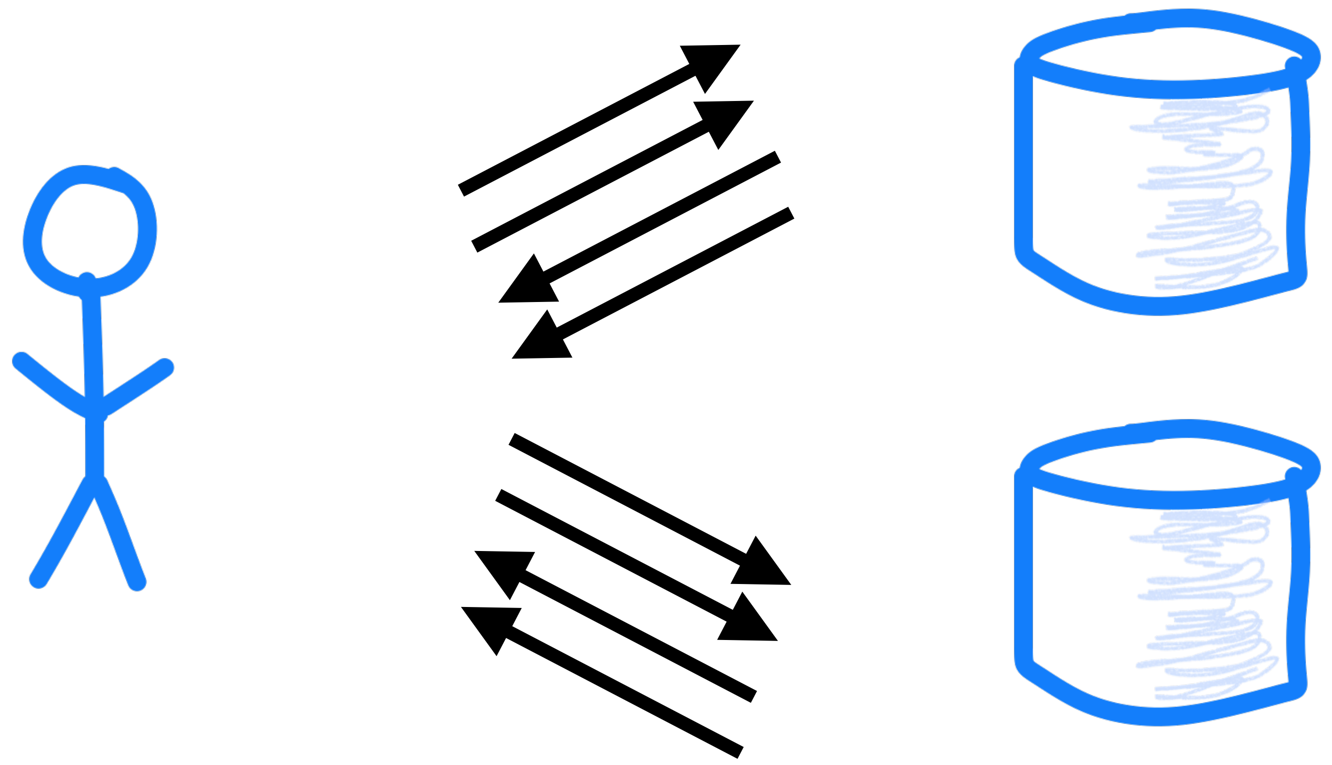

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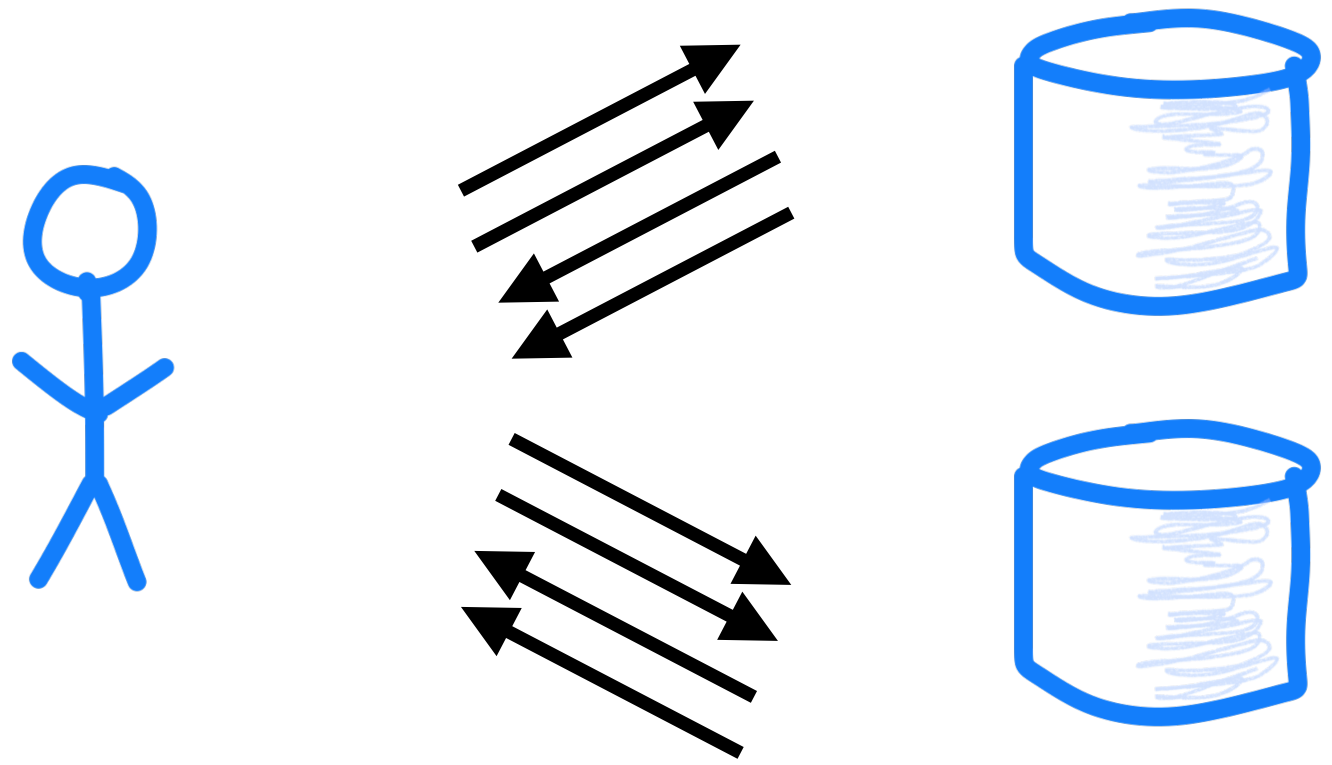

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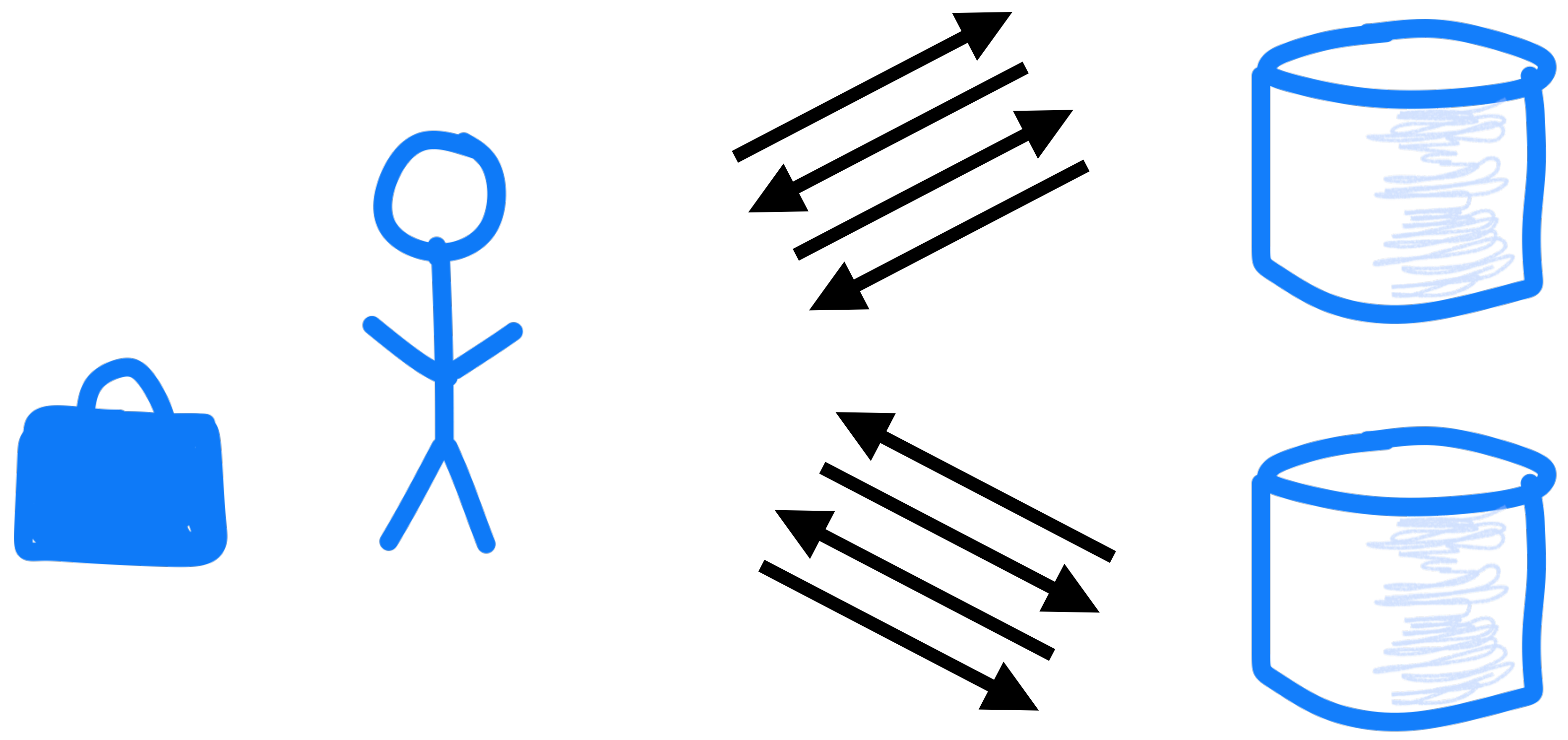
Offline/online PIR with stateful clients + many queries



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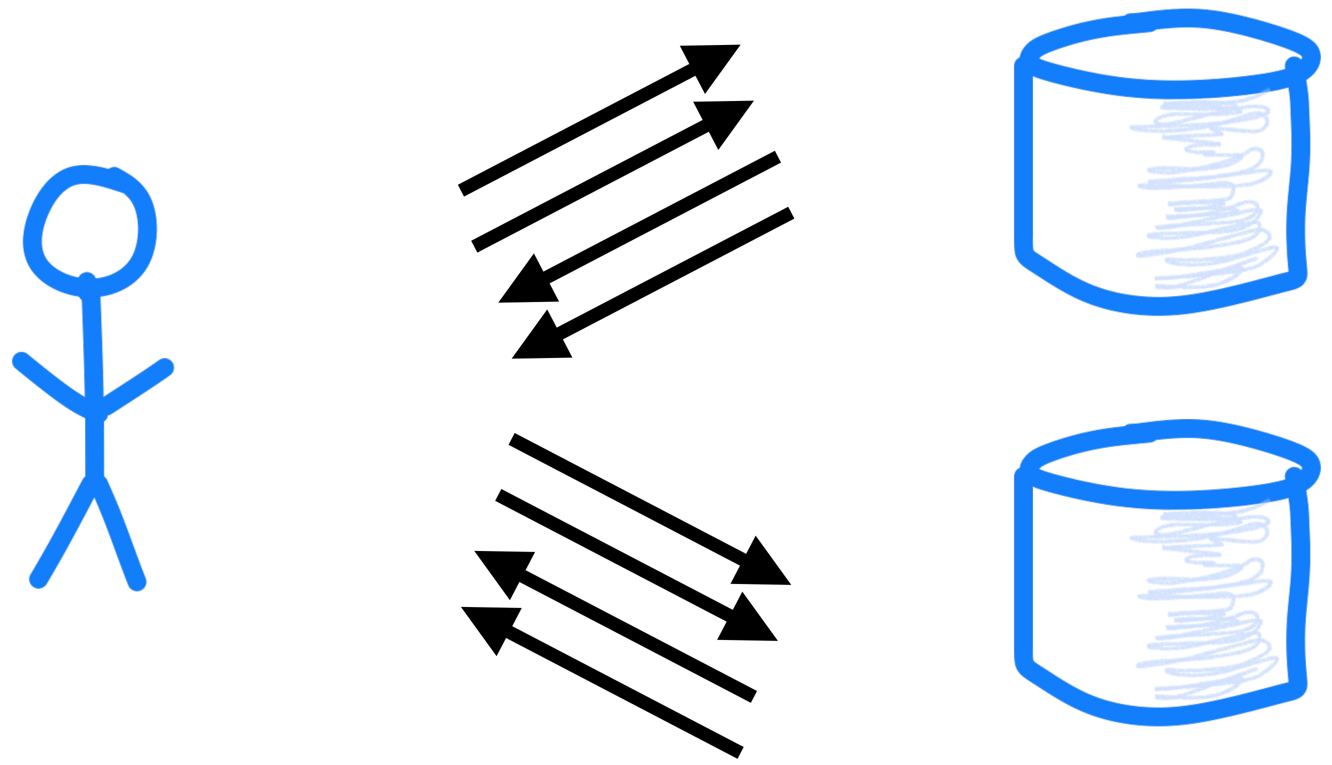

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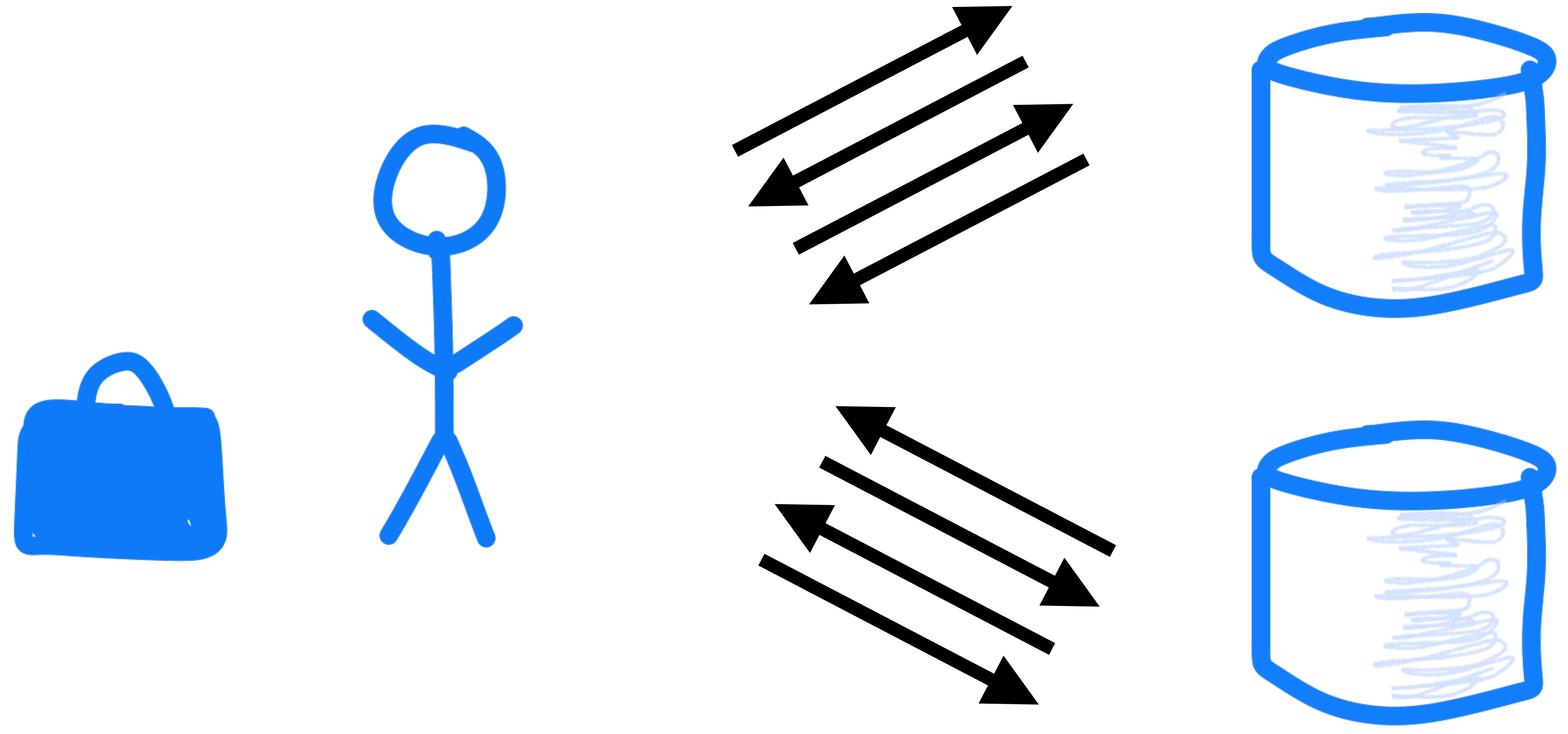

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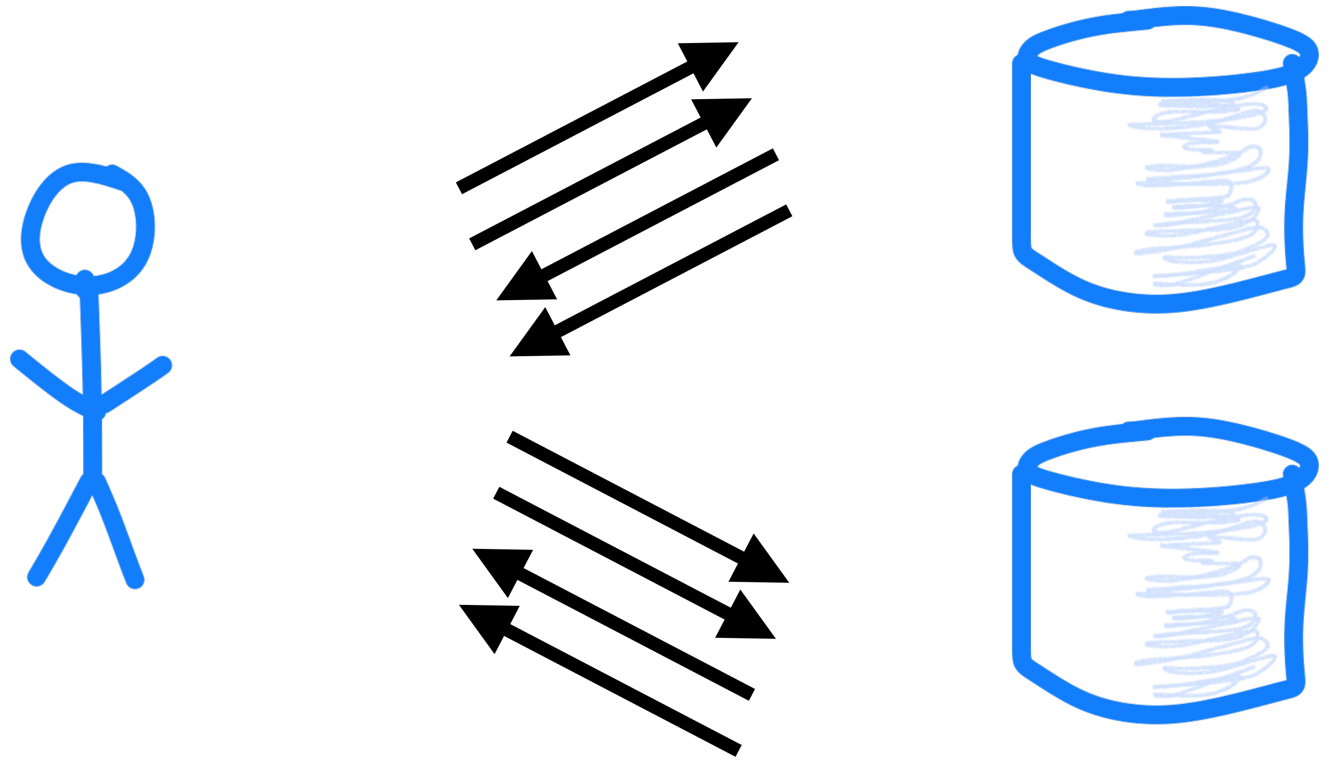

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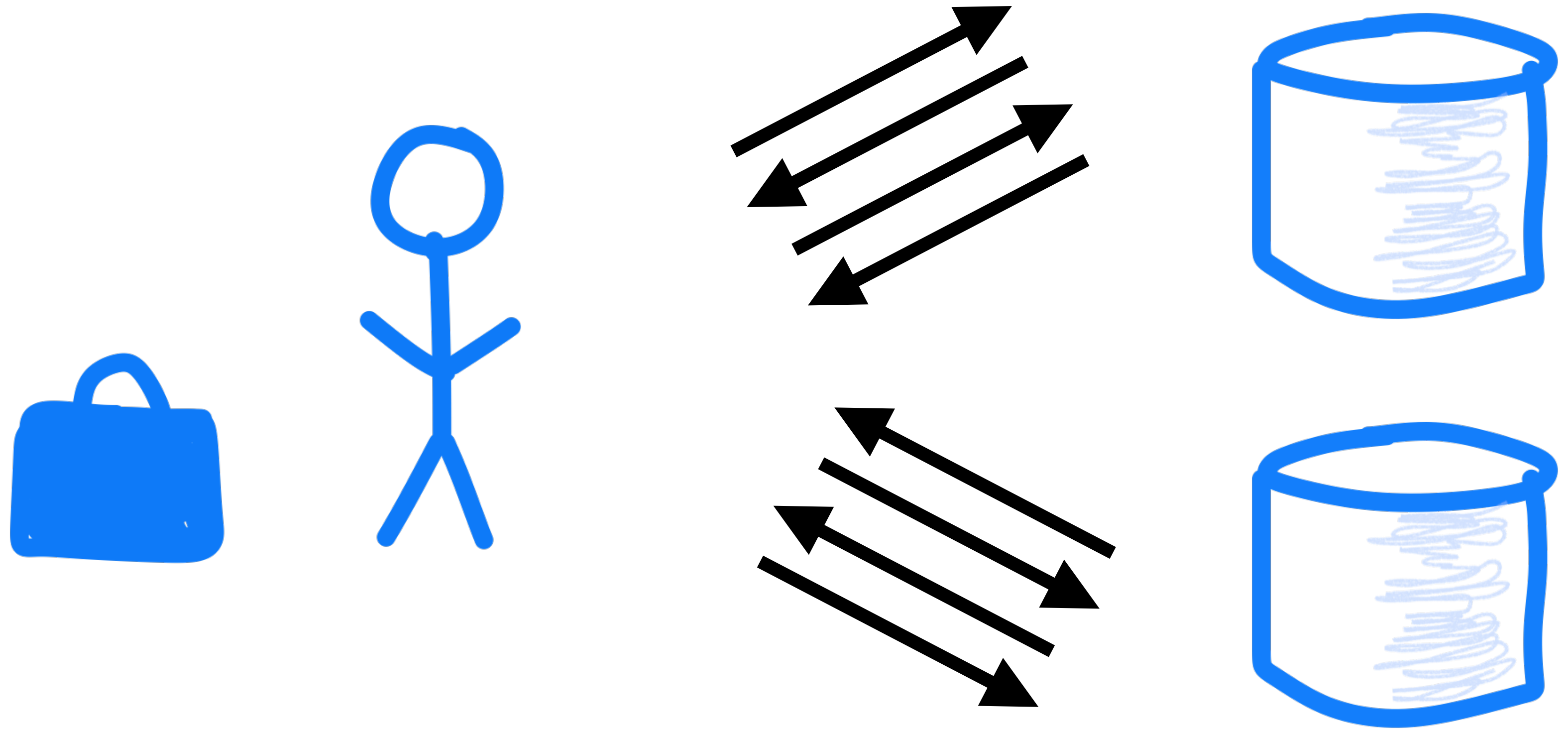

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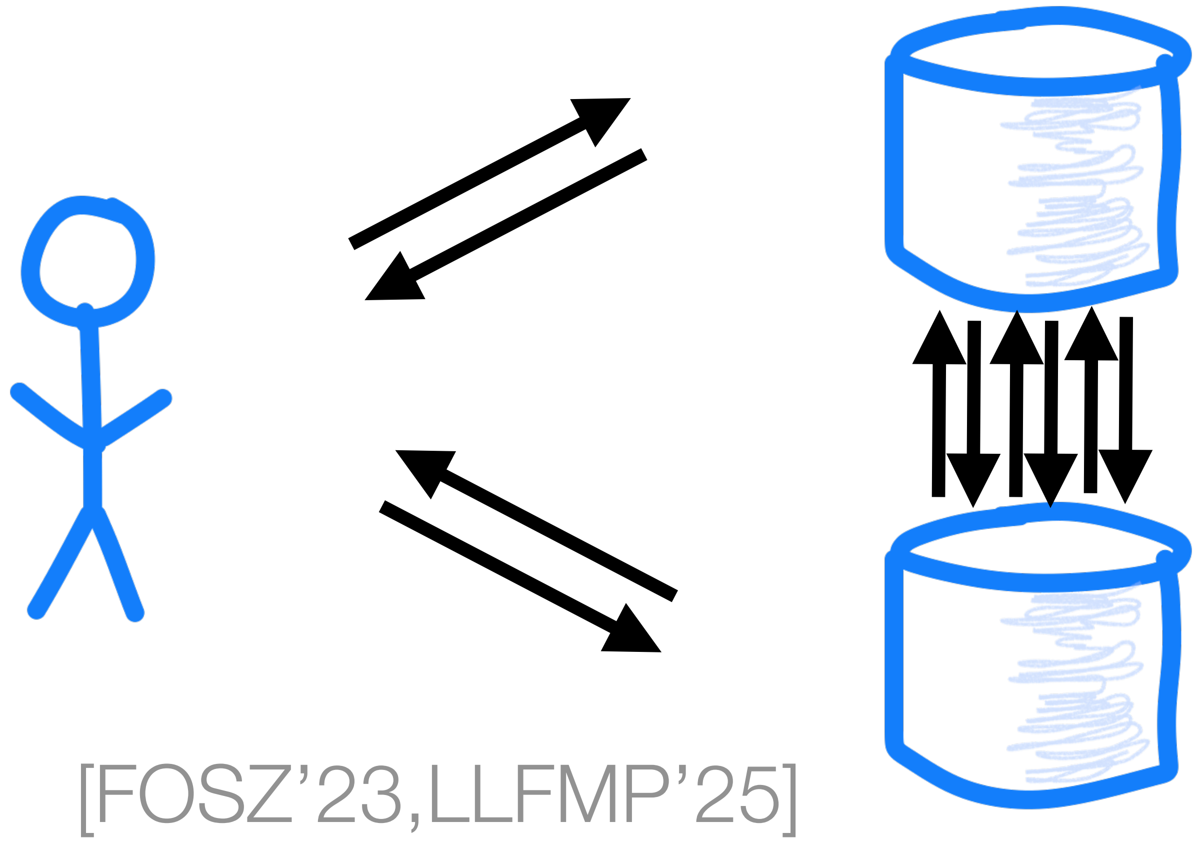
Distributed ORAM with communicating servers



[FOSZ’23,LLFMP’25]

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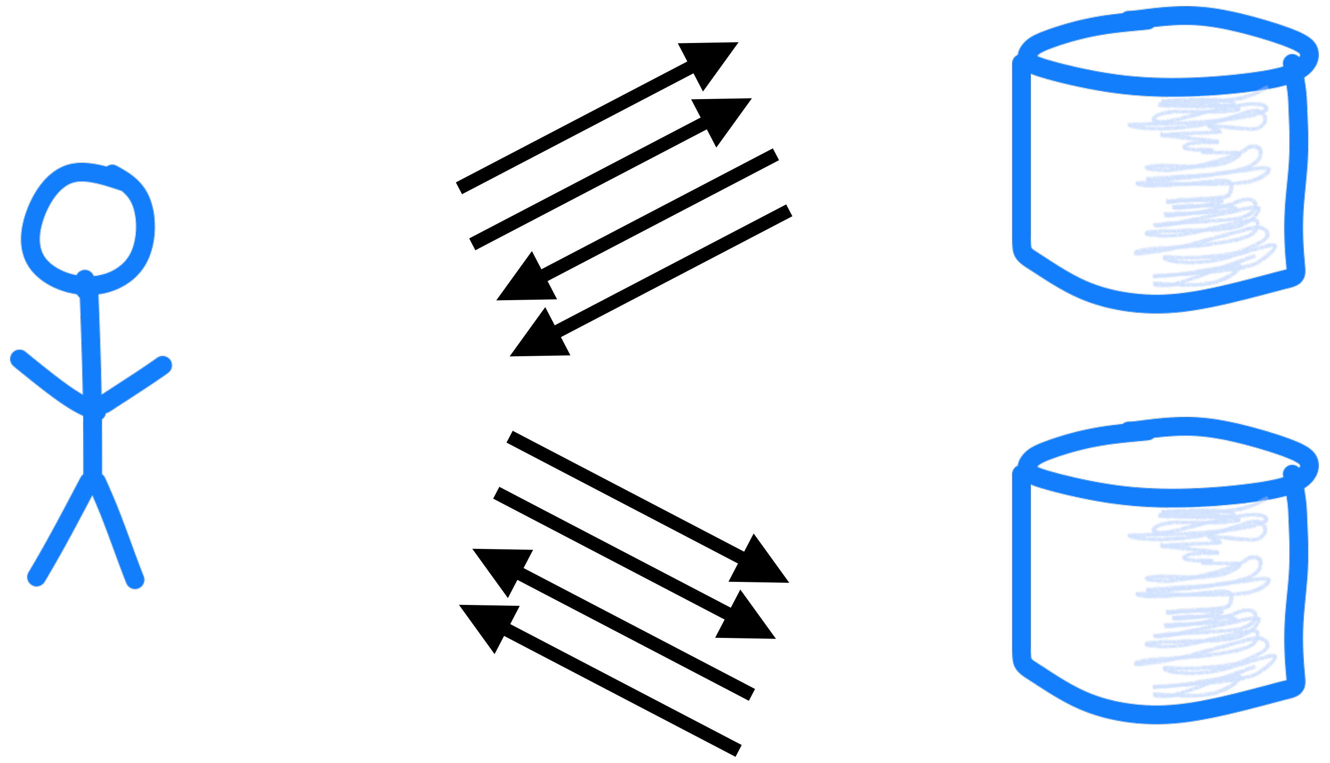

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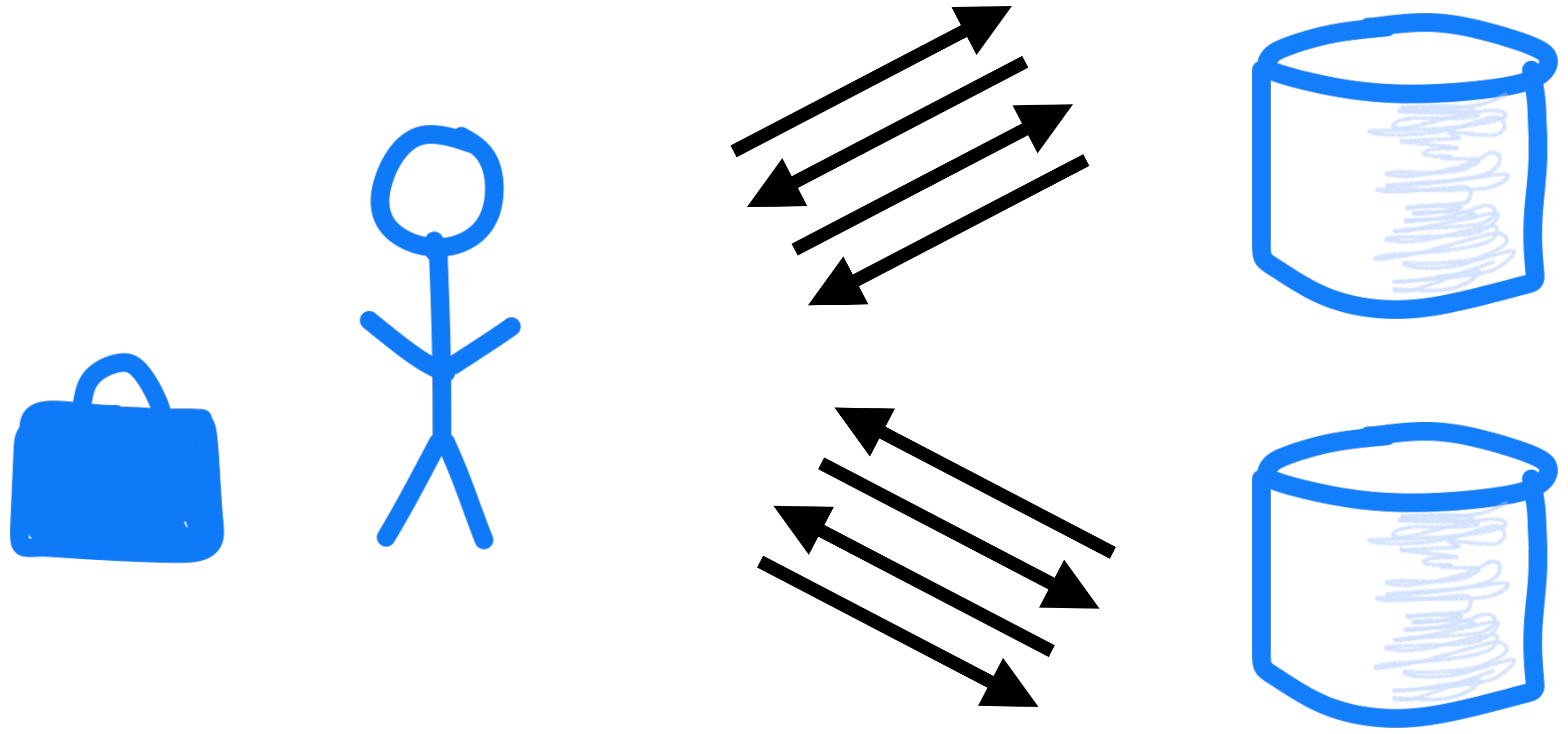

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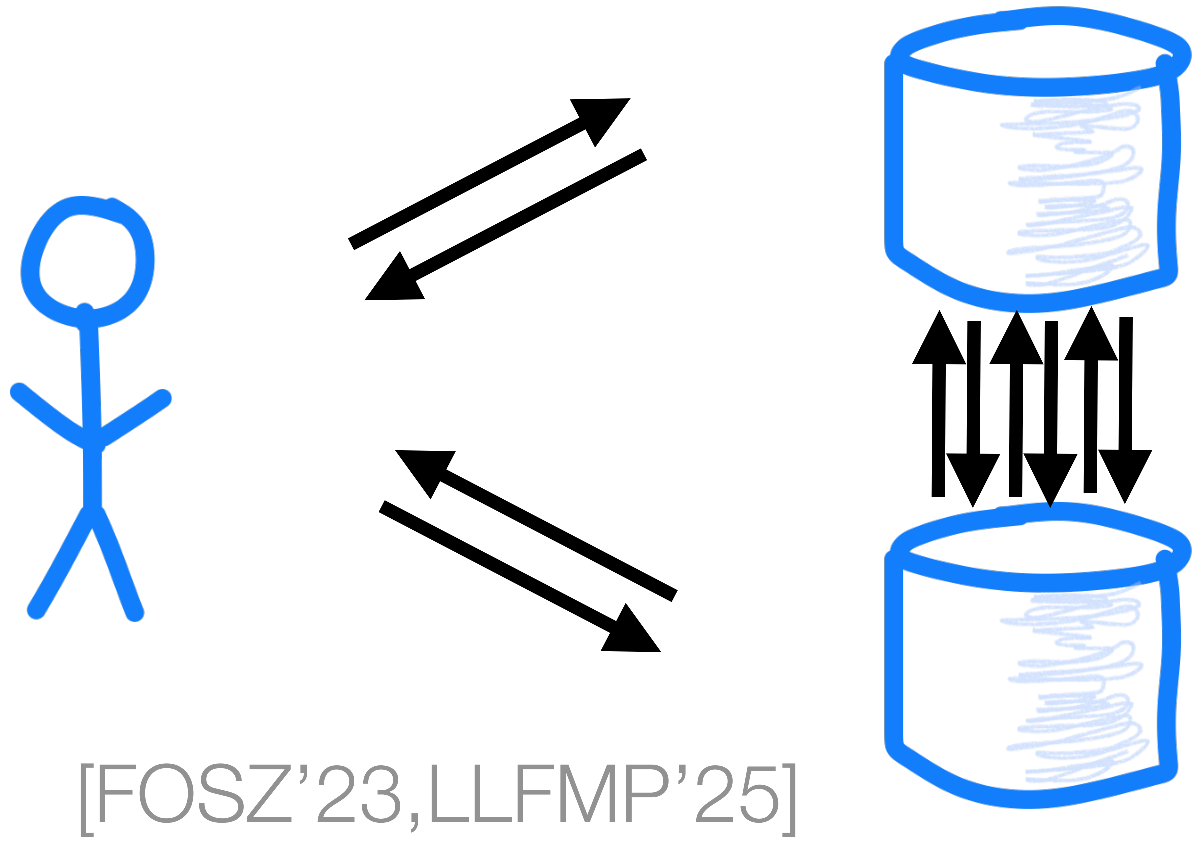

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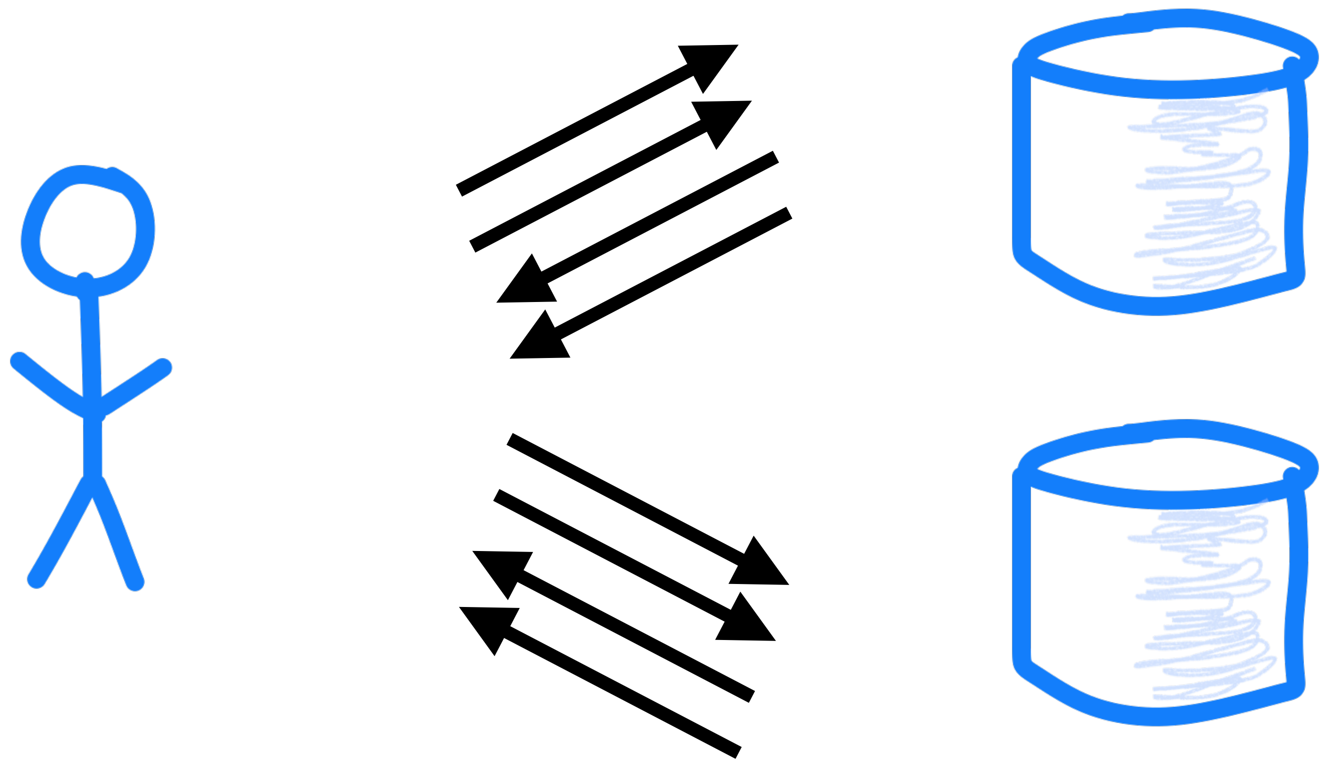

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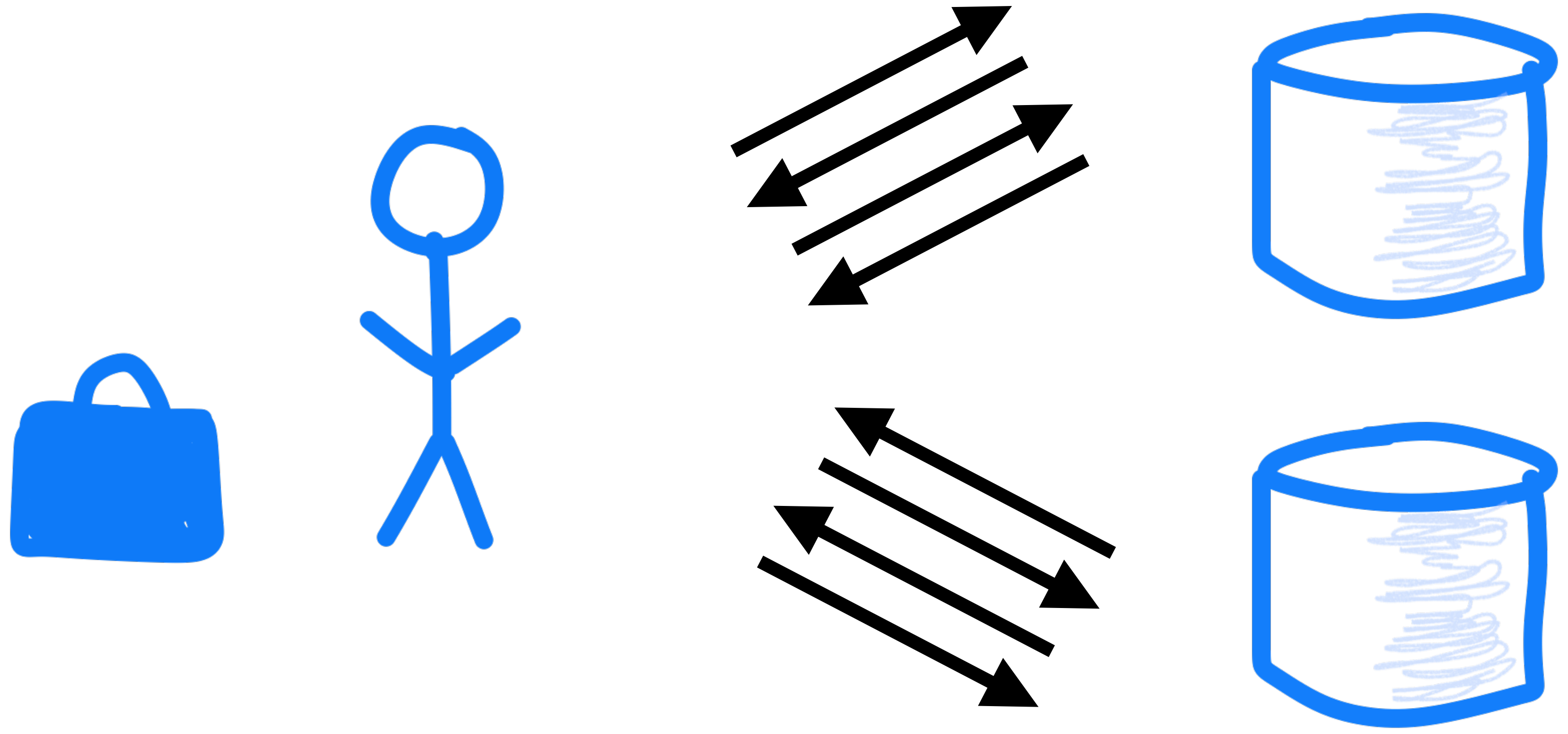

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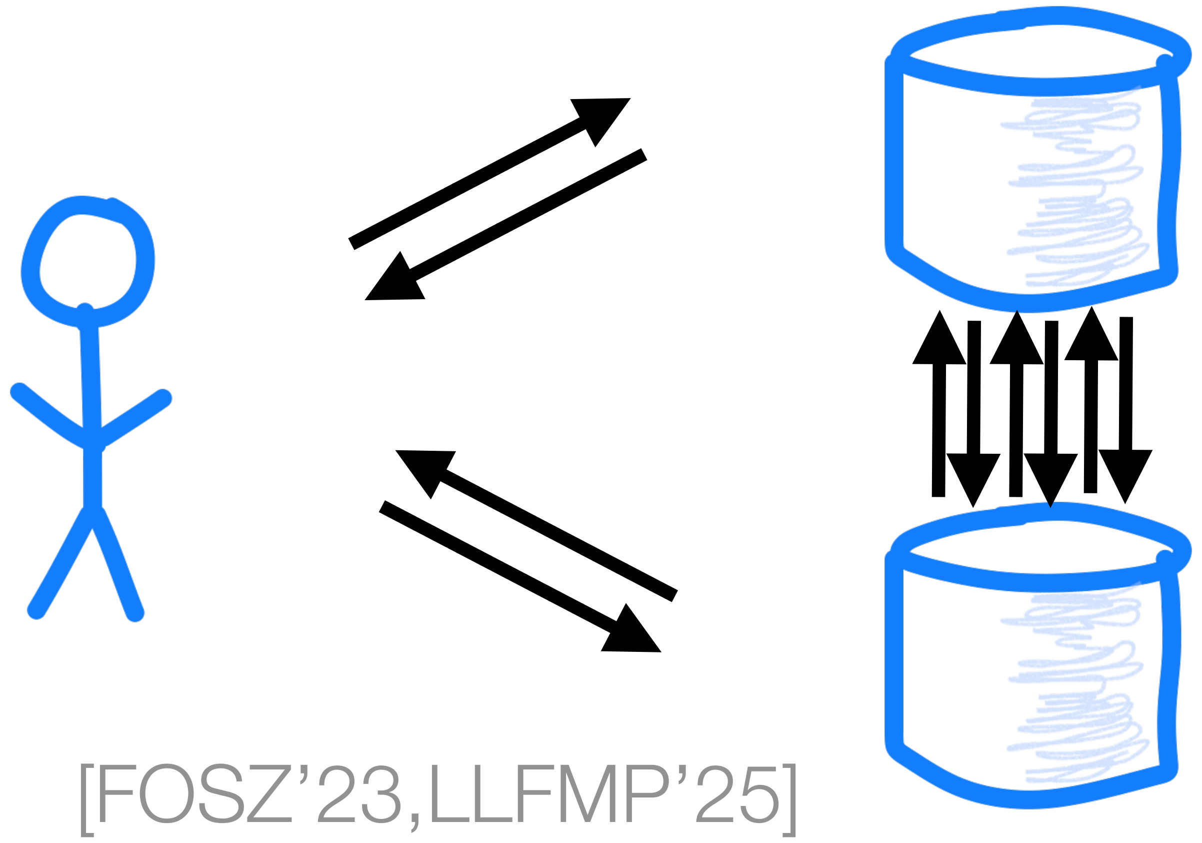

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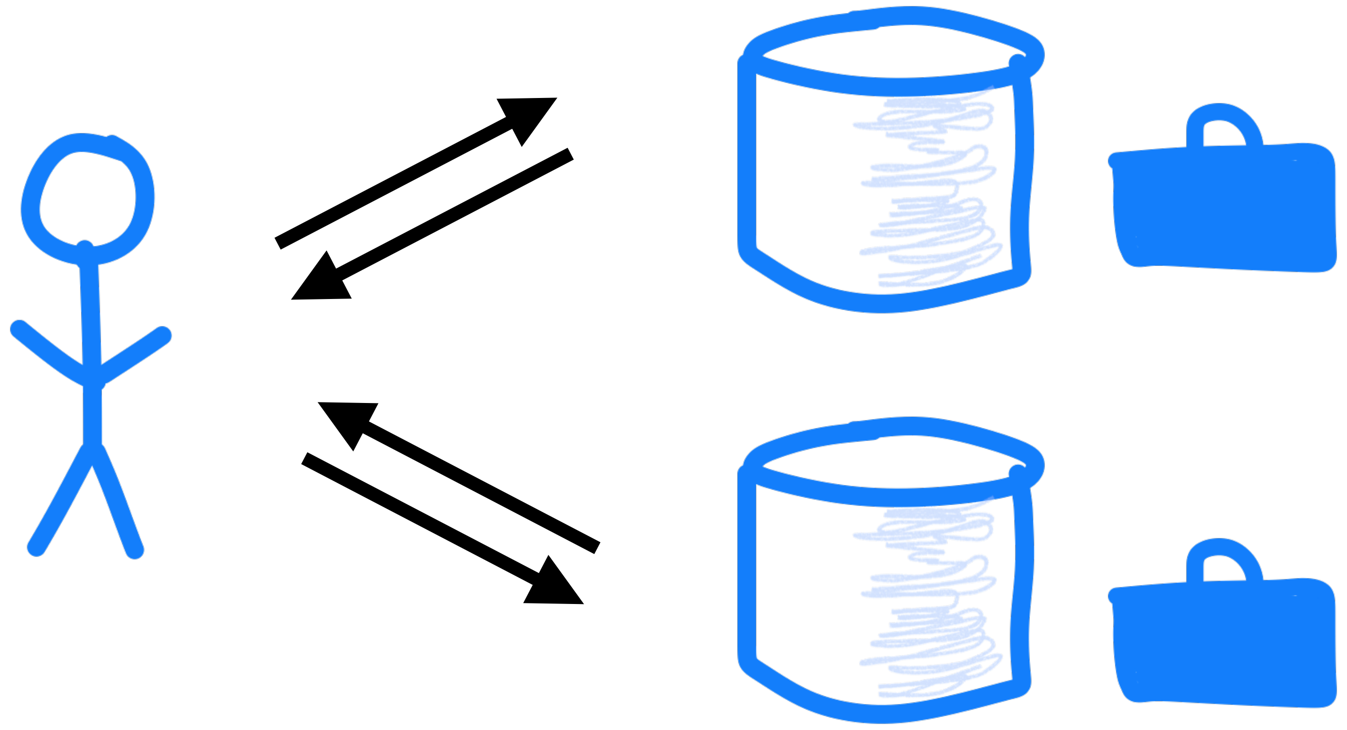
PIR with preprocessing



[BIM'00,BIPW'17,CHR'17,HOWW'18,LMW'23,GLMDS25]

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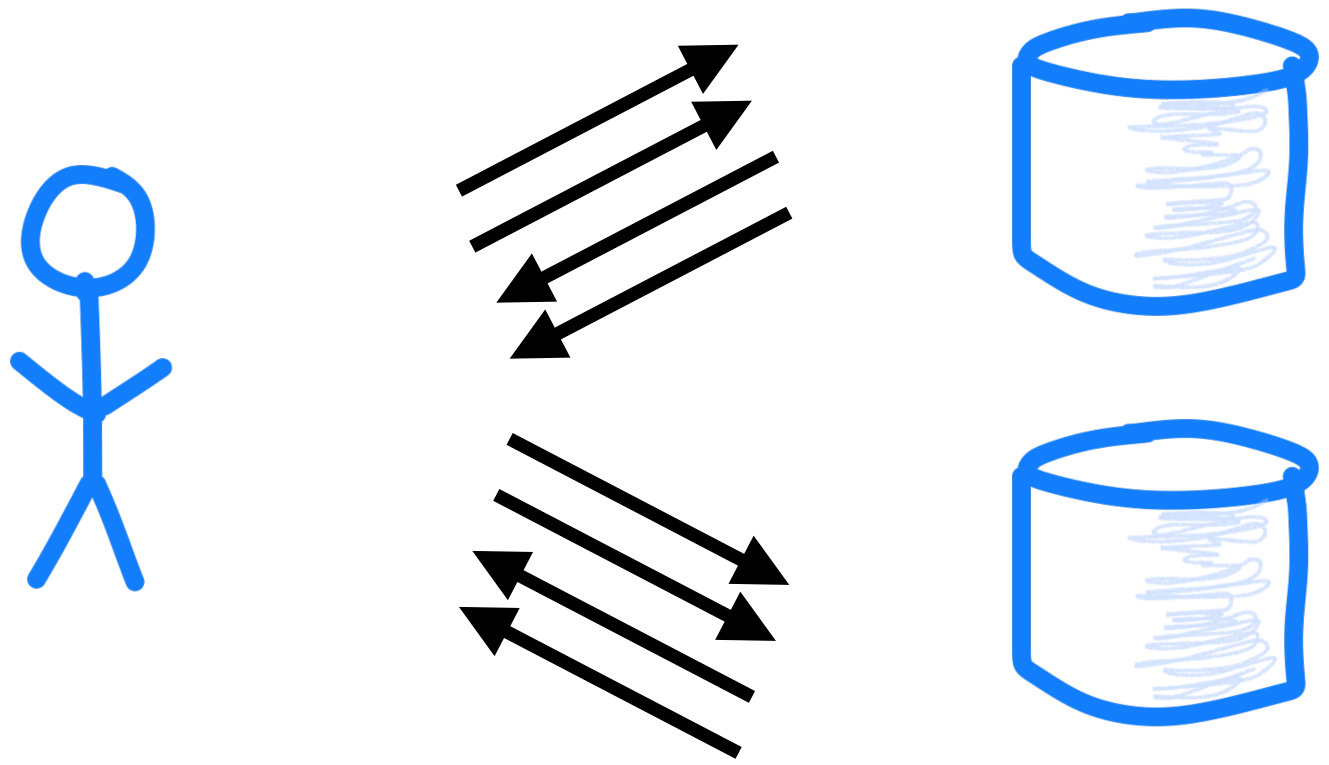

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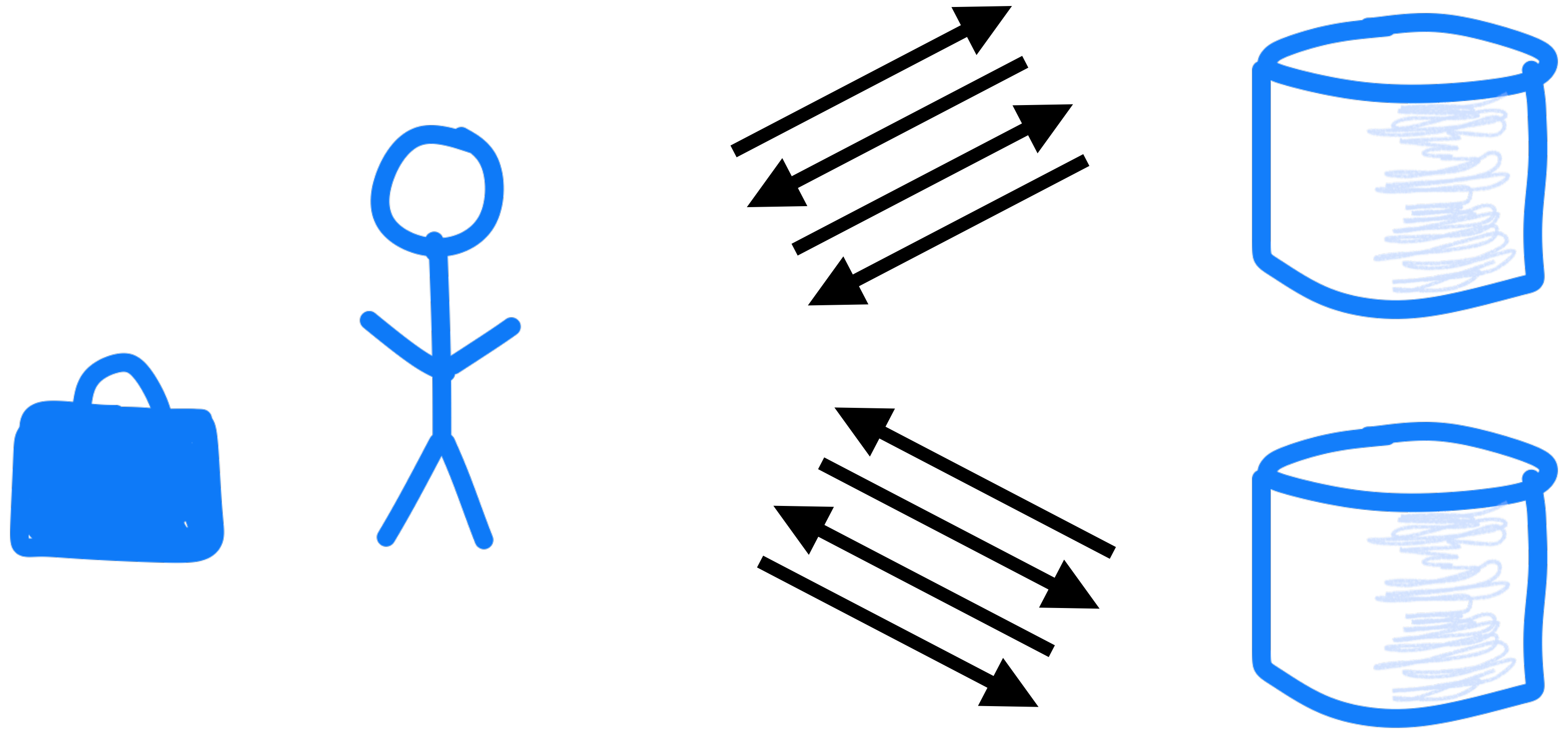

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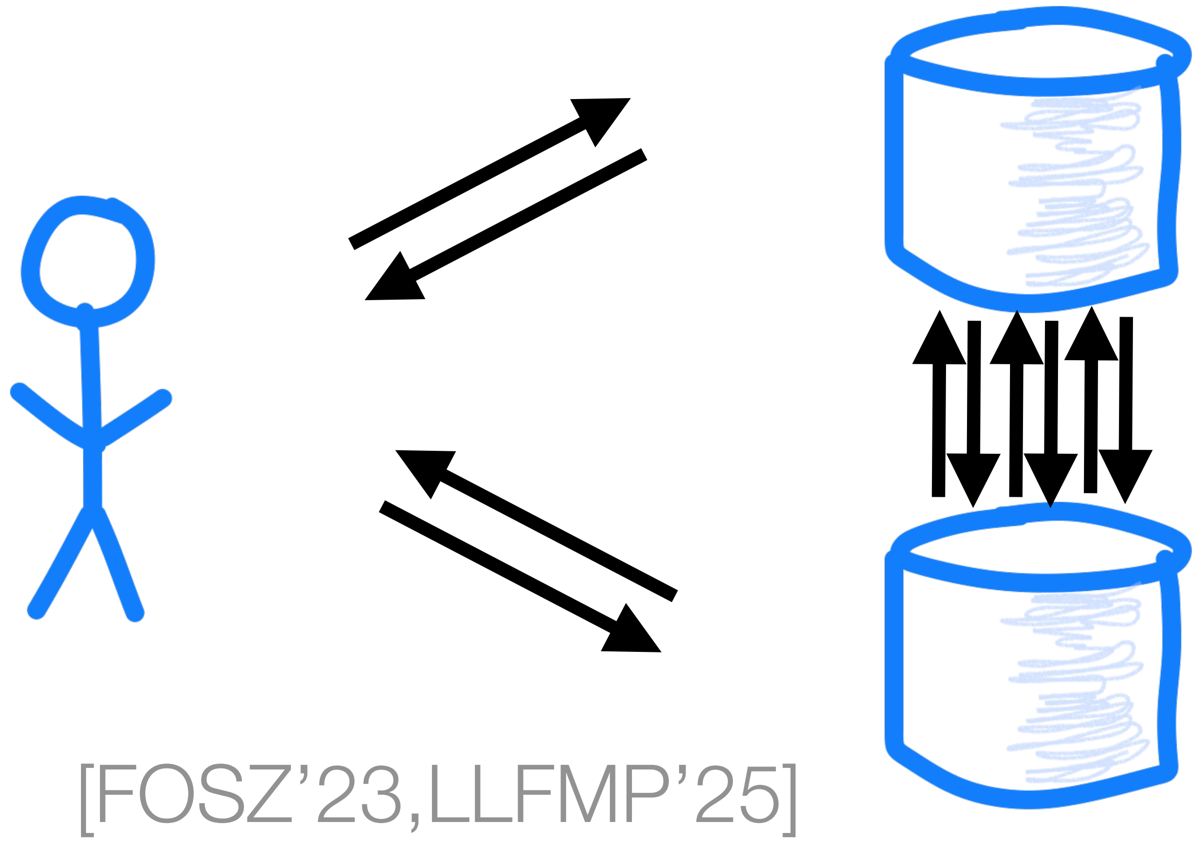

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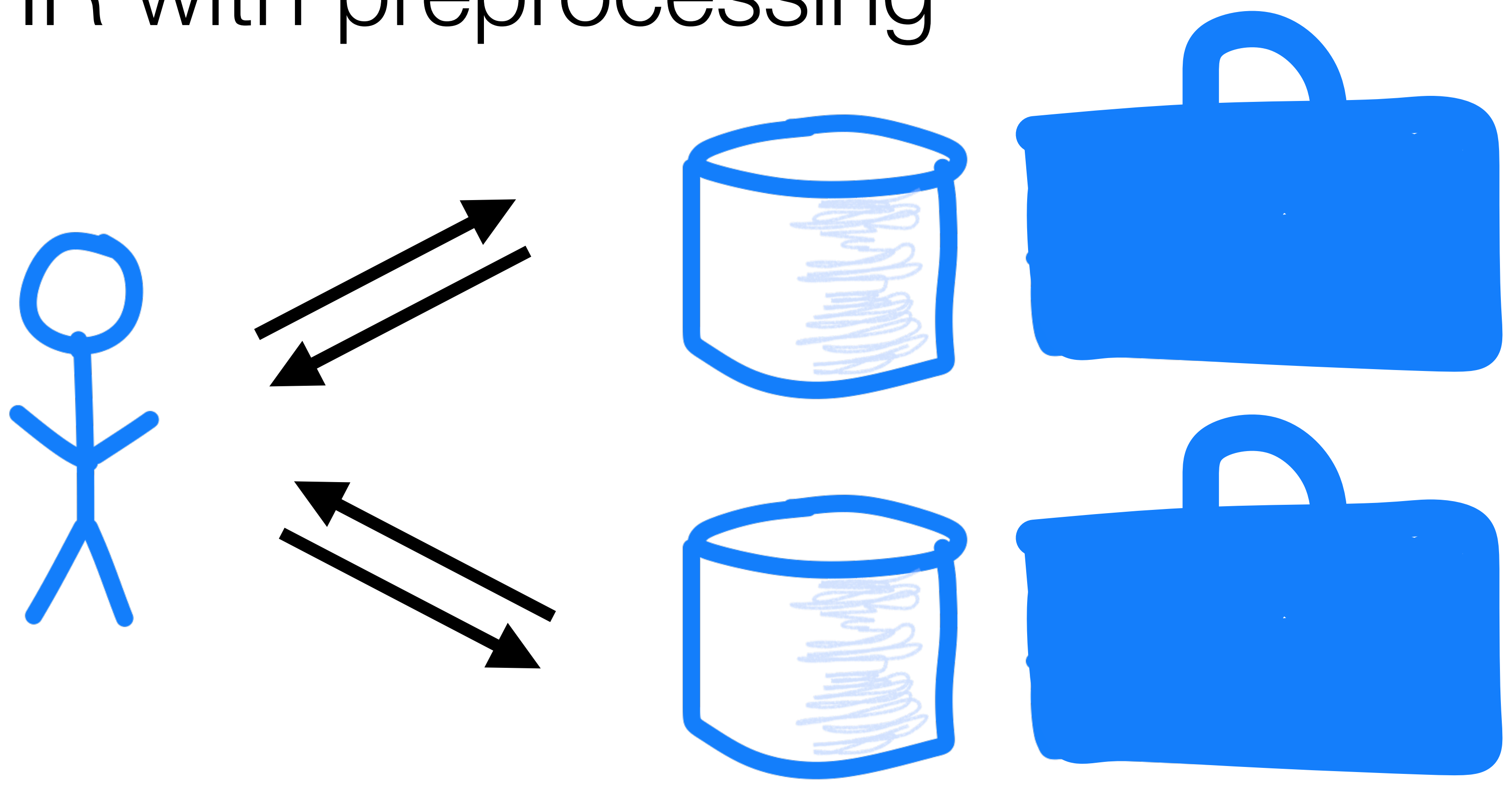

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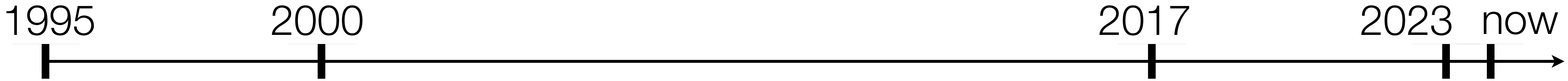
25 years of work on PIR with preprocessing



Info-theoretic
(2 servers)

Computational
(1 server)

Concrete storage
(2 GB database)

*Ignoring polylog savings in time

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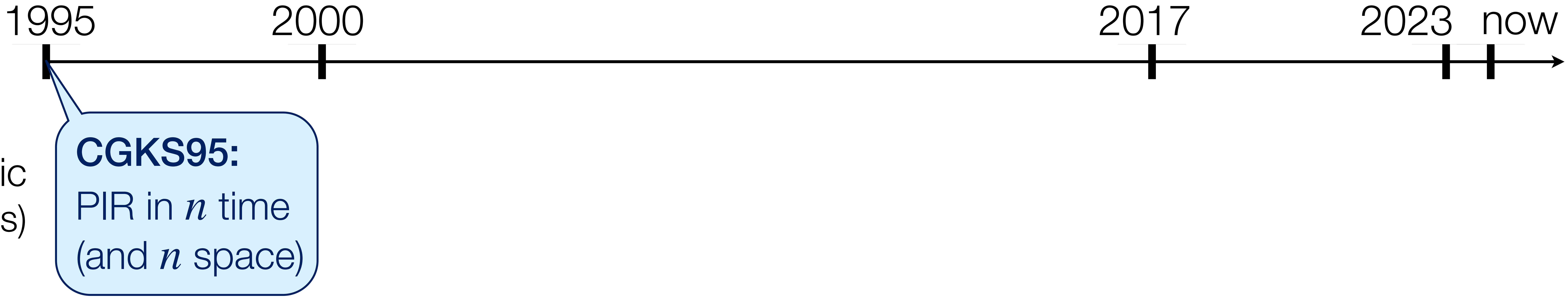

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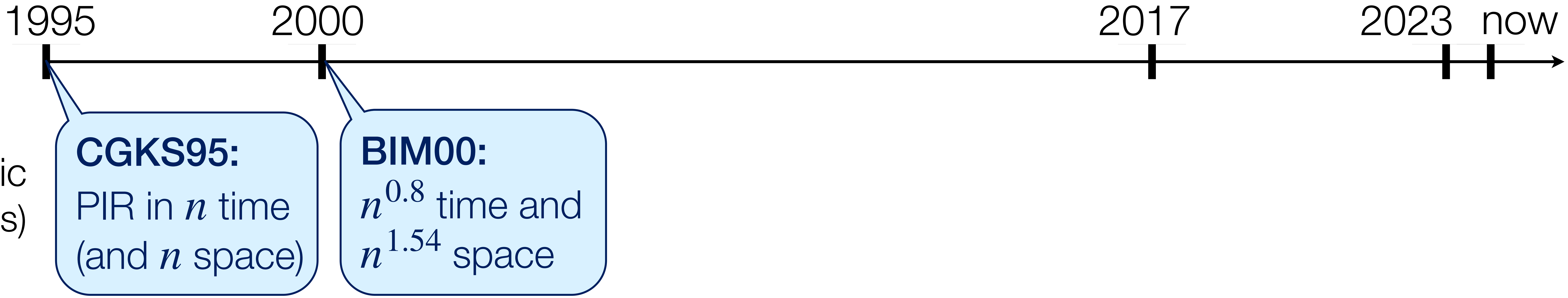

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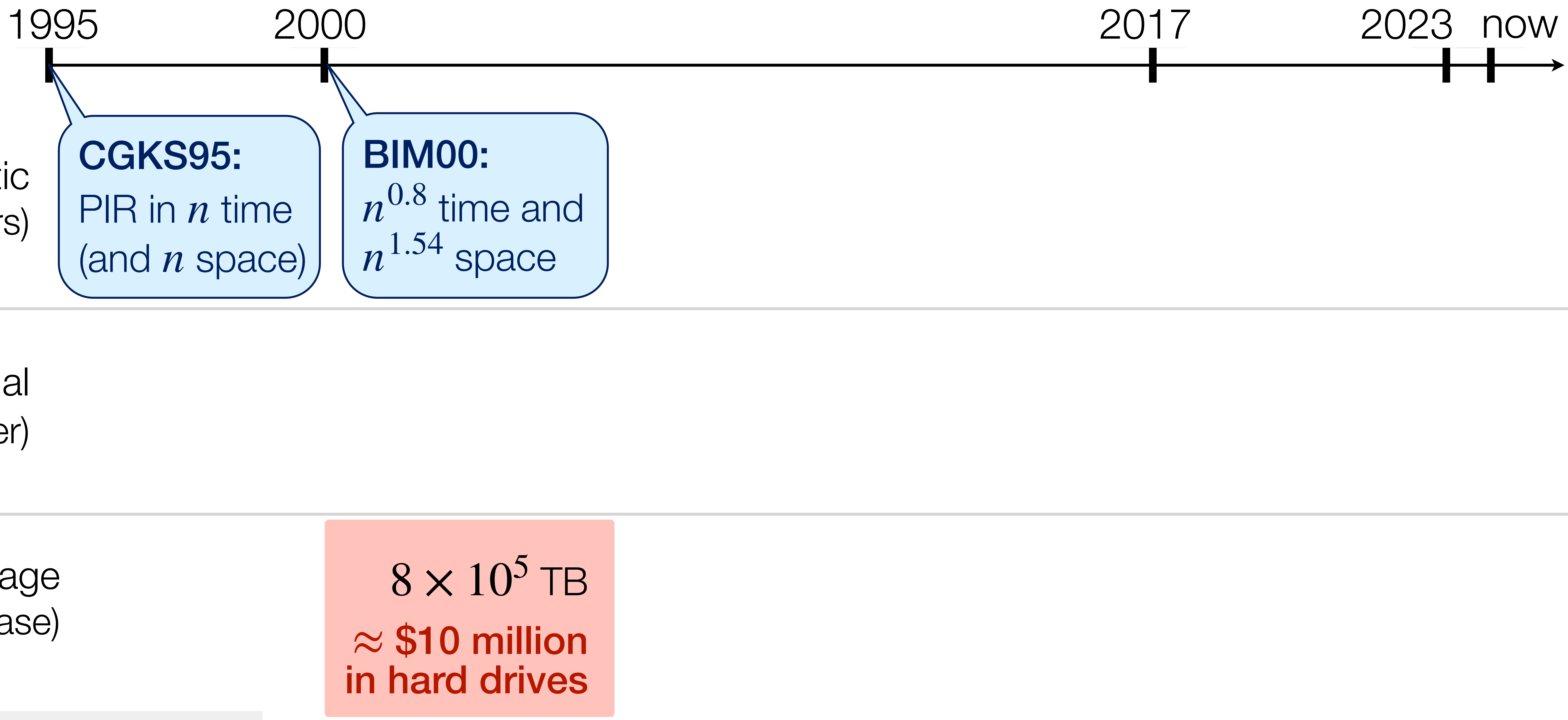

1995 2000 2017 2023 now

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8×10^5 TB

**≈ $10 million
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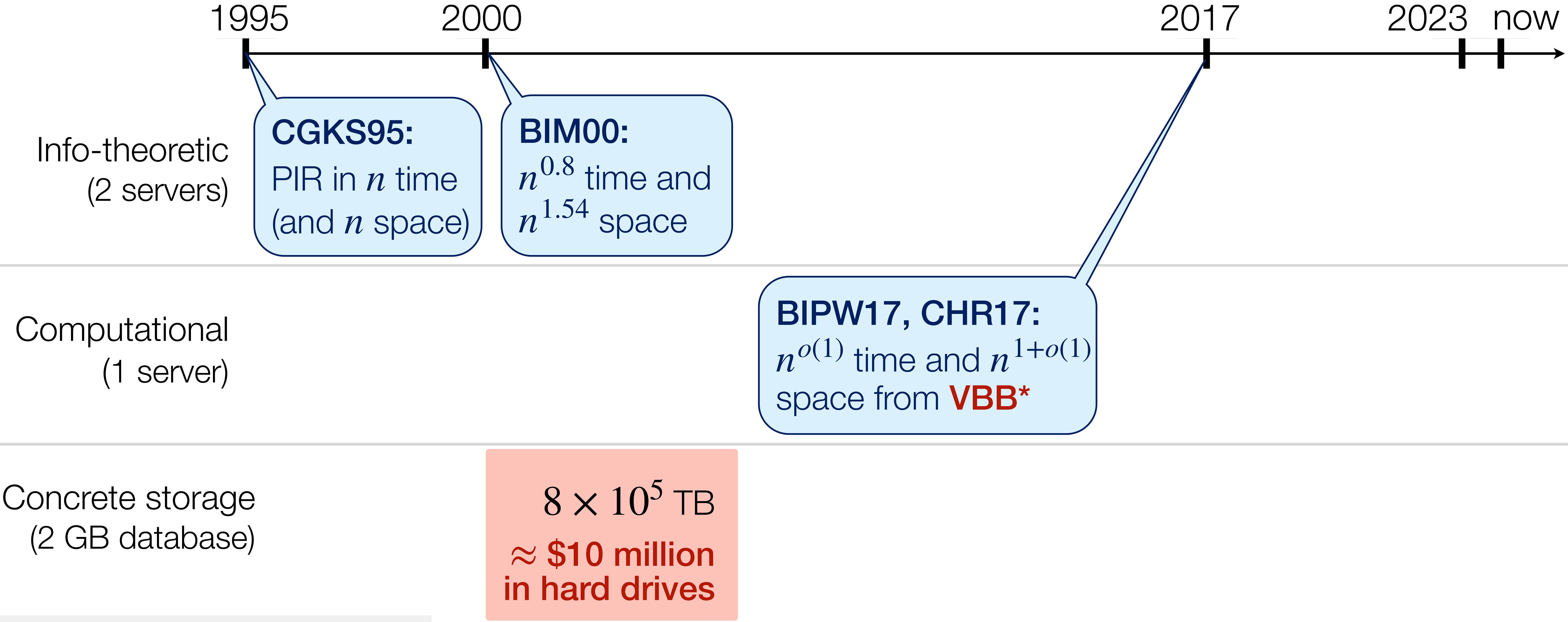

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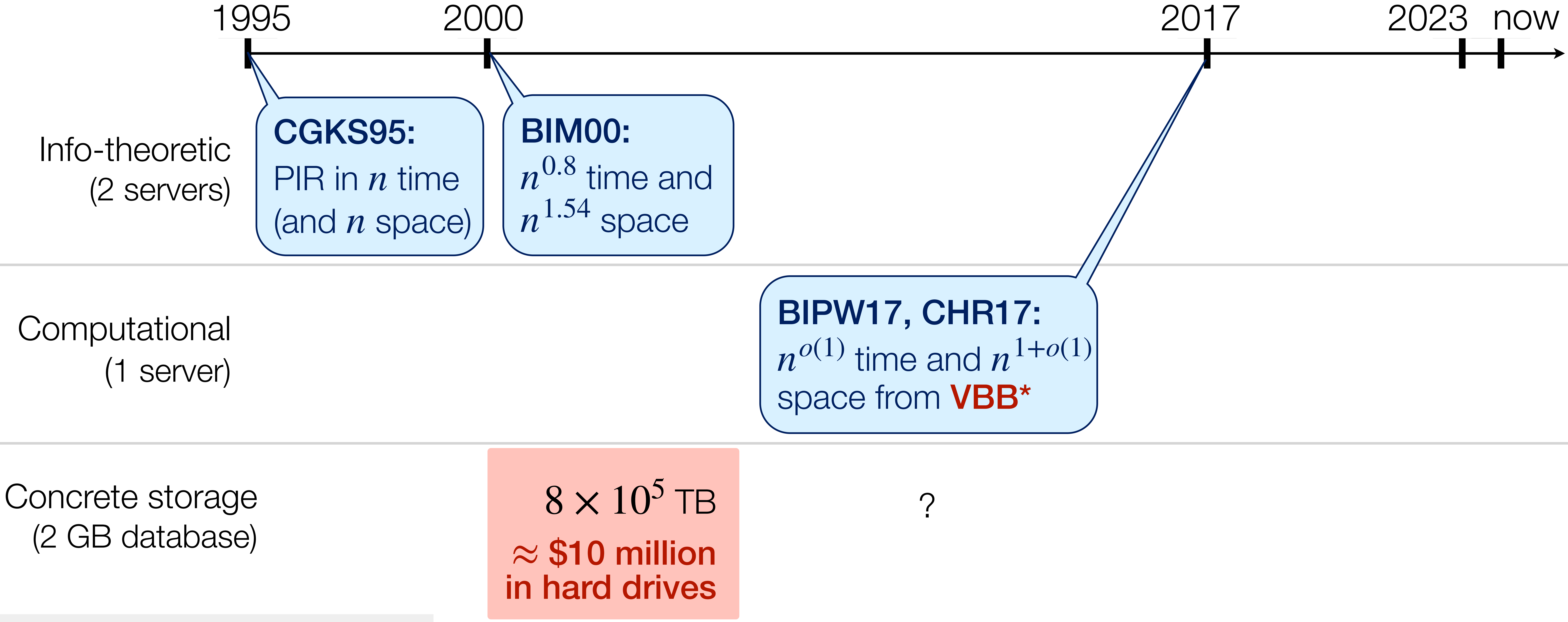

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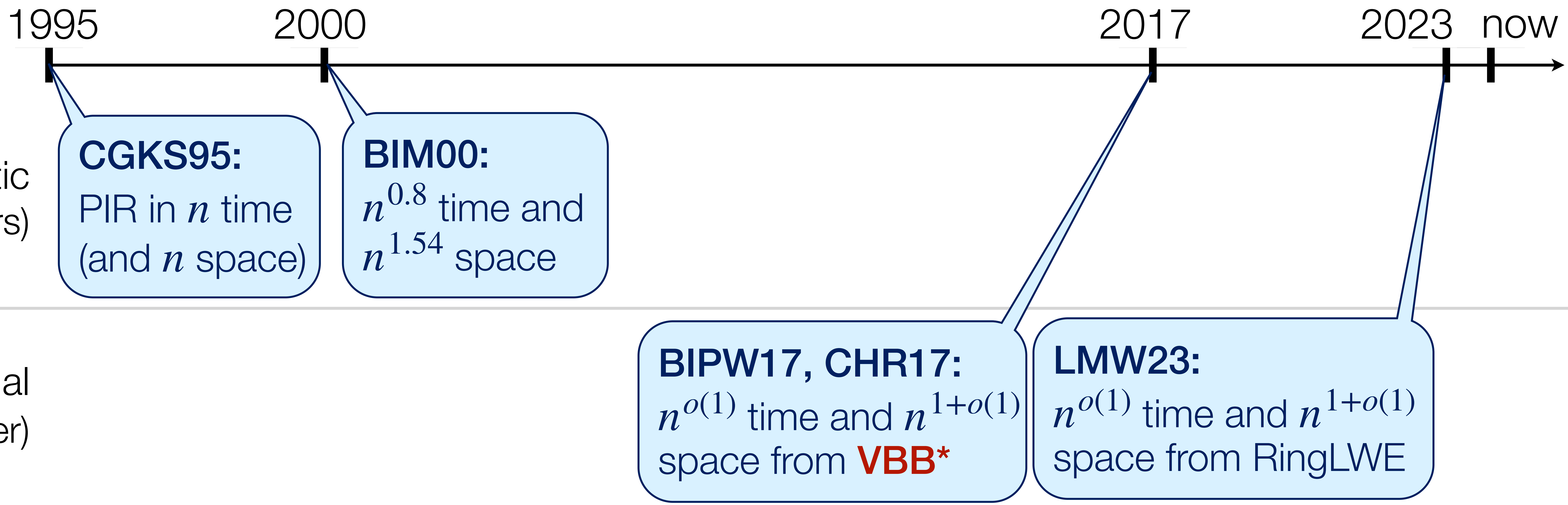

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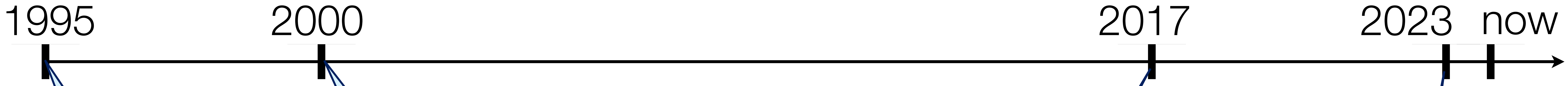

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25 years of work on PIR with preprocessing

1995 | 2000 | 2017 | 2023 | now

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2×10^6 TB

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1 TB

≈ $10 in hard drives

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Theorem. For any database of $n > 10^6$ bits, there exists information-theoretic, two-server PIR with preprocessing with:

- $1.5 \cdot \sqrt{\log_2 n} \cdot n$ bits of storage,
- $12 \cdot n^{0.82}$ server RAM lookups per query,
- $12 \cdot n^{0.82}$ bits of communication.

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1. constant number of servers,
2. quasilinear storage, and
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Corollary 1: with two servers and compact LHE [known from DDH, DCR, QR, LWE], the server time is $n^{0.82} \cdot \mathsf{poly}(\lambda)$ and the communication is $n^{0.31} \cdot \mathsf{poly}(\lambda)$.

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Corollary 2: with two servers and compact FHE*, the server time is $n^{0.82} \cdot$ poly(λ) and the communication is $\log(n) \cdot$ poly(λ).

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Corollary 2: with two servers and compact FHE*, the server time is $n^{0.82} \cdot \mathsf{poly}(\lambda)$ and the communication is $\log(n) \cdot \mathsf{poly}(\lambda)$.

Our schemes support a broader time-space tradeoff, that strictly improves on prior work.

This talk

1. **Background:** PIR with preprocessing
2. **[HR26] New two-server PIR:** sublinear time, quasilinear space
3. **Evaluation:** what does this mean for practice?
4. **Bonuses** 😁
    - Reducing communication using crypto
    - **[HPR26]** Connecting multi-server PIR to complexity theory

This talk

1. **Background:** PIR with preprocessing

2. **[HR26] New two-server PIR:** sublinear time, quasilinear space
3. **Evaluation:** what does this mean for practice?
4. **Bonuses** 😁
    - Reducing communication using crypto
    - **[HPR26]** Connecting multi-server PIR to complexity theory

Prior information-theoretic PIR

[BIM00,GLMDS25]

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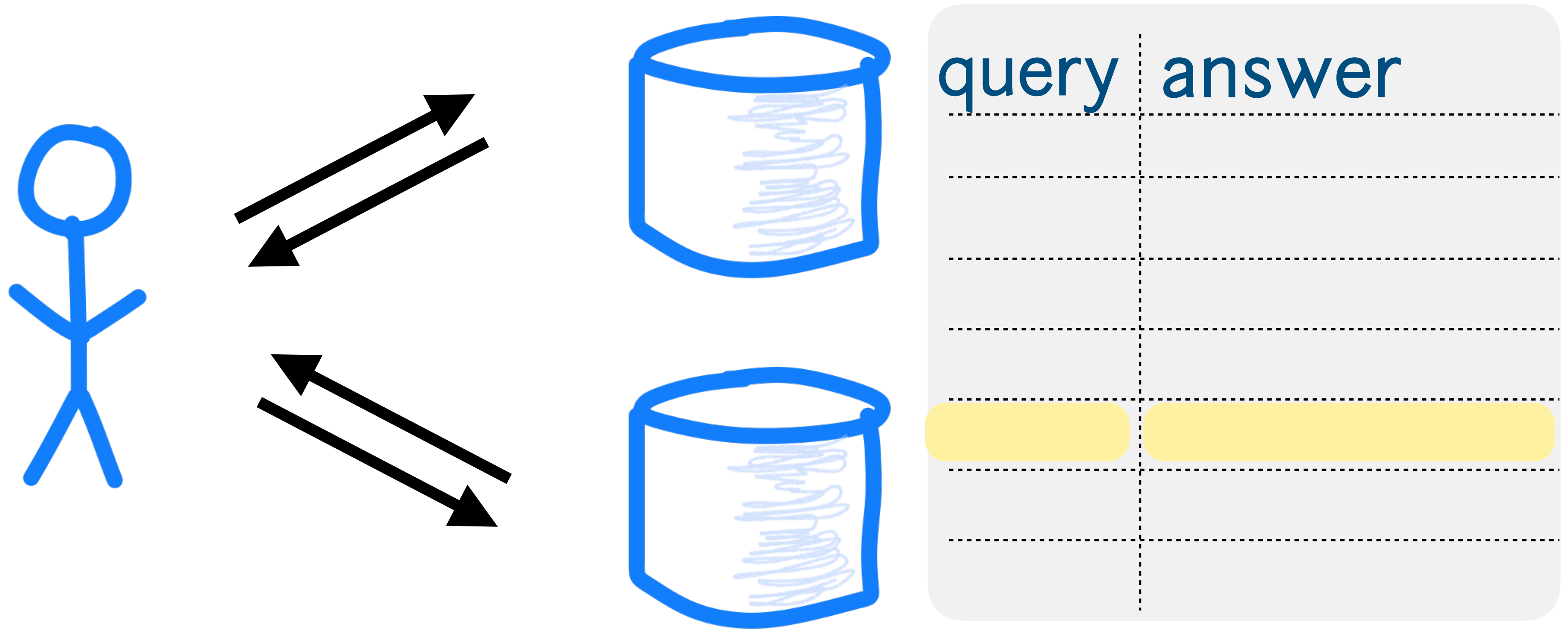
1. Build "imbalanced" PIR with tiny queries
    - Query length $= (1 + o(1)) \cdot \log n$
    - Answer length $= \ell = O(n^{0.82})$

2. Precompute the answer to every query
    - To answer a query: read 1 location of length ℓ

➡ PIR in $n^{1.82+o(1)}$ space and $n^{0.82}$ time



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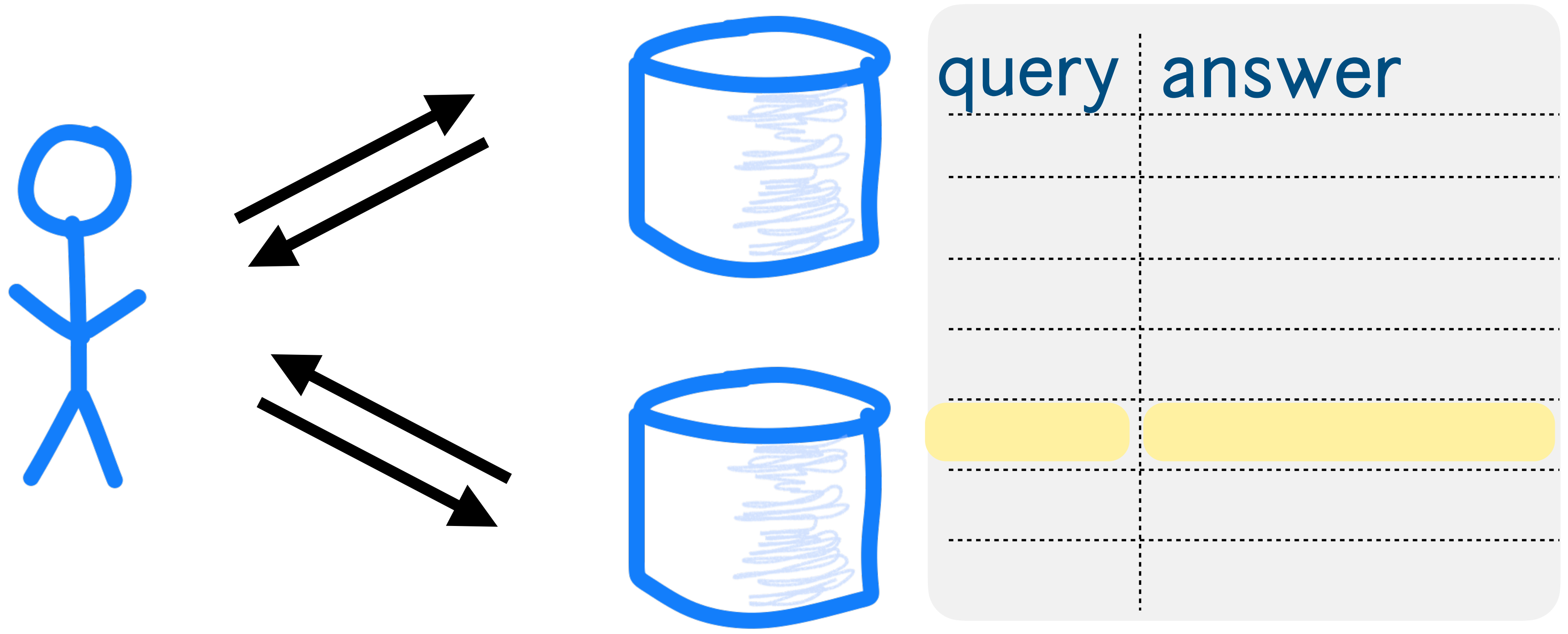

This work

2. New data structure
    - To answer a query: read ℓ locations of length 1

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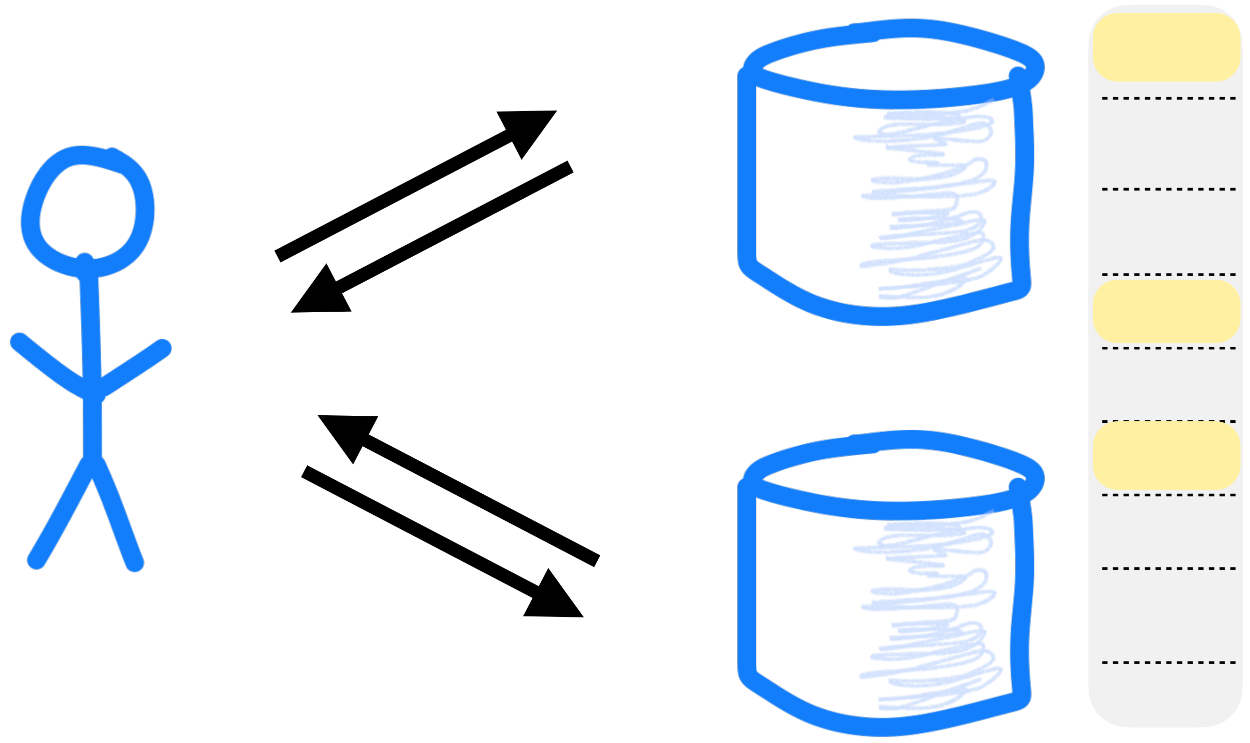
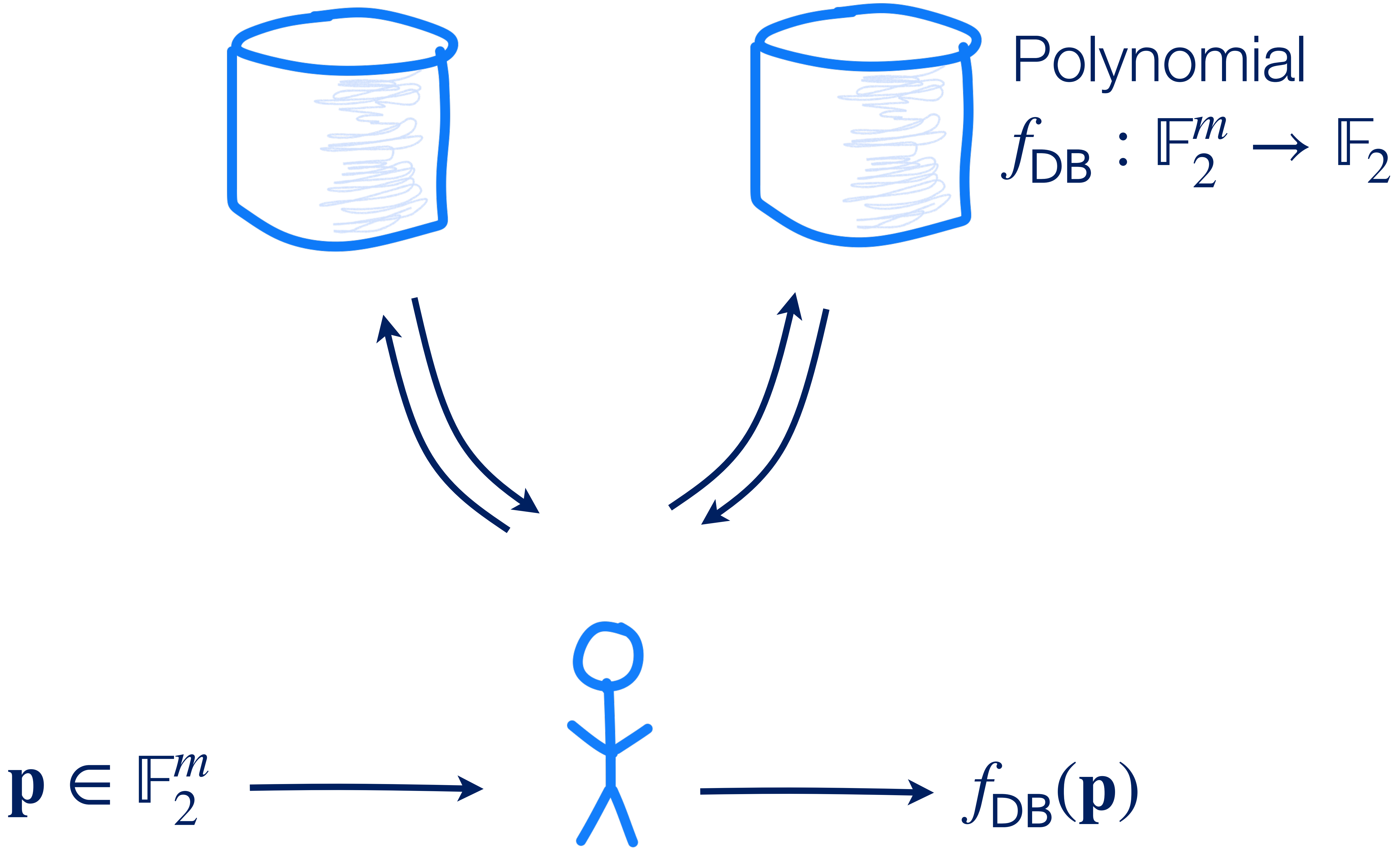


Starting point: PIR from private polynomial evaluation

A common step in [BIM00,BIKR02,BIK05,WY05,BV11…]



Polynomial

$f_{\mathsf{DB}} : \mathbb{F}_2^m \to \mathbb{F}_2$

$\mathbf{p} \in \mathbb{F}_2^m$

$f_{\mathsf{DB}}(\mathbf{p})$

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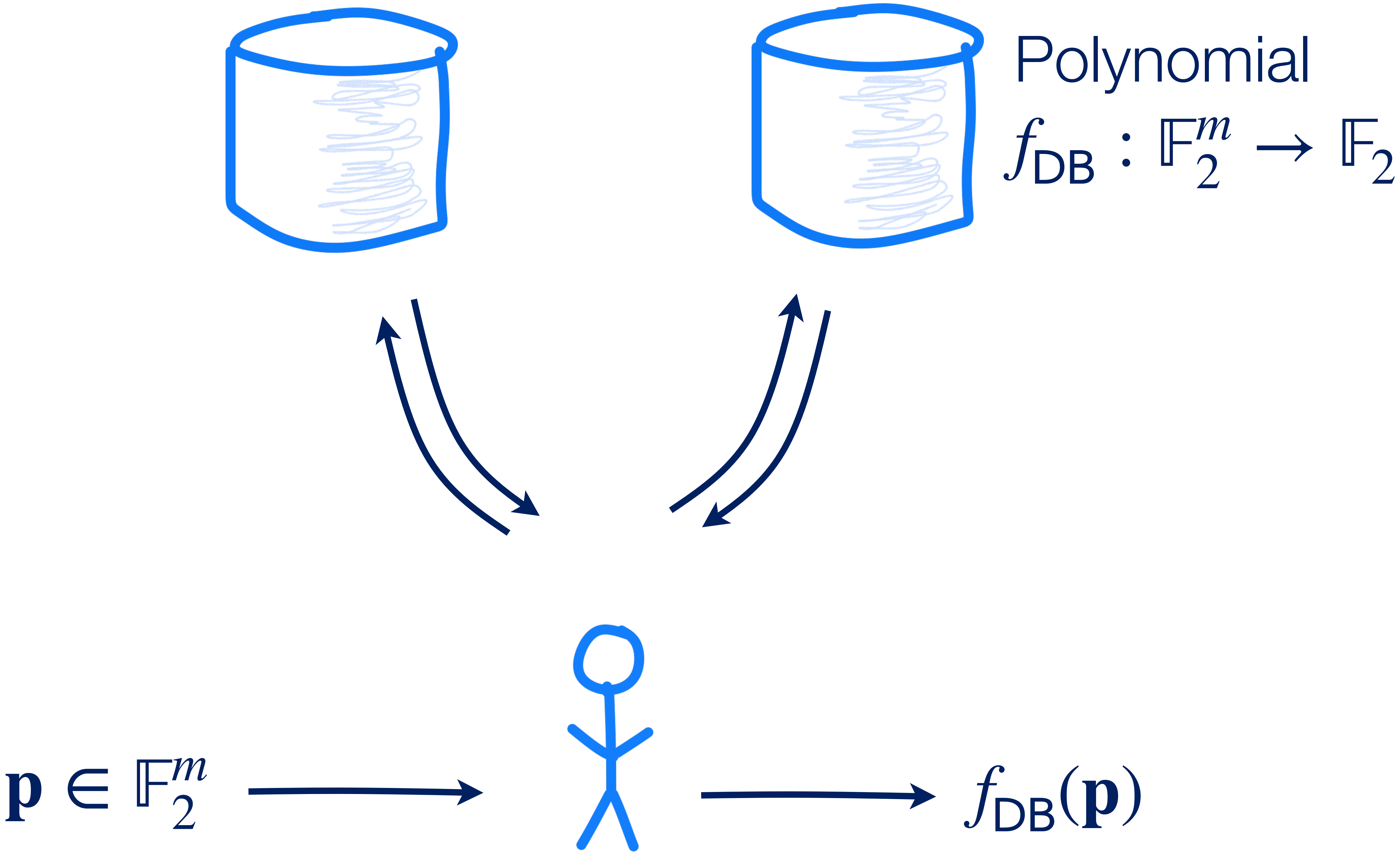
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We need: f_{DB} encodes the database

1. f_{DB} is homogenous and degree-D
2. $\binom{m}{D} \geq n$
3. $\mathbf{E}(j) = j$-th point in $\mathbb{F}_2^m$ of weight D
4. $\forall j \in [n], \quad f_{\mathsf{DB}}(\mathbf{E}(j)) = \mathsf{DB}_j$

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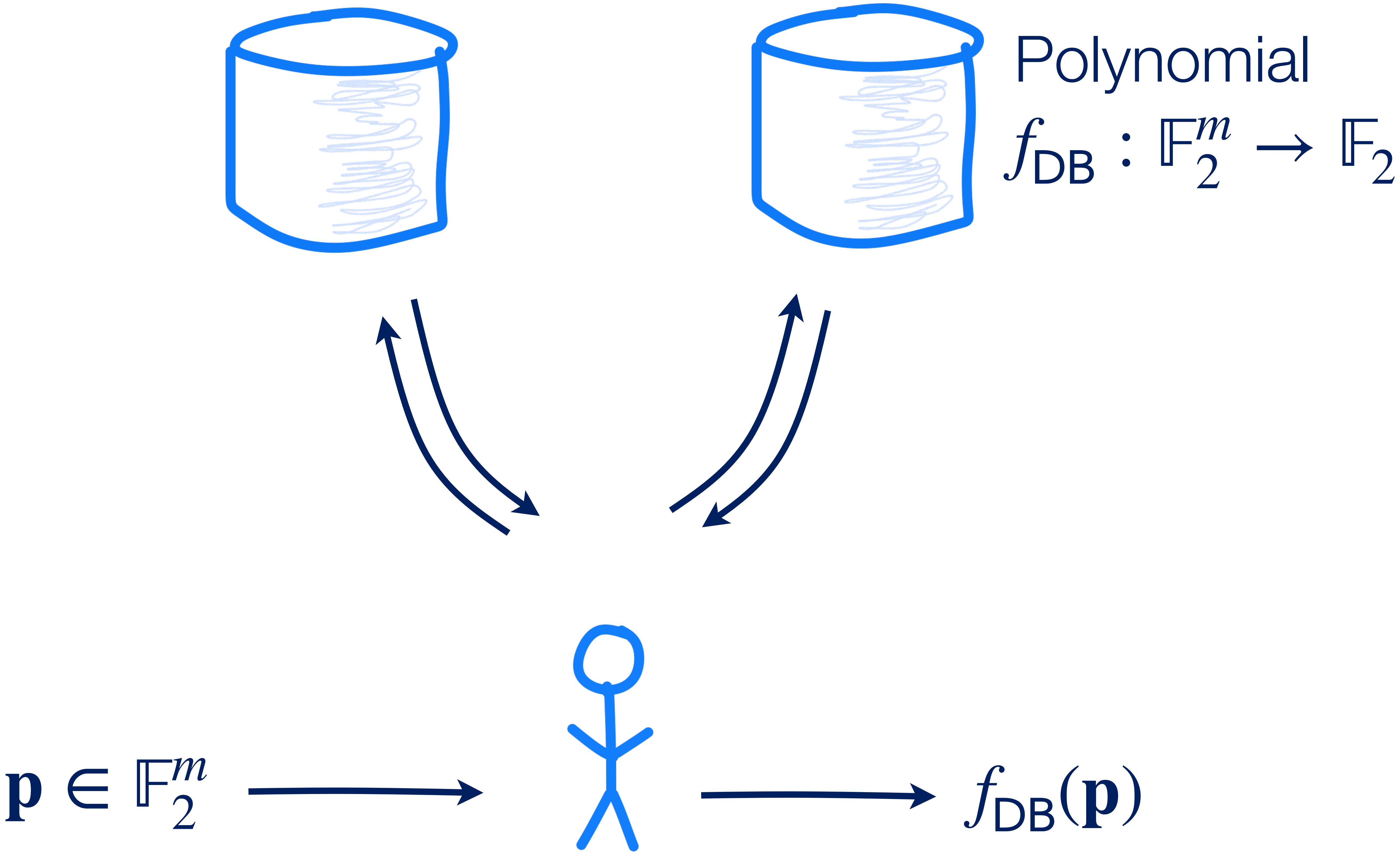

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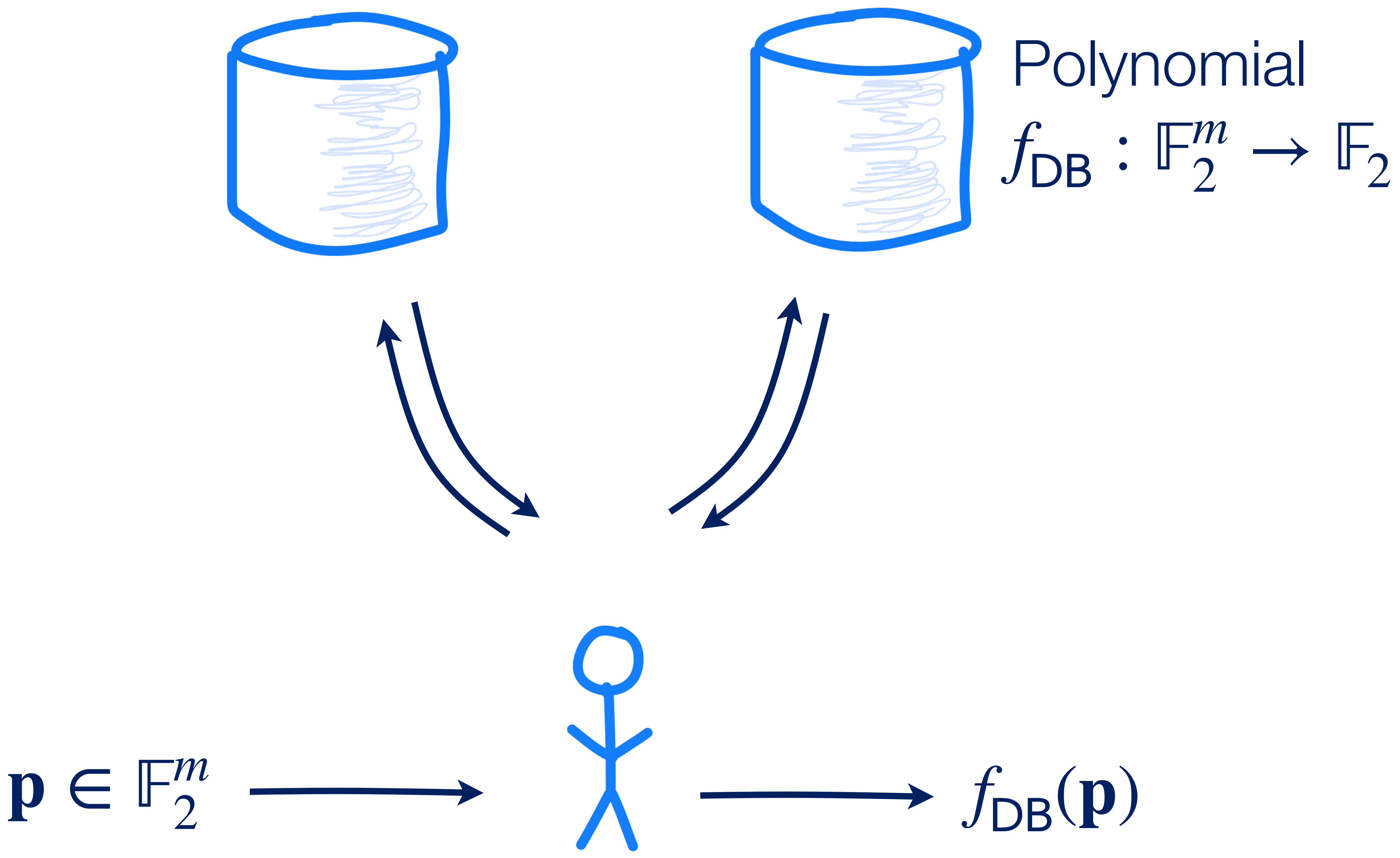
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Privacy: each server learns nothing about $\mathbf{p}$, even if the server is malicious.

Starting point: PIR from private polynomial evaluation

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More explicitly:

$$f_{\mathsf{DB}}(\mathbf{y} \in \mathbb{F}_2^m) = \sum_{j=1}^{n} \mathsf{DB}_j \cdot \mathbf{y}^{\mathbf{E}(j)}, \text{ where}$$

$$\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^{m} a_i^{b_i}.$$

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PIR from private polynomial evaluation

Scheme 1a: from Lagrange Interpolation [Sha79]

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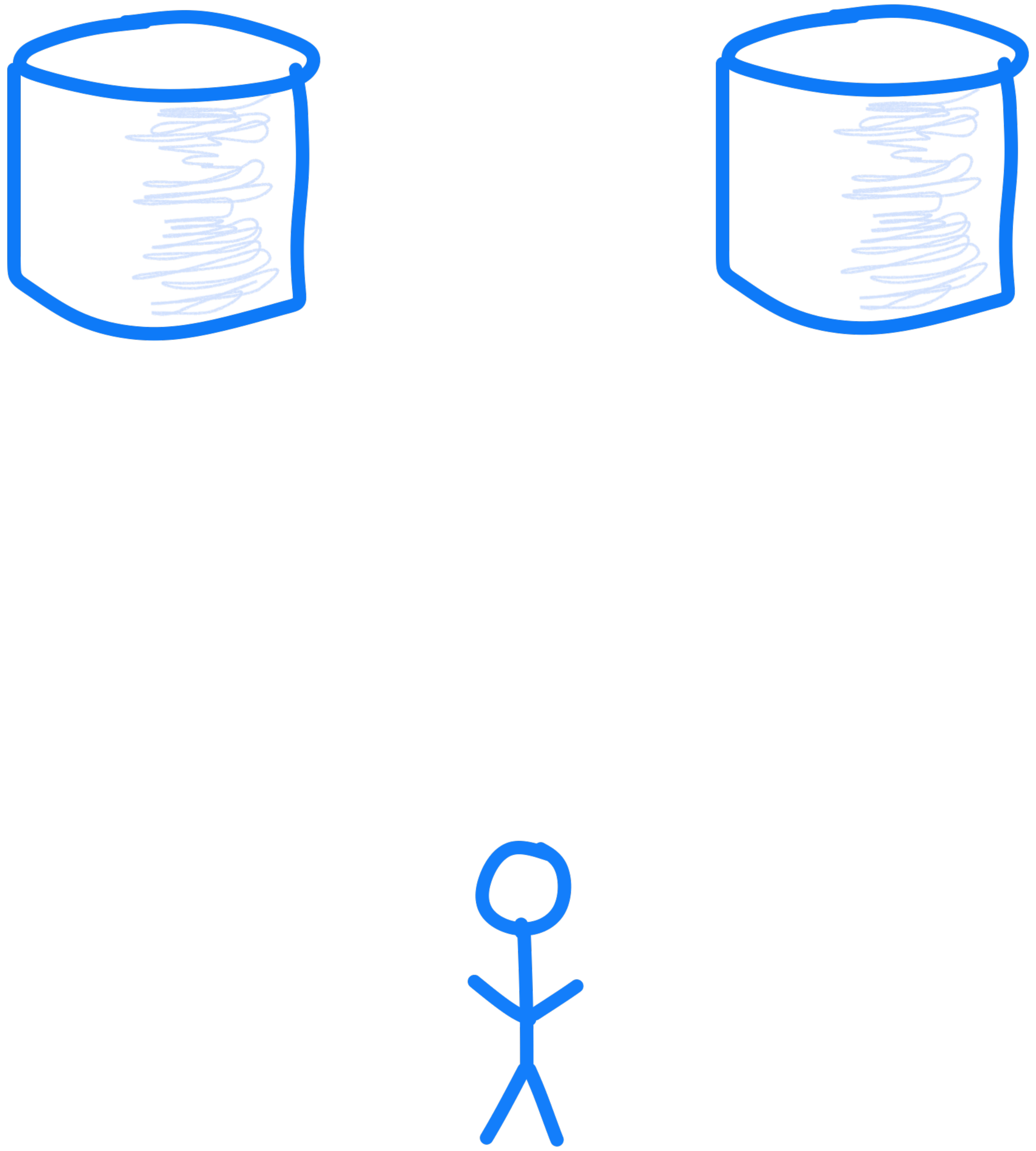

Homogenous degree-D

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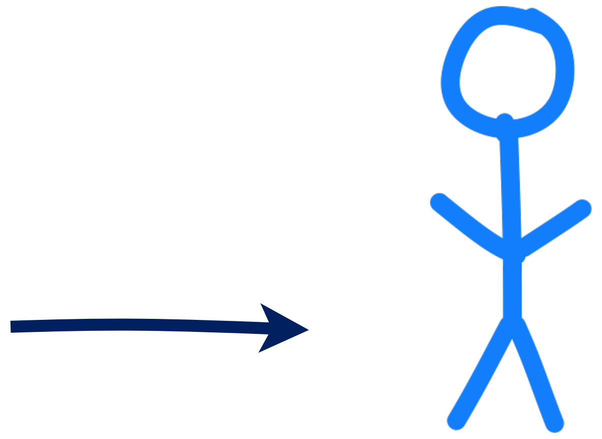
Sample line

$\mathbf{L}(t) = \mathbf{r} + t \cdot \mathbf{p}$



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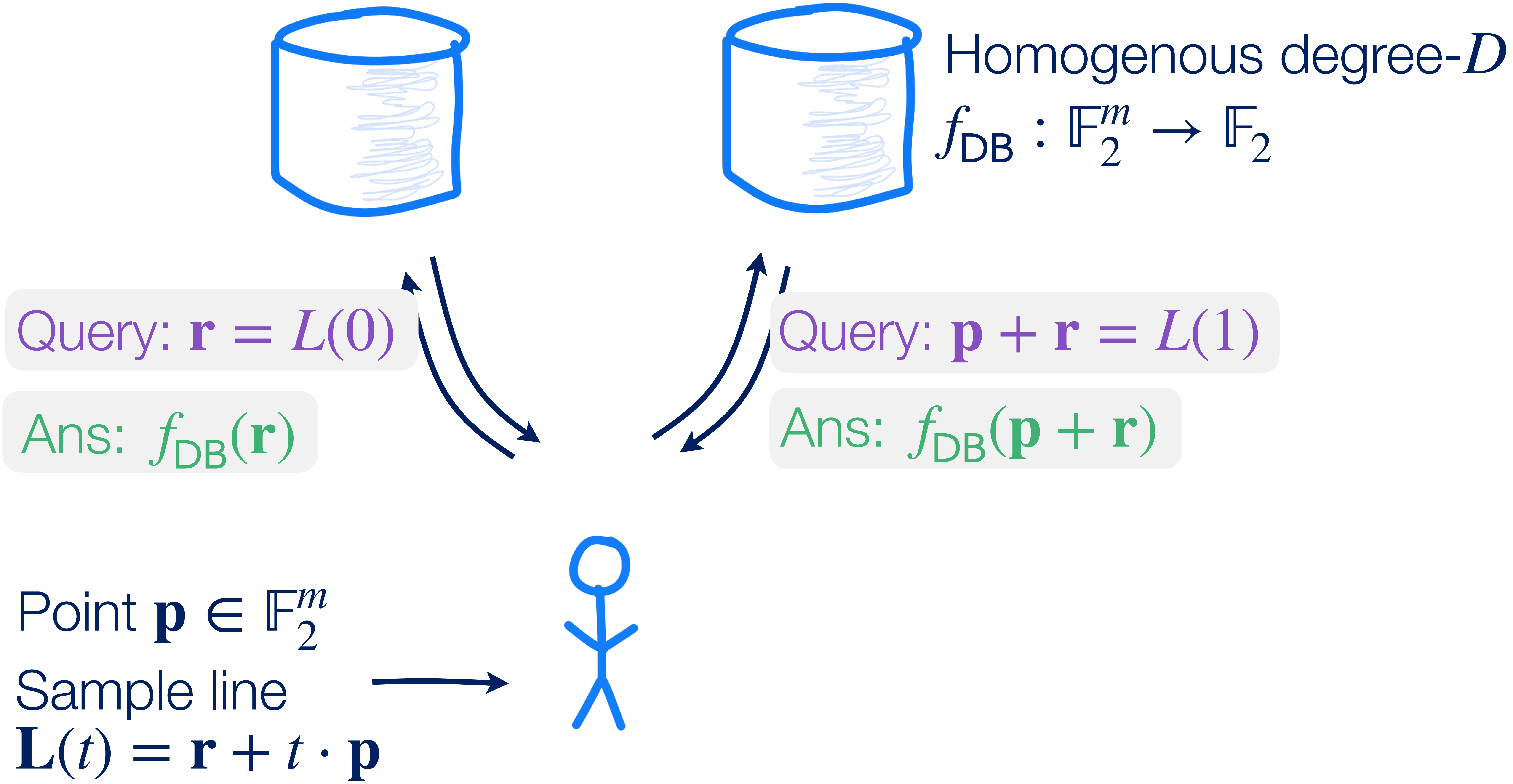

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Query: $\mathbf{r} = L(0)$

Ans: $f_{\mathsf{DB}}(\mathbf{r})$

Query: $\mathbf{p} + \mathbf{r} = L(1)$

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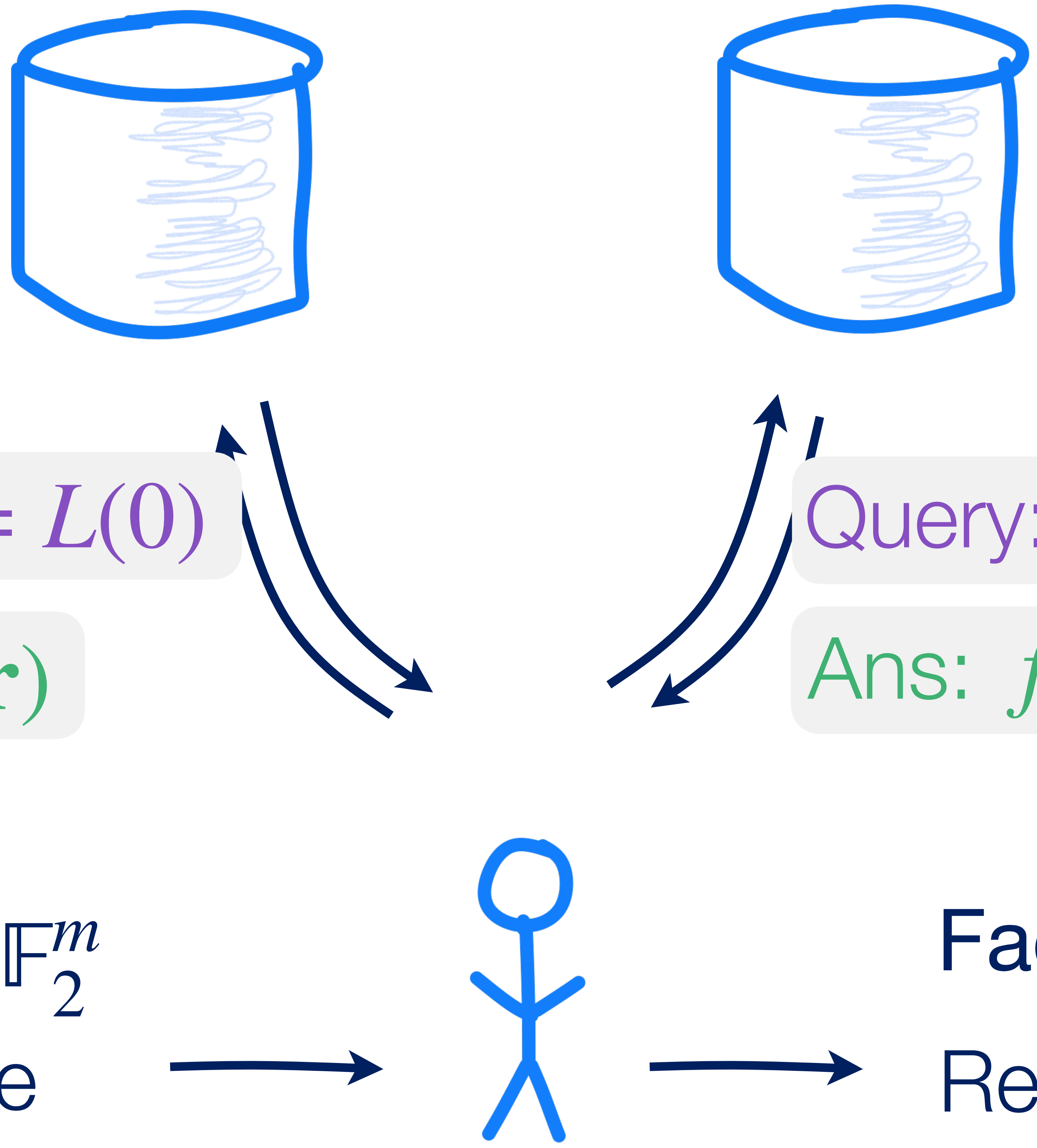

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Fact: leading coefficient of $f_{\text{DB}}(\mathbf{L}(t))$ is $f_{\text{DB}}(\mathbf{p})$

Recover $f_{\text{DB}} \circ \mathbf{L}$ from its evaluations at $D + 1$ points, via Lagrange interpolation

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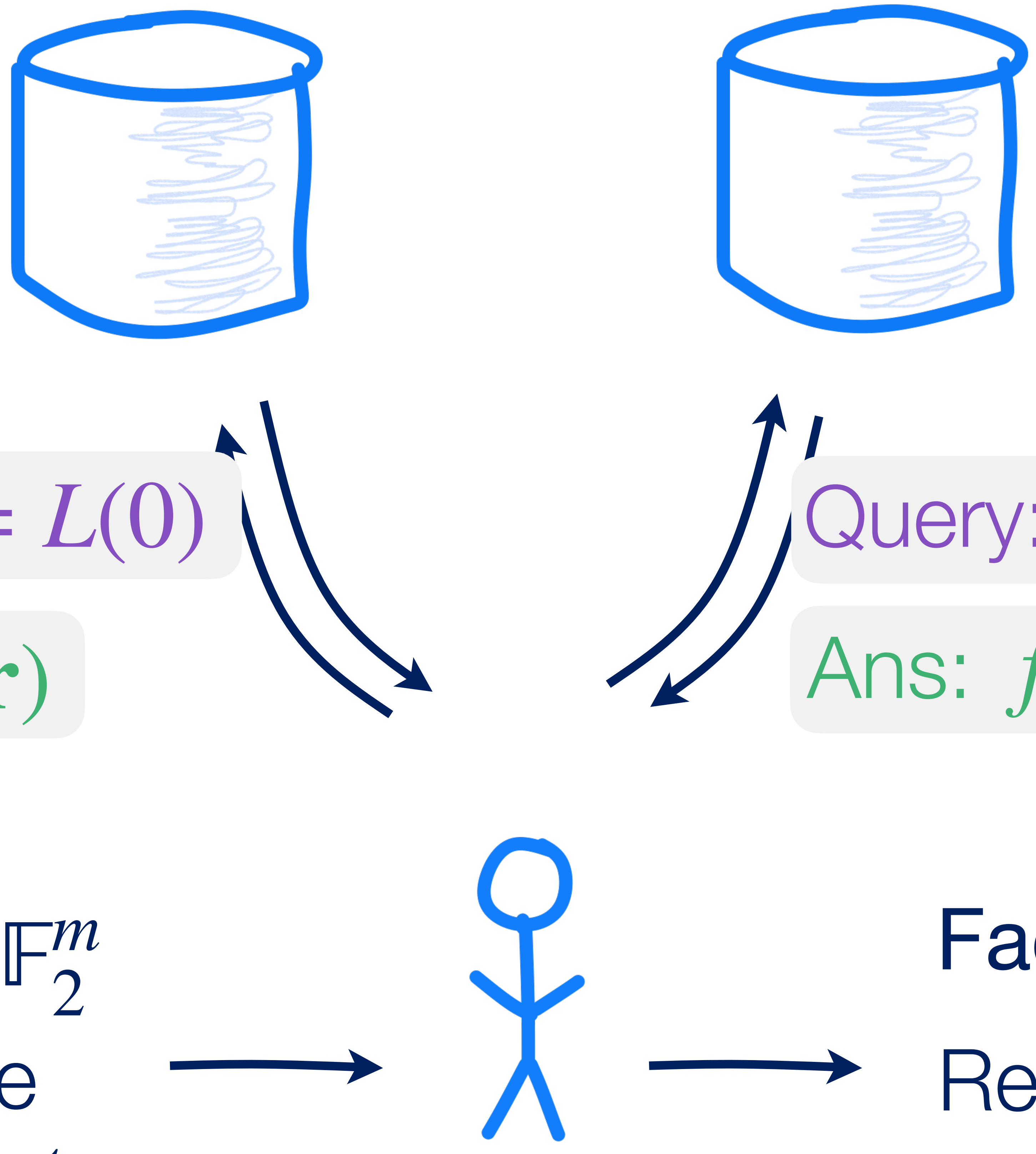

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With 2 servers, forces $D = 1$!

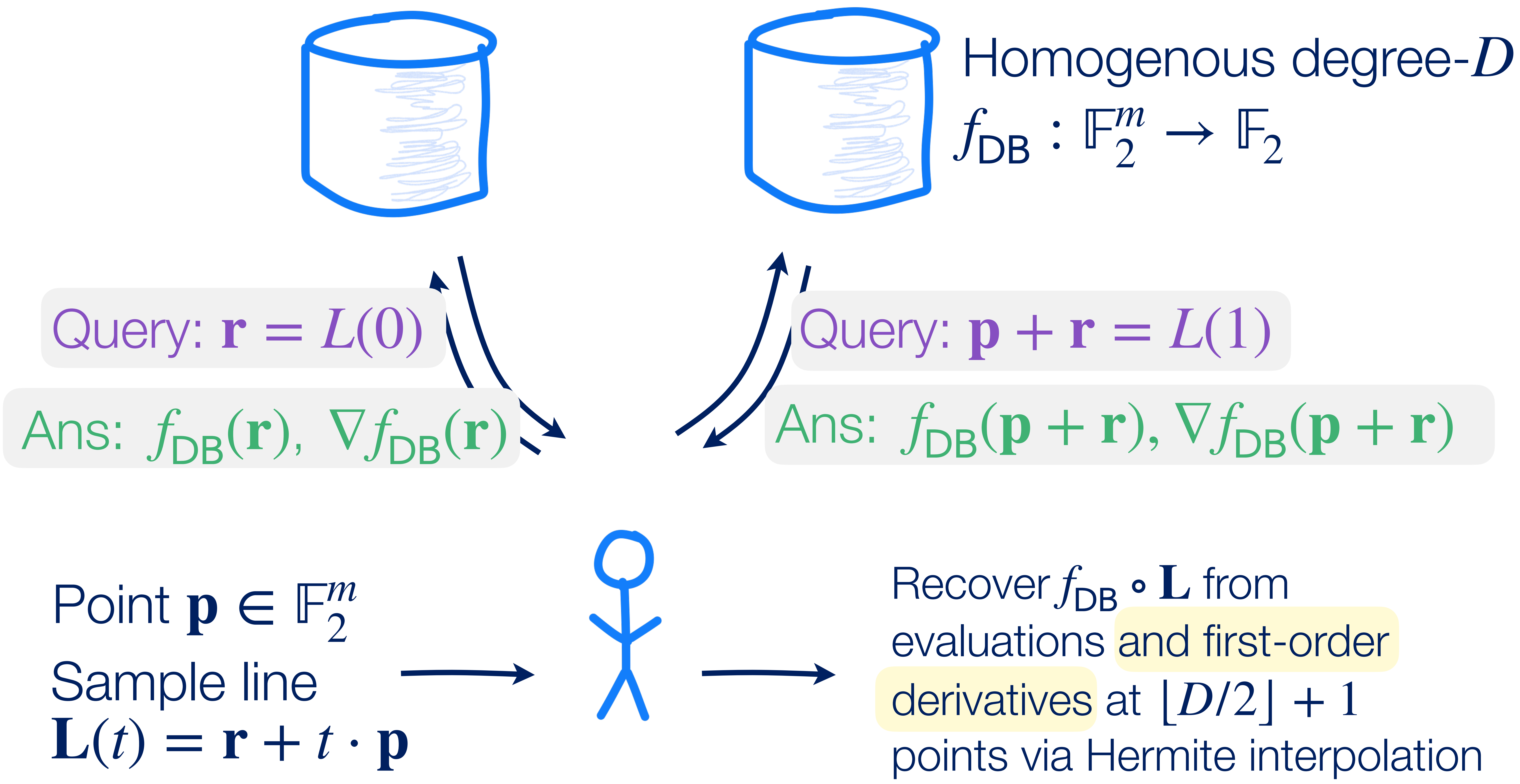
→ gives "trivial" PIR with

➡ $\binom{m}{D} \geq n \implies m = n$

➡ upload n and download 1

PIR from private polynomial evaluation

Scheme 1b: add derivatives [WY05]



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Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line $\mathbf{L}(t) = \mathbf{r} + t \cdot \mathbf{p}$

Recover $f_{\mathsf{DB}} \circ \mathbf{L}$ from evaluations and first-order derivatives at $\lfloor D/2 \rfloor + 1$ points via Hermite interpolation

PIR from private polynomial evaluation

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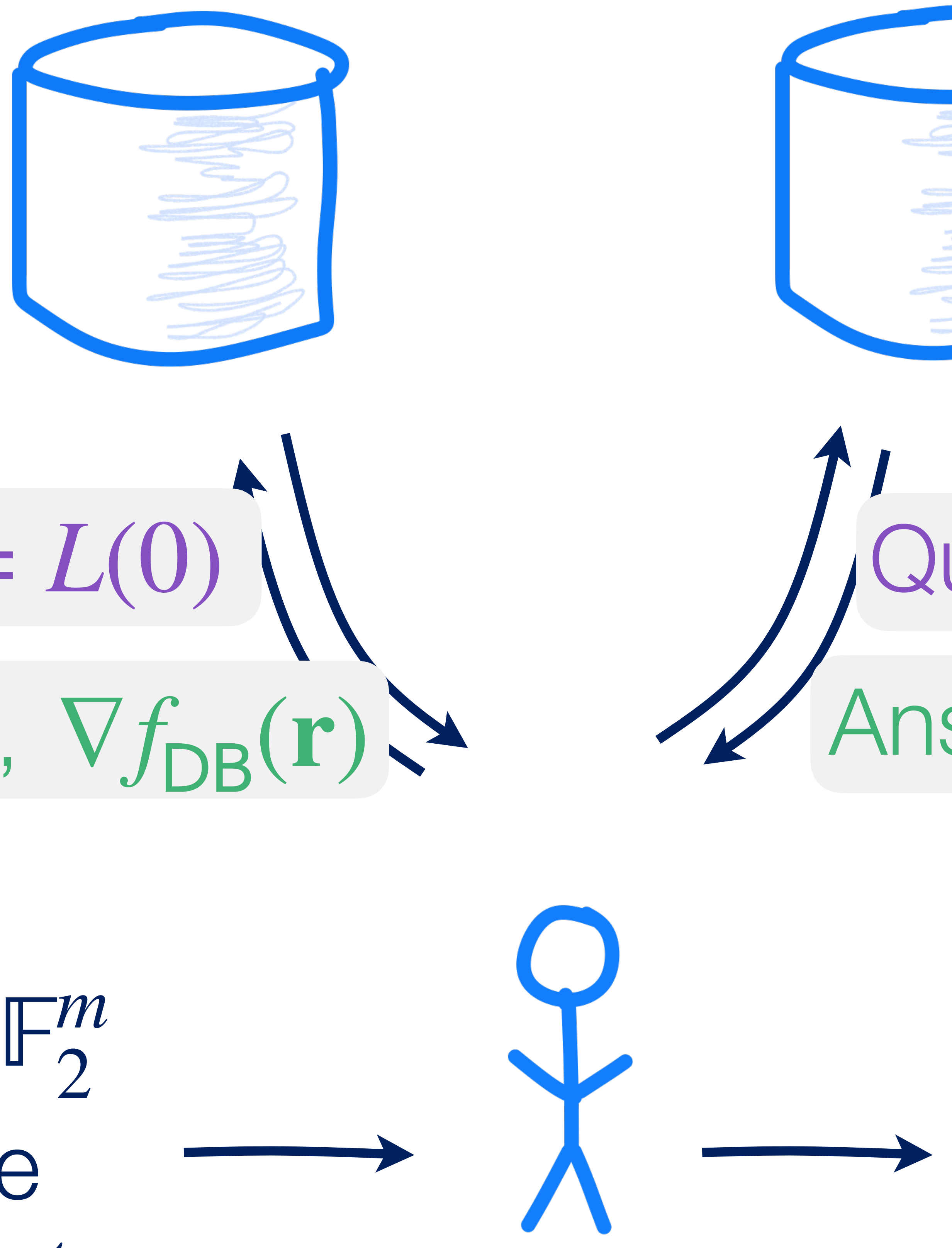

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Warning! Servers can't know the slope of $\mathbf{L}$ so they can't send derivatives of $f_{\mathsf{DB}} \circ \mathbf{L}$

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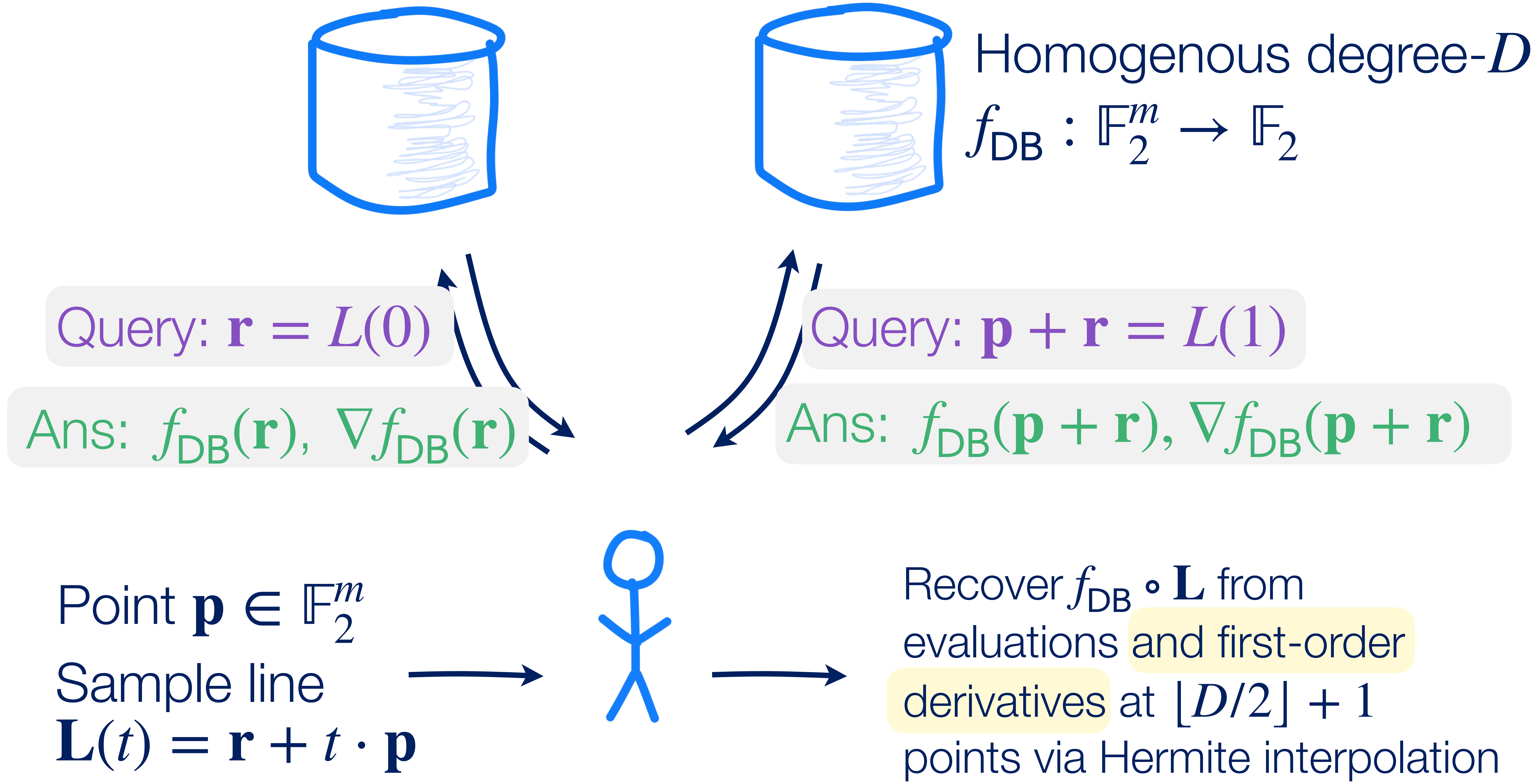

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With 2 servers, gives "balanced" PIR with

➡ $D = 3$

➡ $\binom{m}{D} \geq n \implies m = n^{1/3}$

➡ upload $n^{1/3}$ and download $n^{1/3}$

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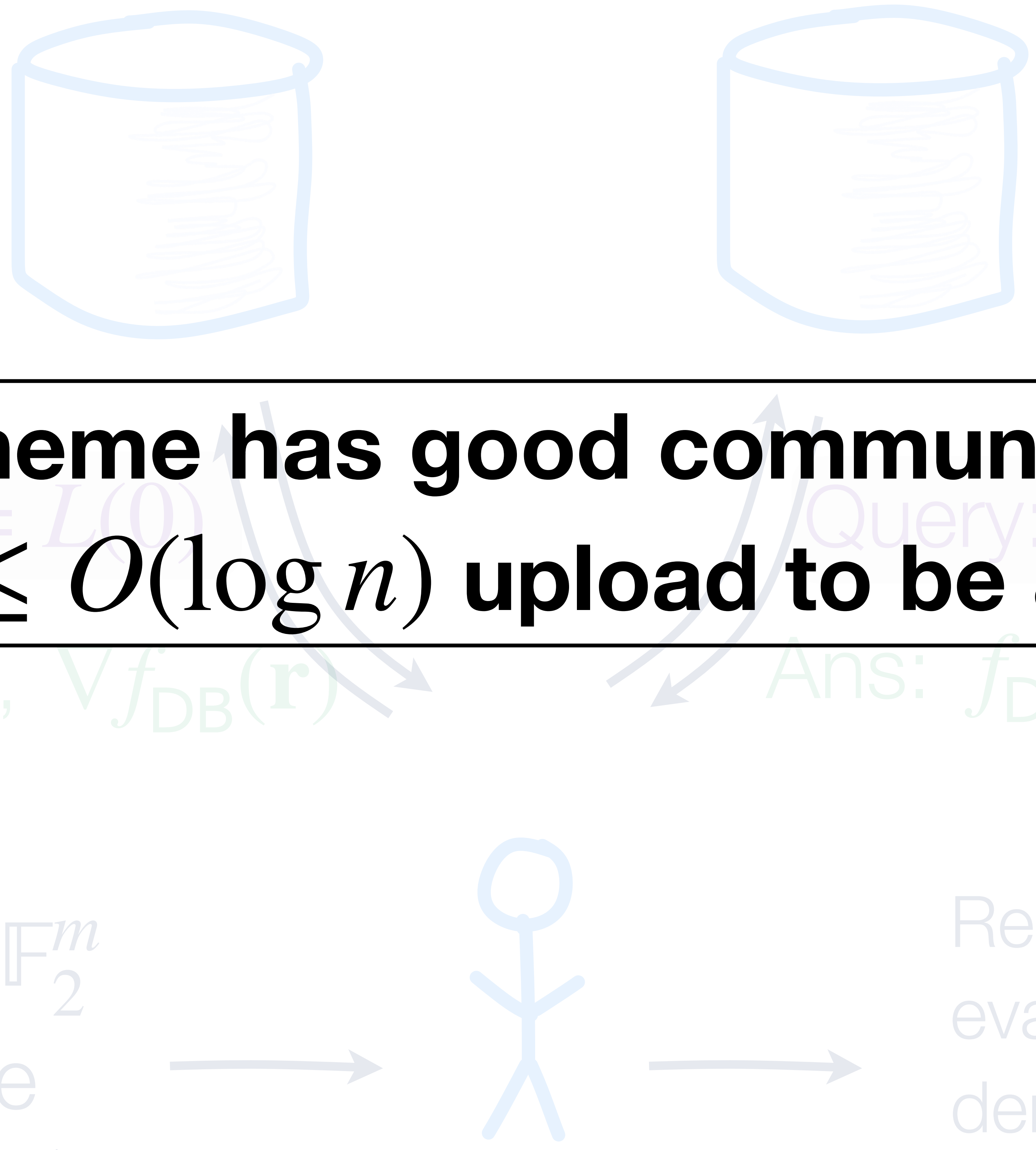

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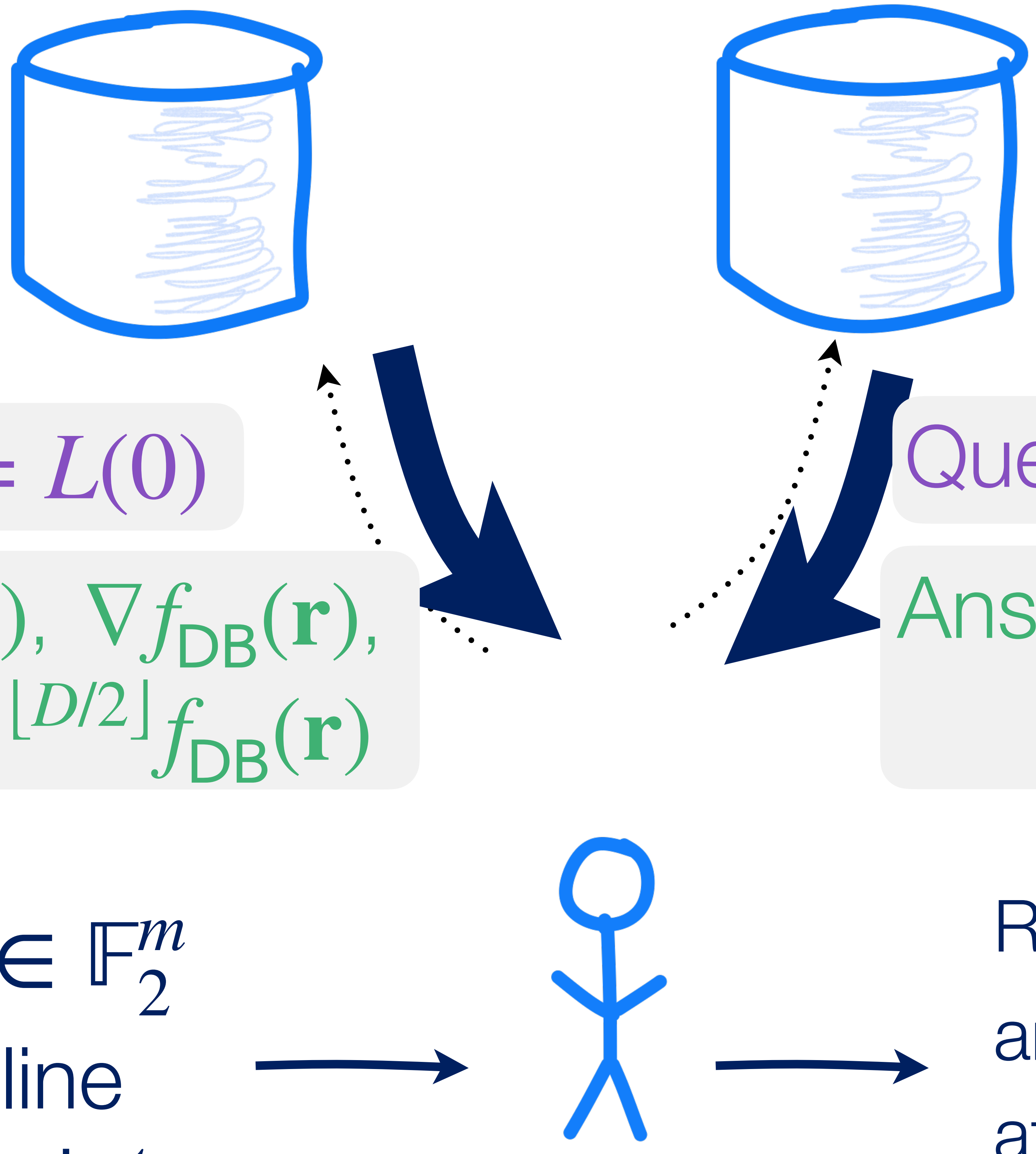
This scheme has good communication but it is not preprocessing-friendly!

Need $\leq O(\log n)$ upload to be able to precompute answers to all queries

PIR from private polynomial evaluation

Make it “imbalanced”: more derivatives [BIM00,GLM+25]



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Recover $f_{\text{DB}} \circ \mathbf{L}$ from evaluations and $\leq \lfloor D/2 \rfloor$-order derivatives at 2 points via Hermite interpolation

Warning! Servers can’t know the slope of $\mathbf{L}$ so they can’t send derivatives of $f_{\text{DB}} \circ \mathbf{L}$

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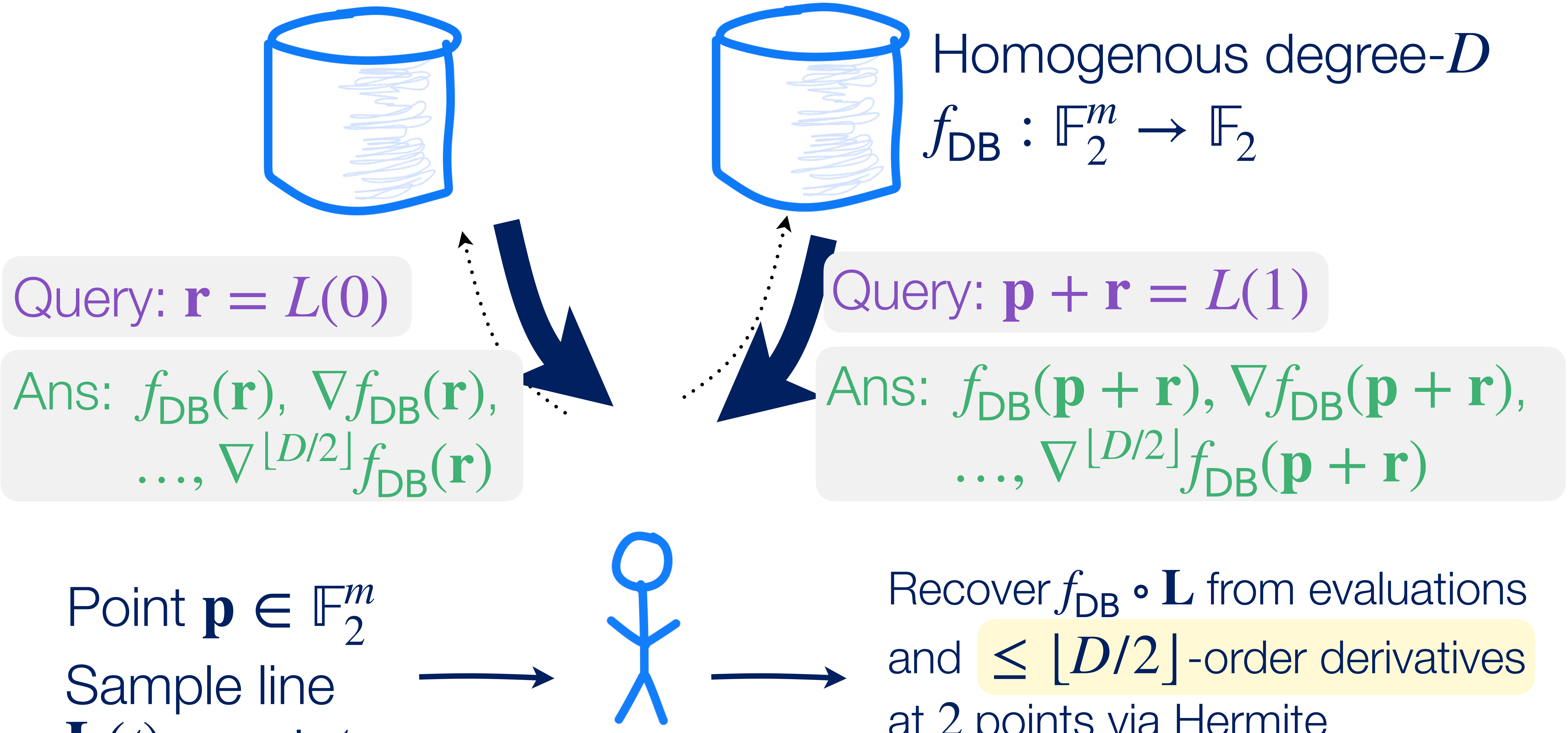

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With 2 servers, gives “imbalanced” PIR with

➡ $m = (1 + o(1)) \cdot \log n$

➡ $D = m/2$

➡ upload $m \approx \log n$ and download $\binom{m}{\lfloor D/2 \rfloor} \approx n^{0.82}$

Warning! Servers can’t know the slope of $\mathbf{L}$ so they can’t send derivatives of $f_{\text{DB}} \circ \mathbf{L}$

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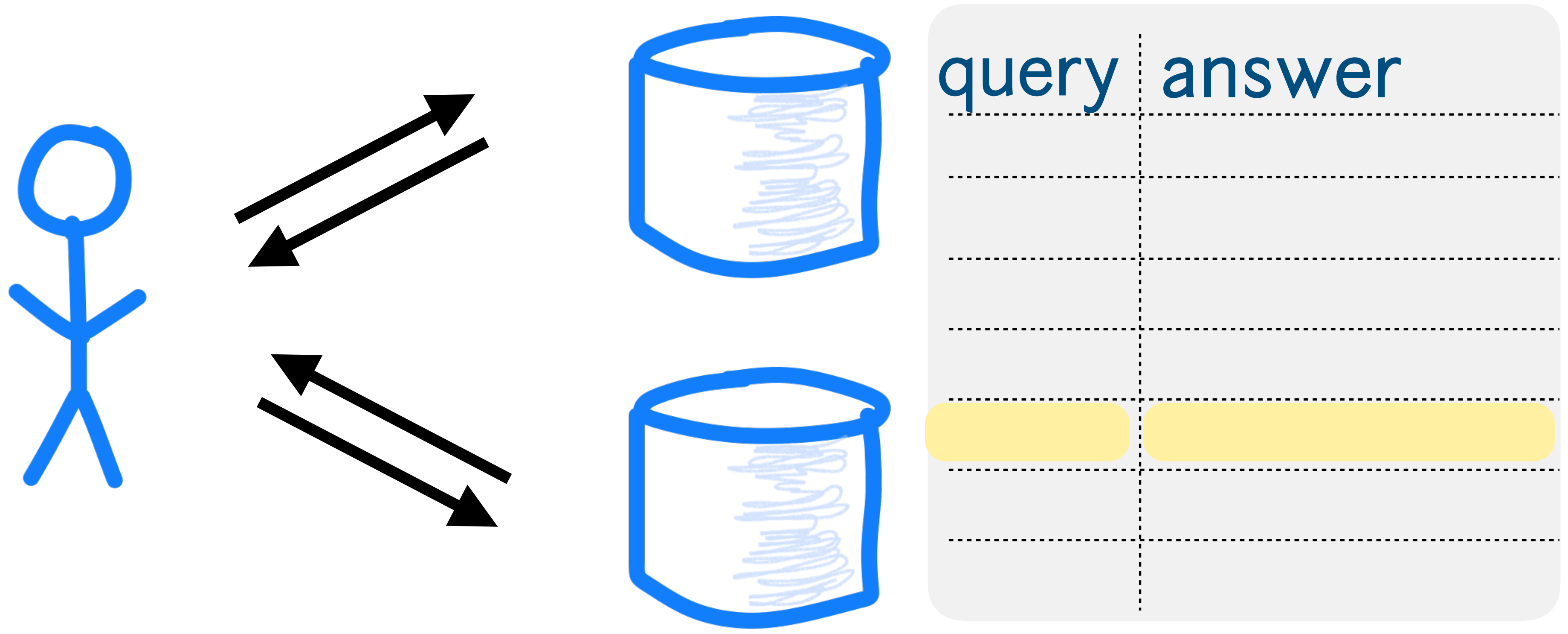

This work

2. New data structure

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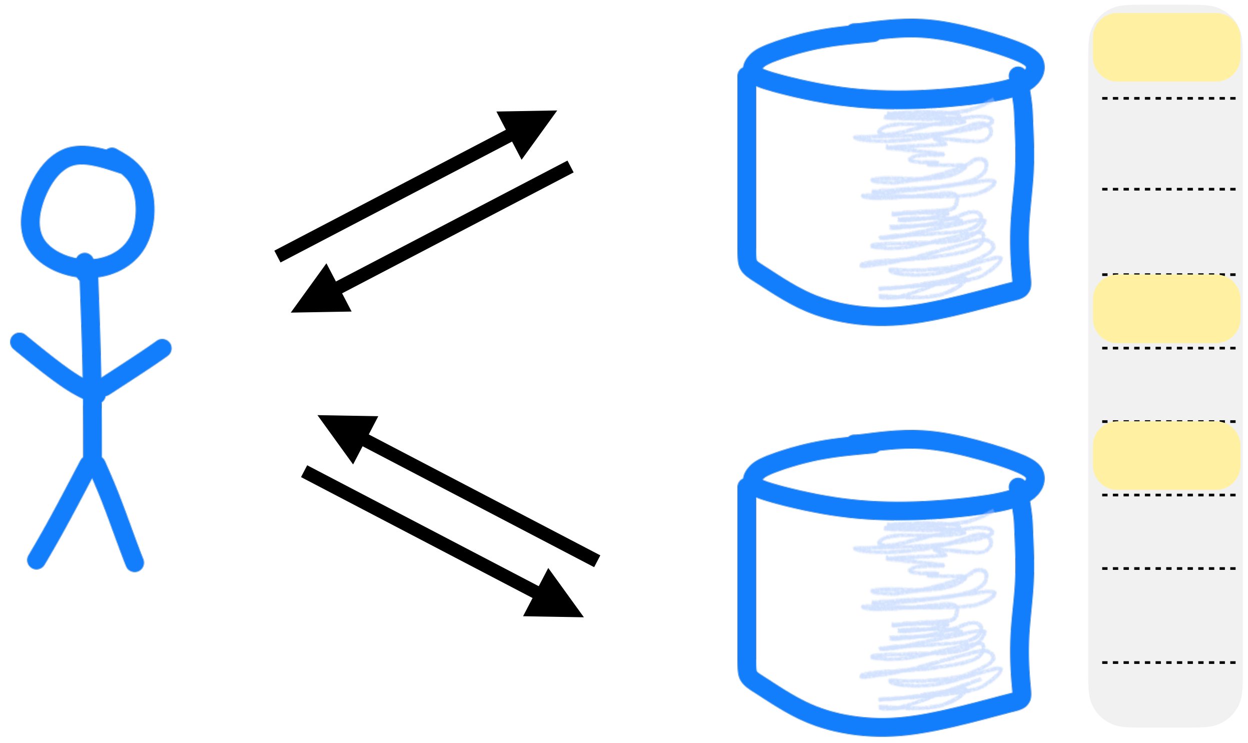
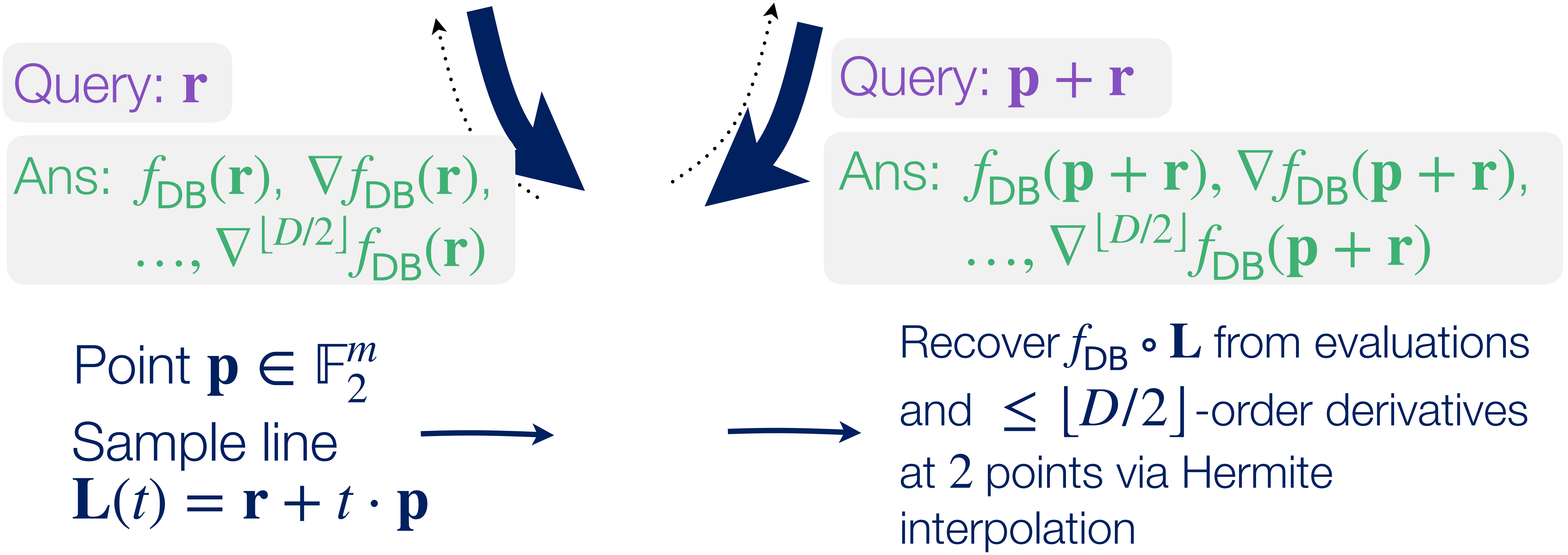


PIR with Preprocessing

Prior work: Precompute every answer [BIM00,GLM+25]



Query: $\mathbf{r}$

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Query: $\mathbf{p} + \mathbf{r}$

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Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line

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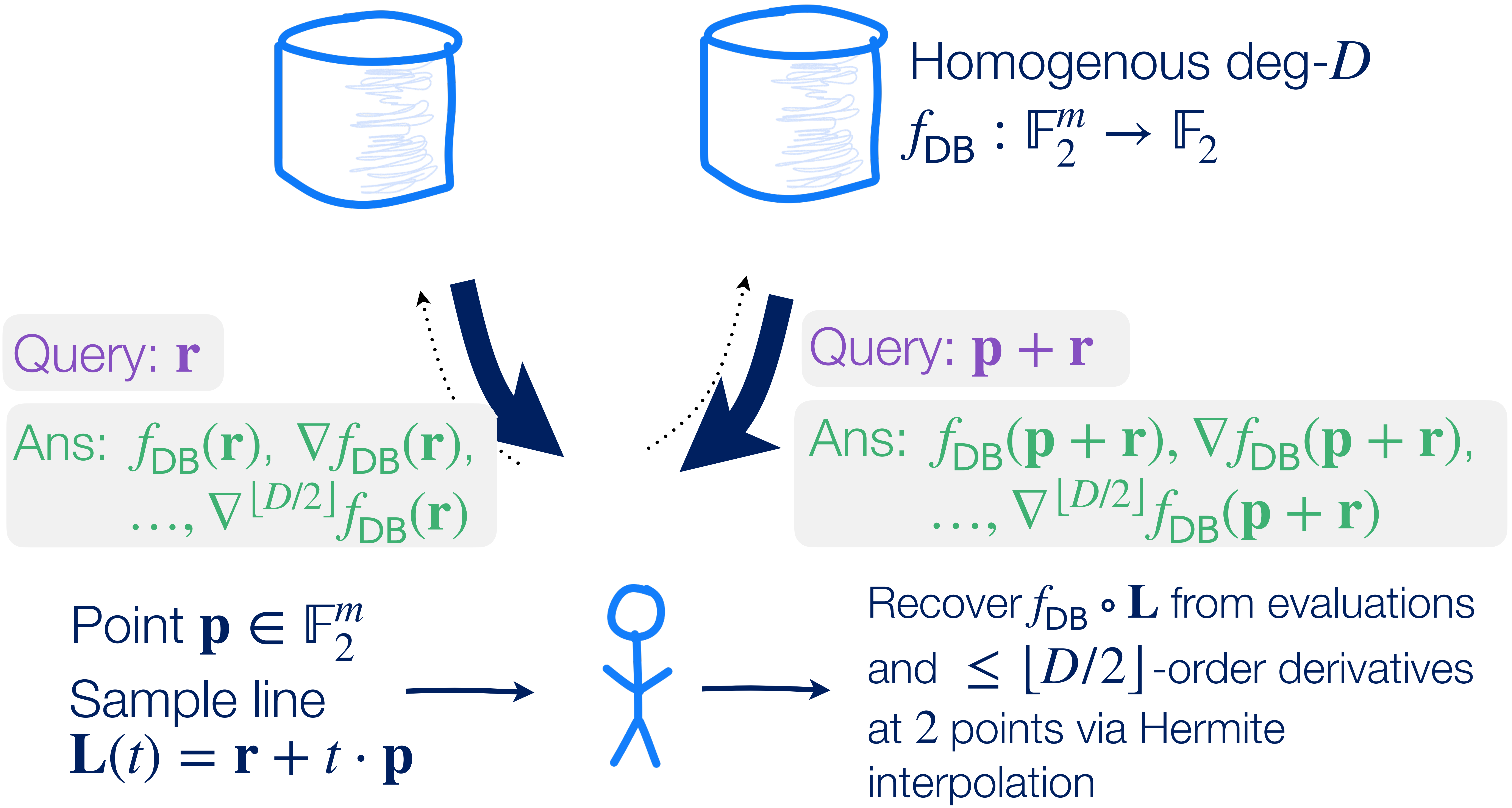

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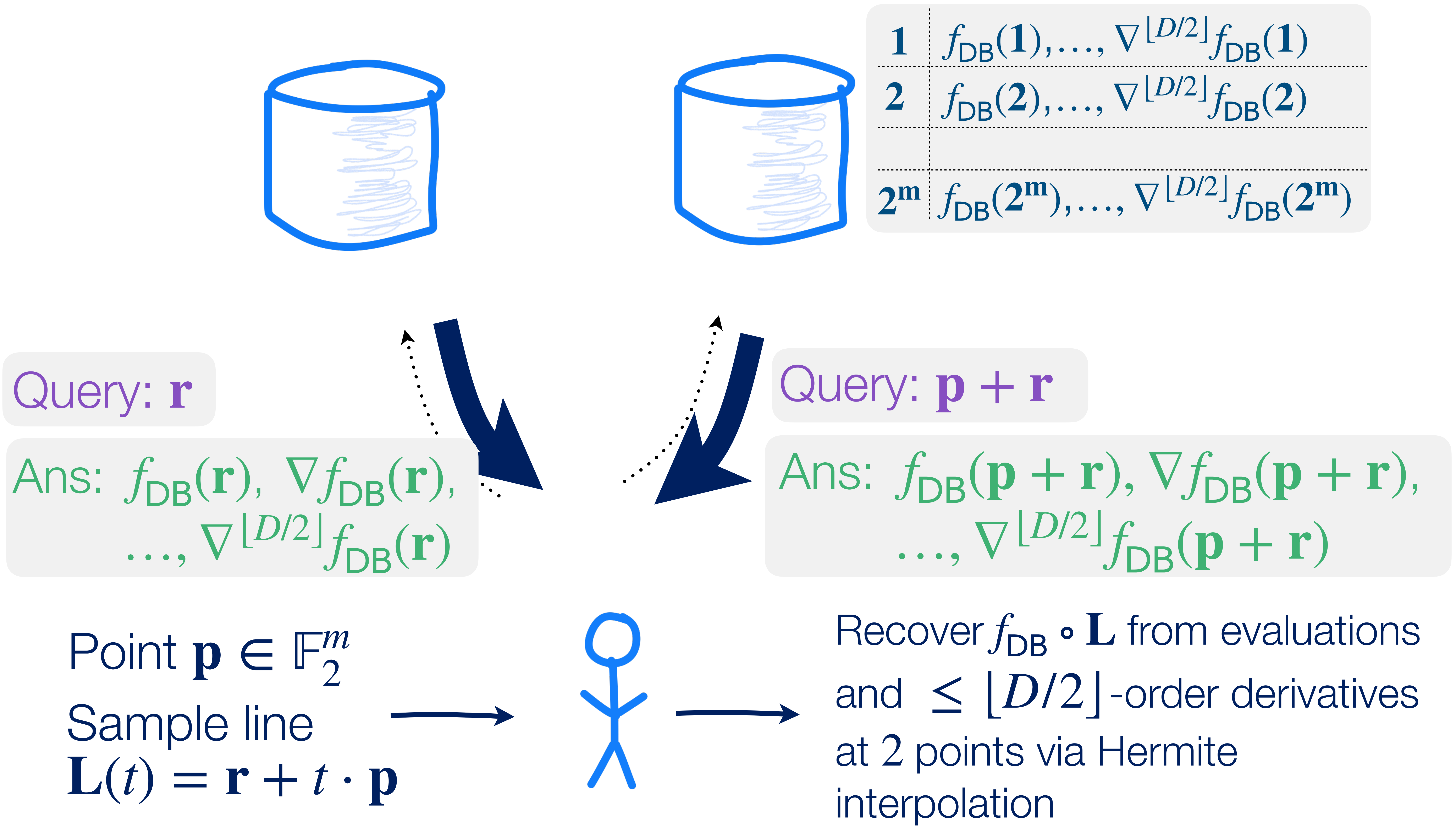

$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1}),\ldots,\nabla^{\lfloor D/2\rfloor}f_{\mathsf{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\mathsf{DB}}(\mathbf{2}),\ldots,\nabla^{\lfloor D/2\rfloor}f_{\mathsf{DB}}(\mathbf{2})$
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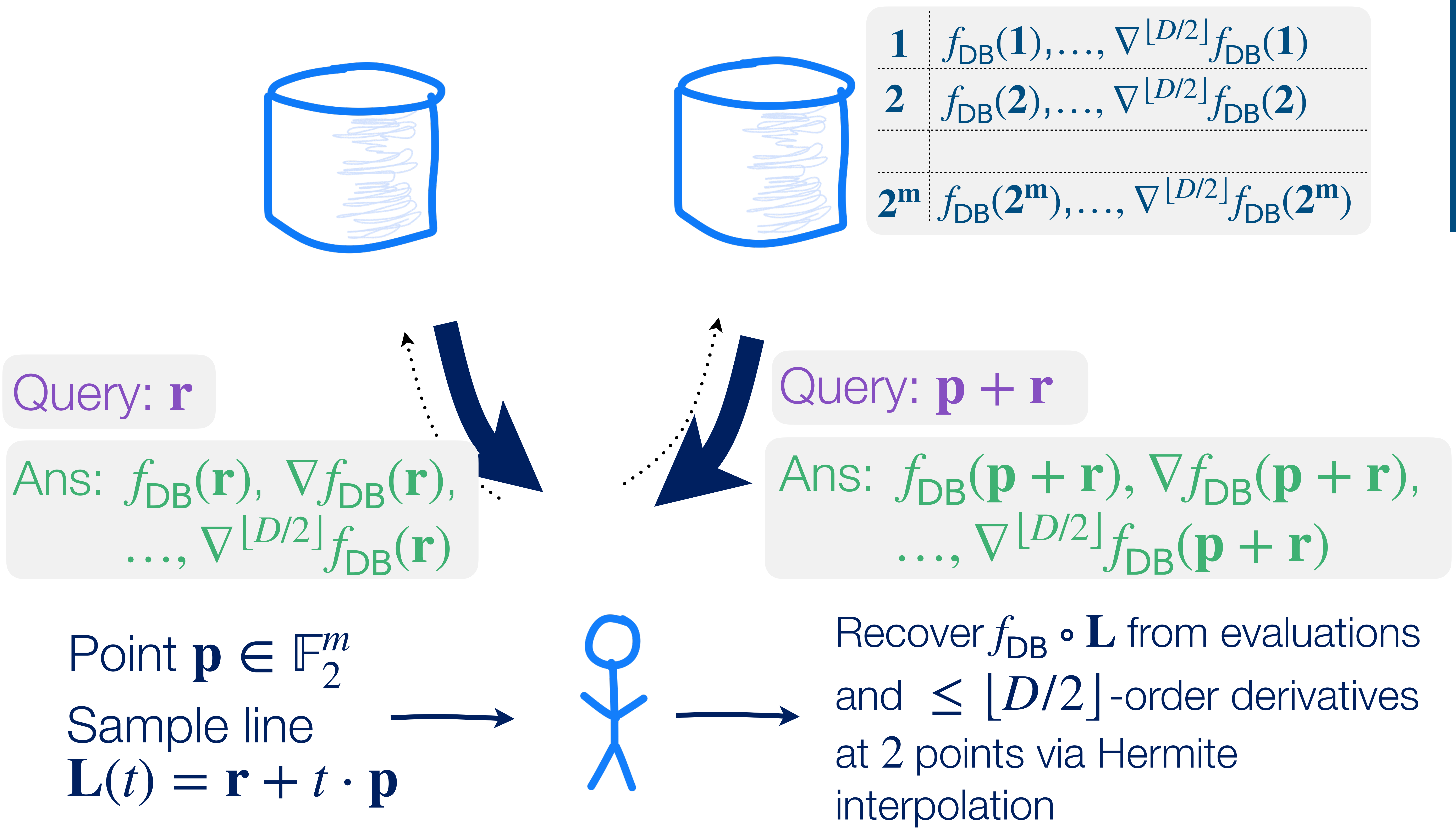
Point $\mathbf{p}\in\mathbb{F}_2^m$

Sample line $\mathbf{L}(t)=\mathbf{r}+t\cdot\mathbf{p}$

Recover $f_{\mathsf{DB}}\circ\mathbf{L}$ from evaluations and $\leq\lfloor D/2\rfloor$-order derivatives at 2 points via Hermite interpolation

PIR with Preprocessing

Prior work: Precompute every answer [BIM00,GLM+25]



$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1}),\ldots,\nabla^{\lfloor D/2\rfloor}f_{\mathsf{DB}}(\mathbf{1})$
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With 2 servers, gives preprocessing PIR with

- ➡ $O(\log n)$ upload and $n^{0.82}$ download
- ➡ $2^m\cdot n^{0.82}=n^{1.82+o(1)}$ server storage
- ➡ $n^{0.82}$ server time

Prior information-theoretic PIR

[BIM00,GLMDS25]

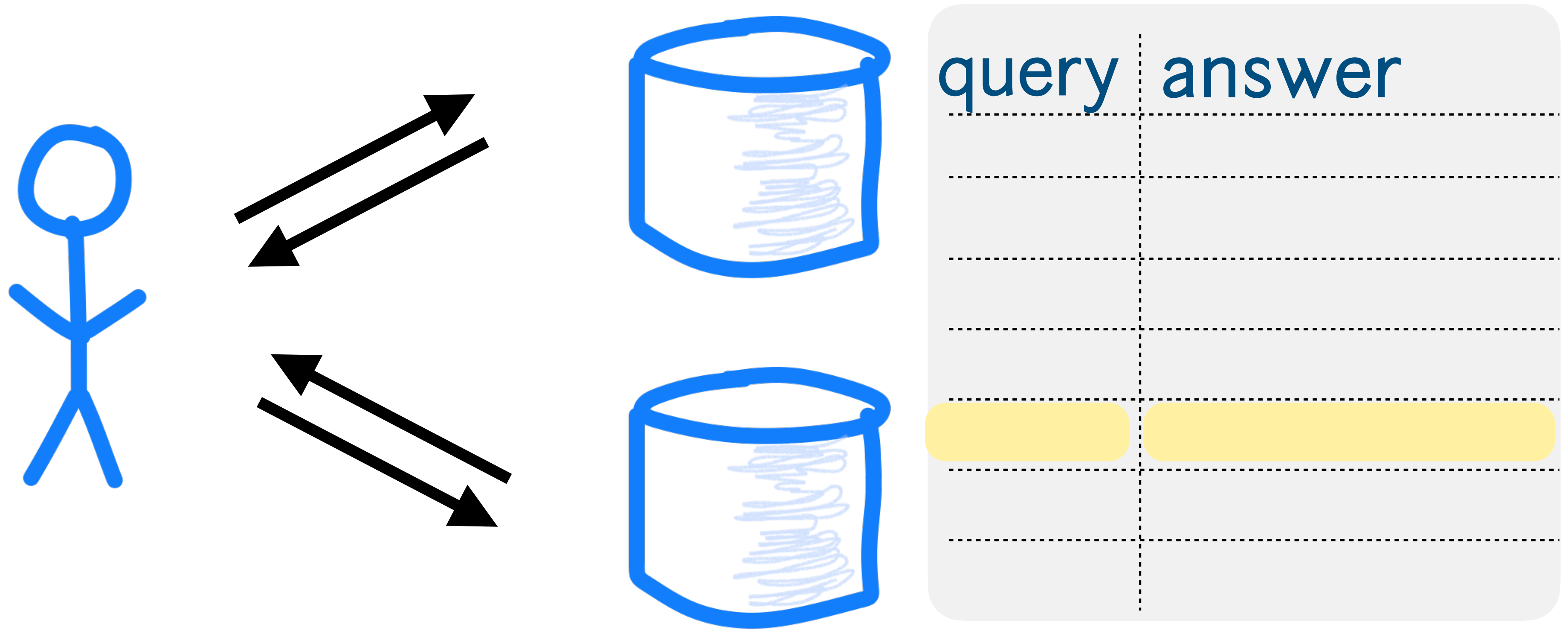
✓ 1. Build “imbalanced” PIR with tiny queries

- Query length = $(1 + o(1)) \cdot \log n$
- Answer length = $\ell = O(n^{0.82})$

✓ 2. Precompute the answer to every query

- To answer a query: read 1 location of length ℓ

➡ PIR in $n^{1.82+o(1)}$ space and $n^{0.82}$ time

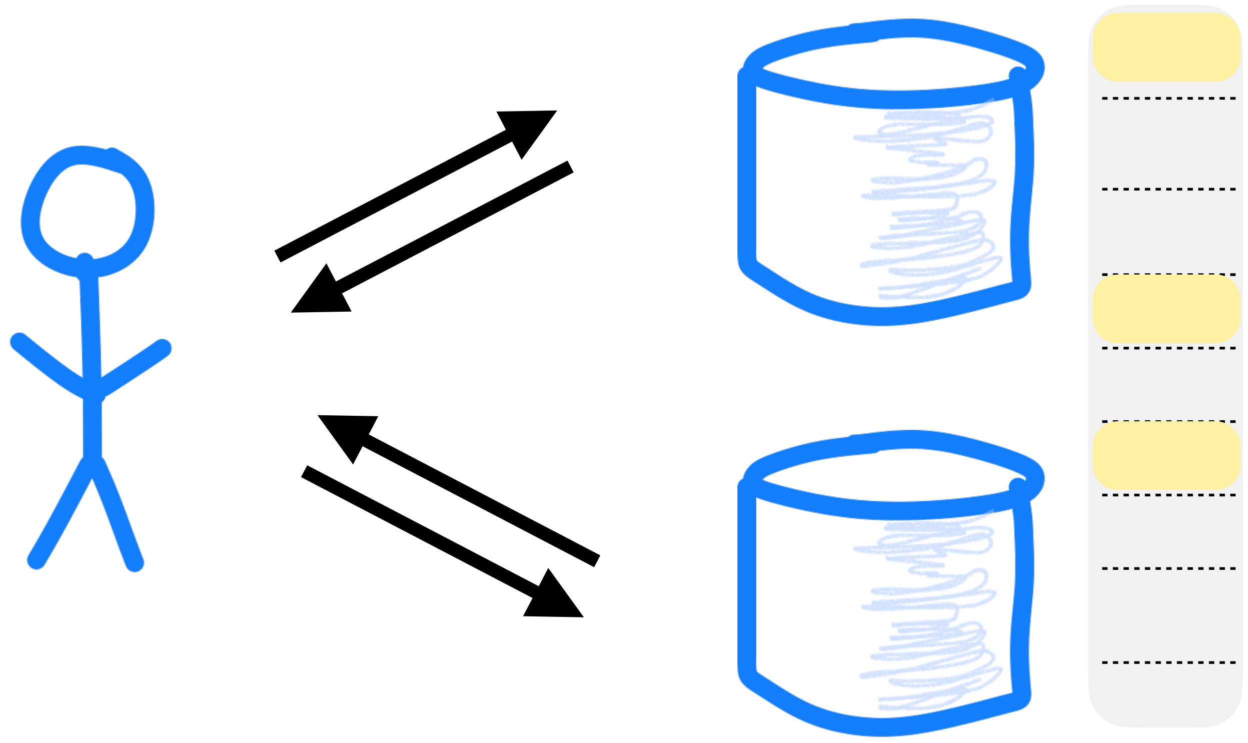


This work

2. New data structure

- To answer a query: read ℓ locations of length 1

➡ PIR in $n^{1+o(1)}$ space and $n^{0.82}$ time



This talk

1. **Background:** PIR with preprocessing
2. **[HR26] New two-server PIR:** sublinear time, quasilinear space
3. **Evaluation:** what does this mean for practice?
4. **Bonuses** 😁
    - Reducing communication using crypto
    - **[HPR26]** Connecting multi-server PIR to complexity theory

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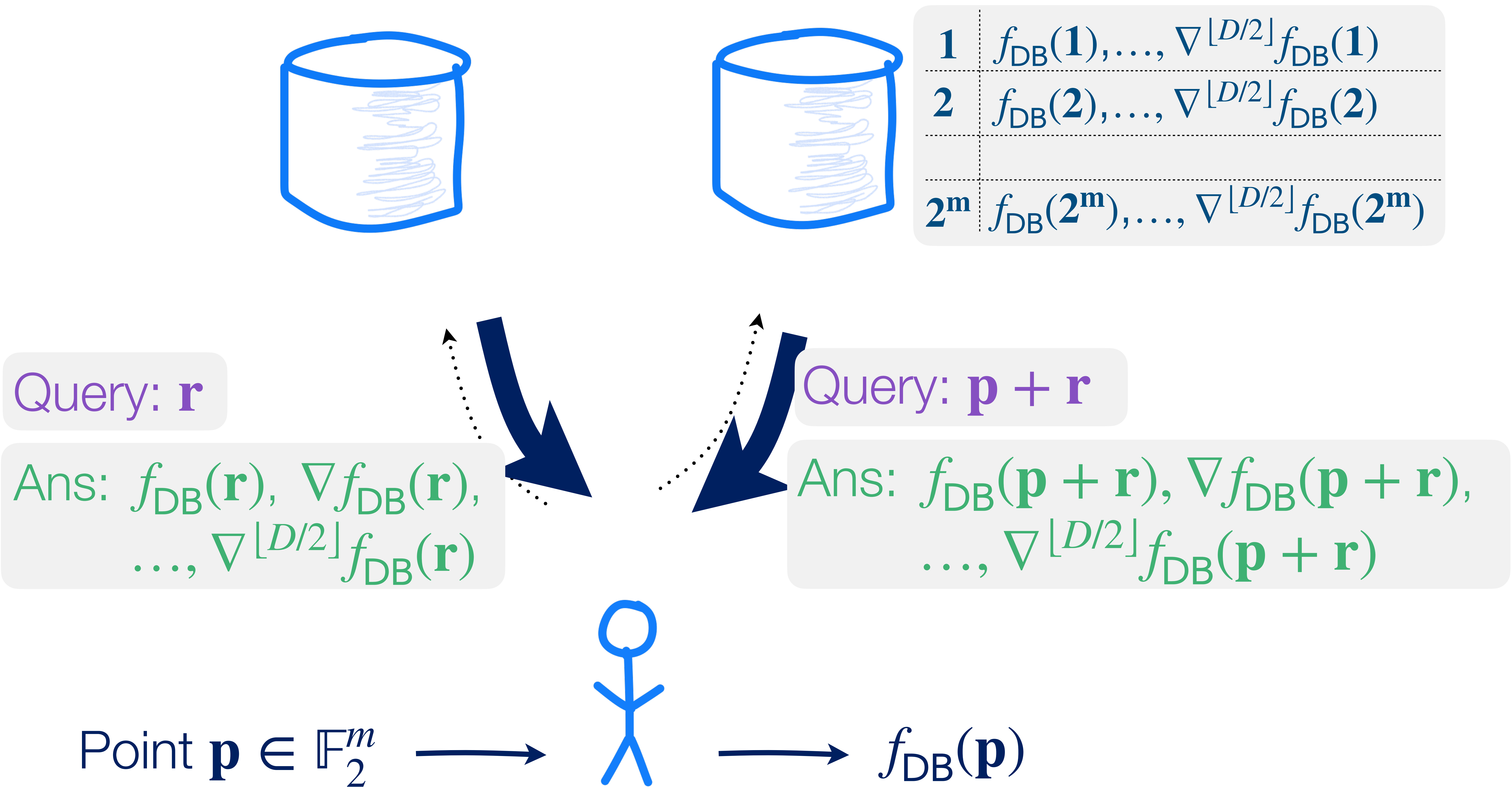
Prior work: Precompute every possible PIR answer

[BIM00,GLMDS25]



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Point $\mathbf{p}\in\mathbb{F}_2^m \longrightarrow$ $\longrightarrow f_{\mathsf{DB}}(\mathbf{p})$

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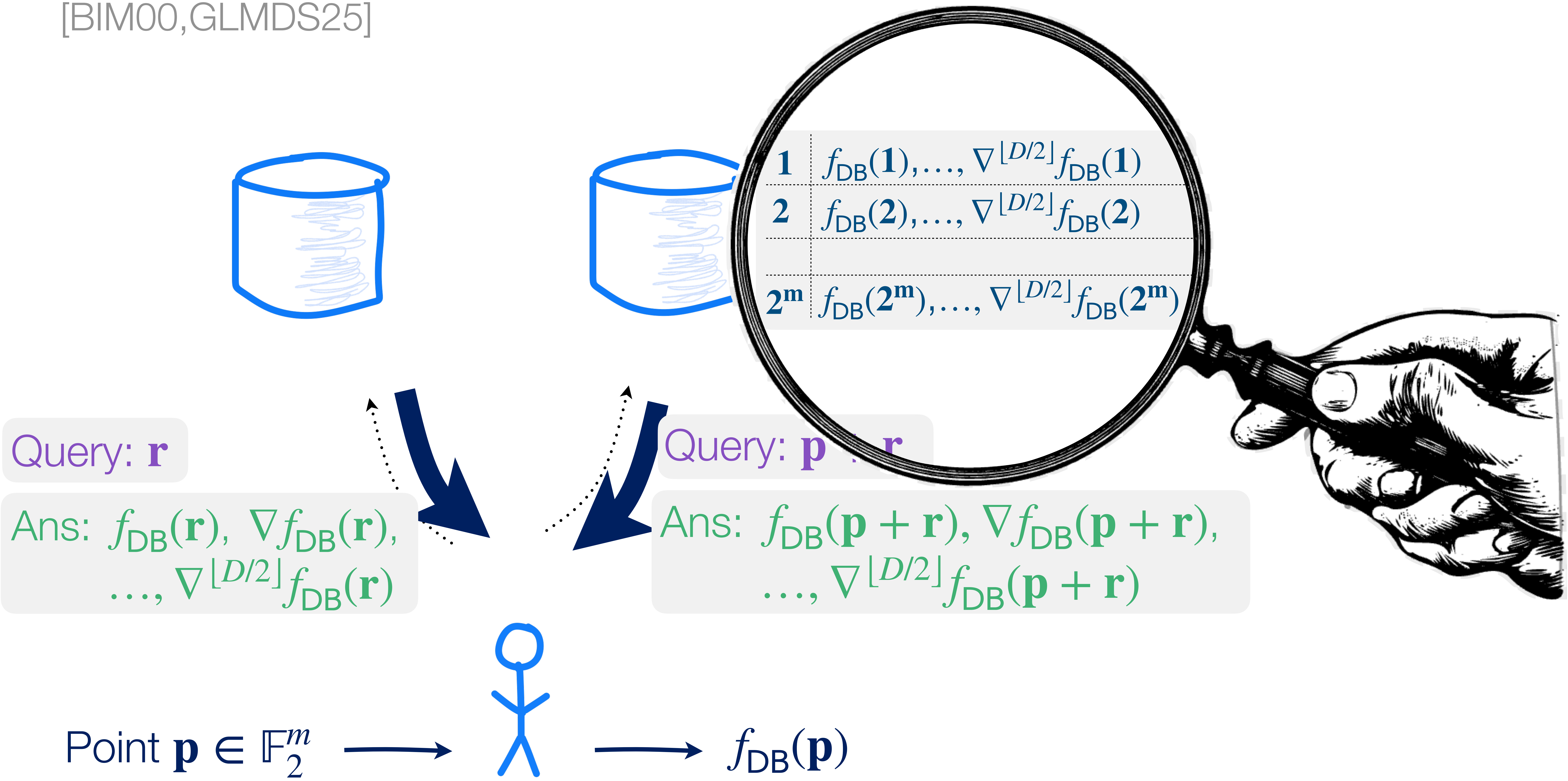
[BIM00,GLMDS25]



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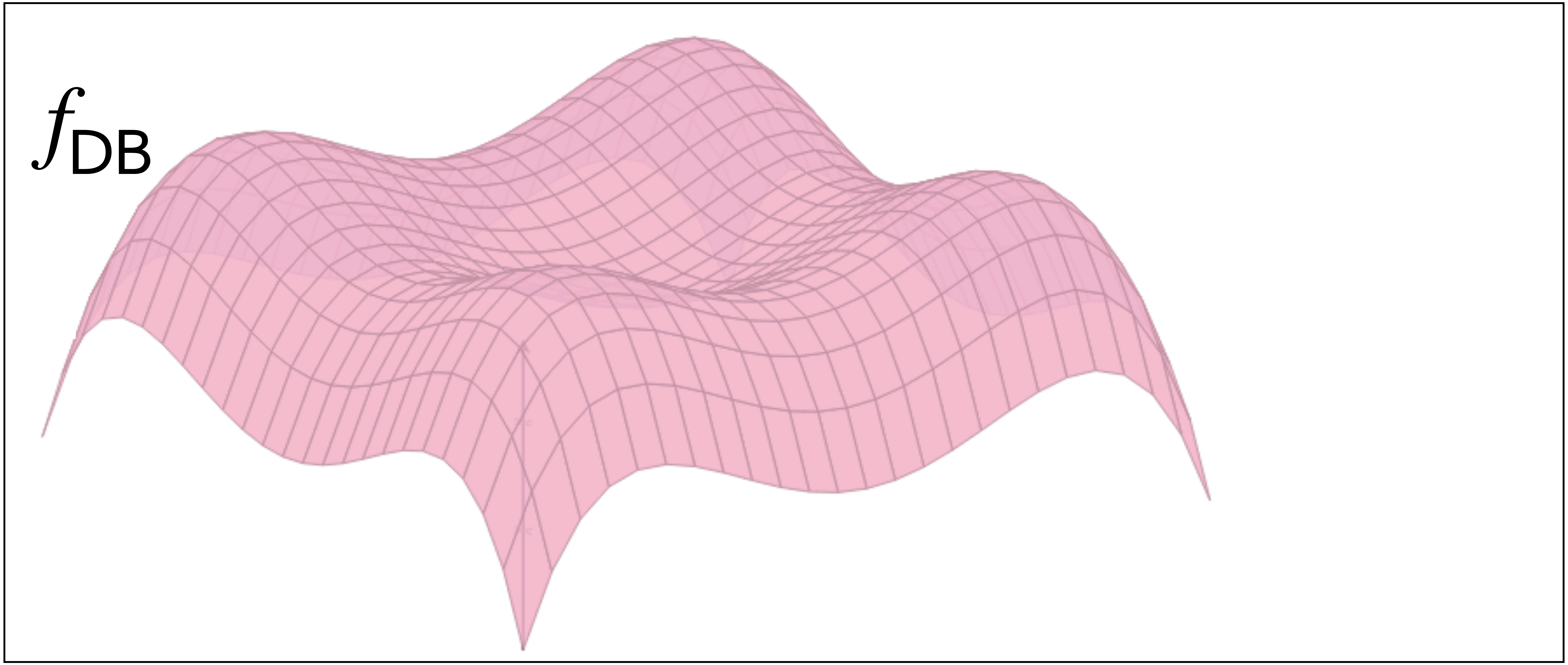

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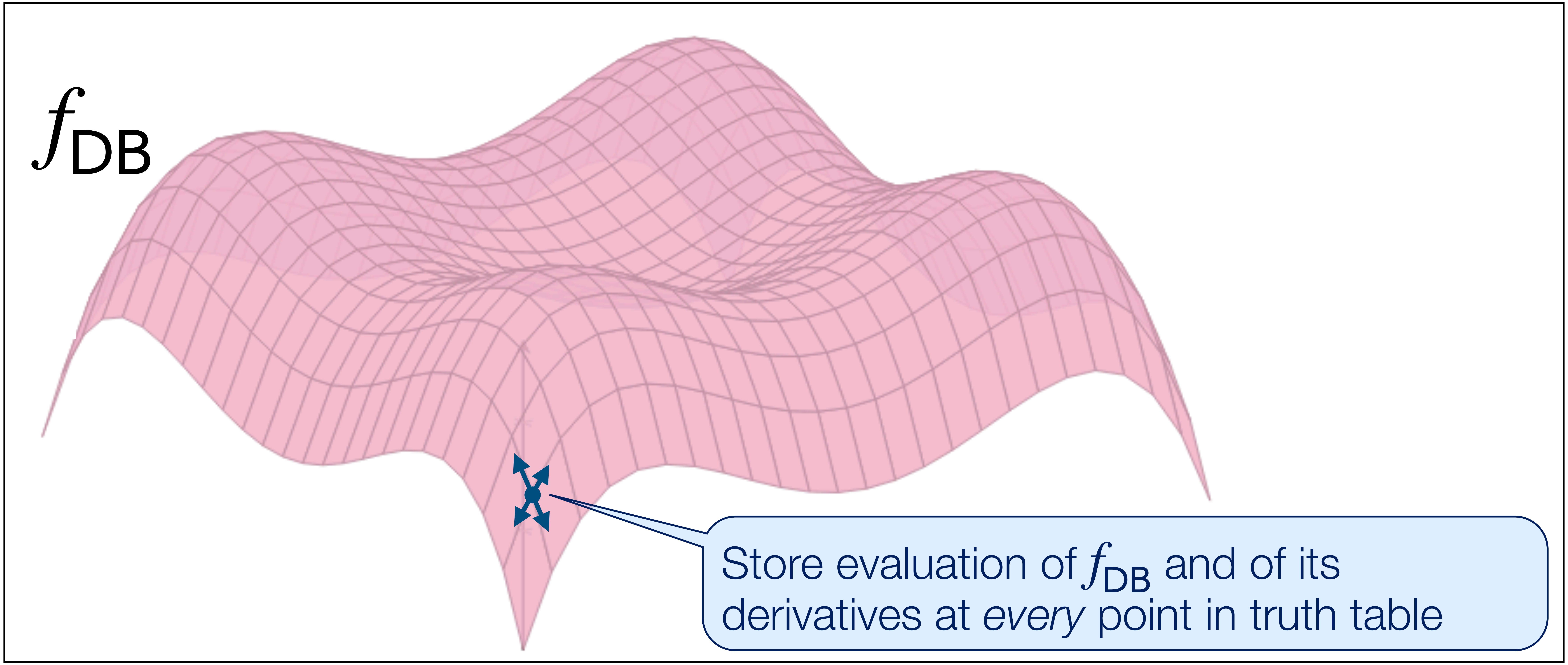
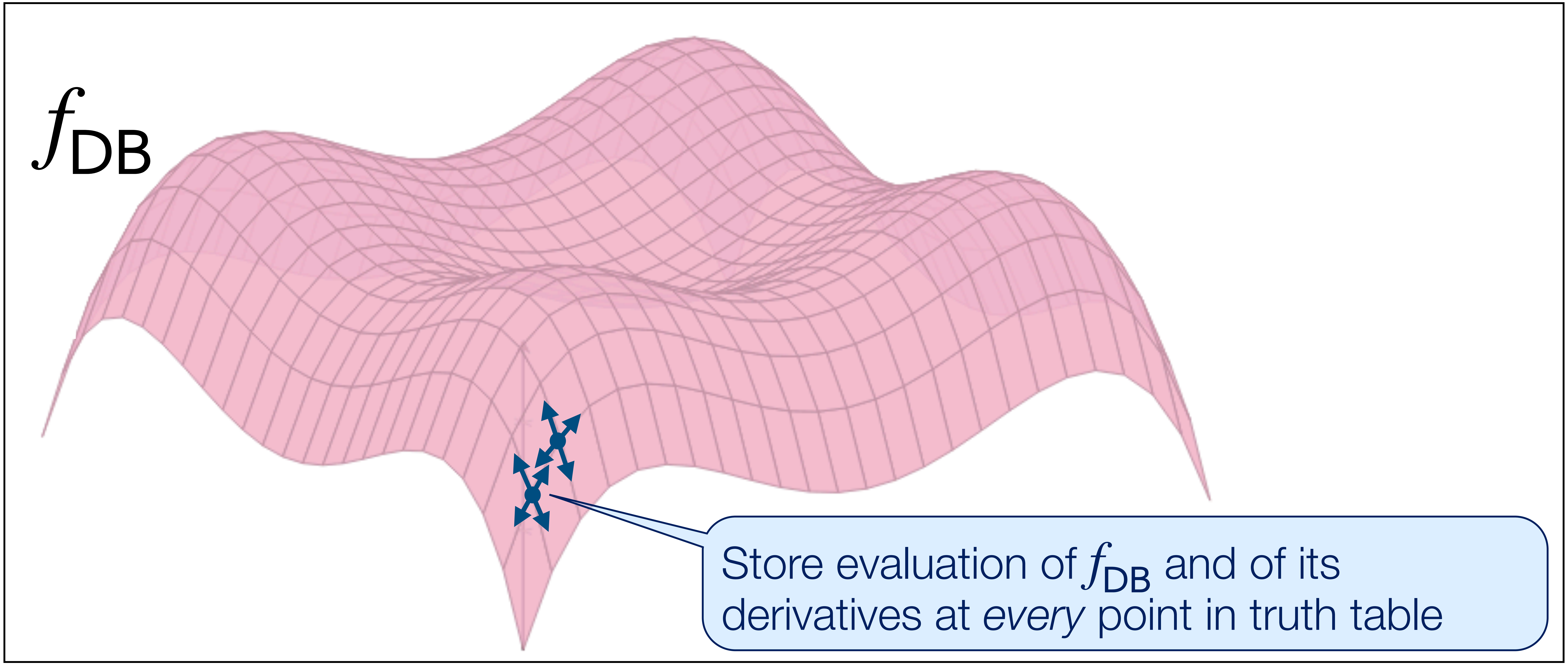


Store evaluation of f_{DB} and of its derivatives at *every* point in truth table

Prior work: Precompute every possible PIR answer

[BIM00,GLMDS25]



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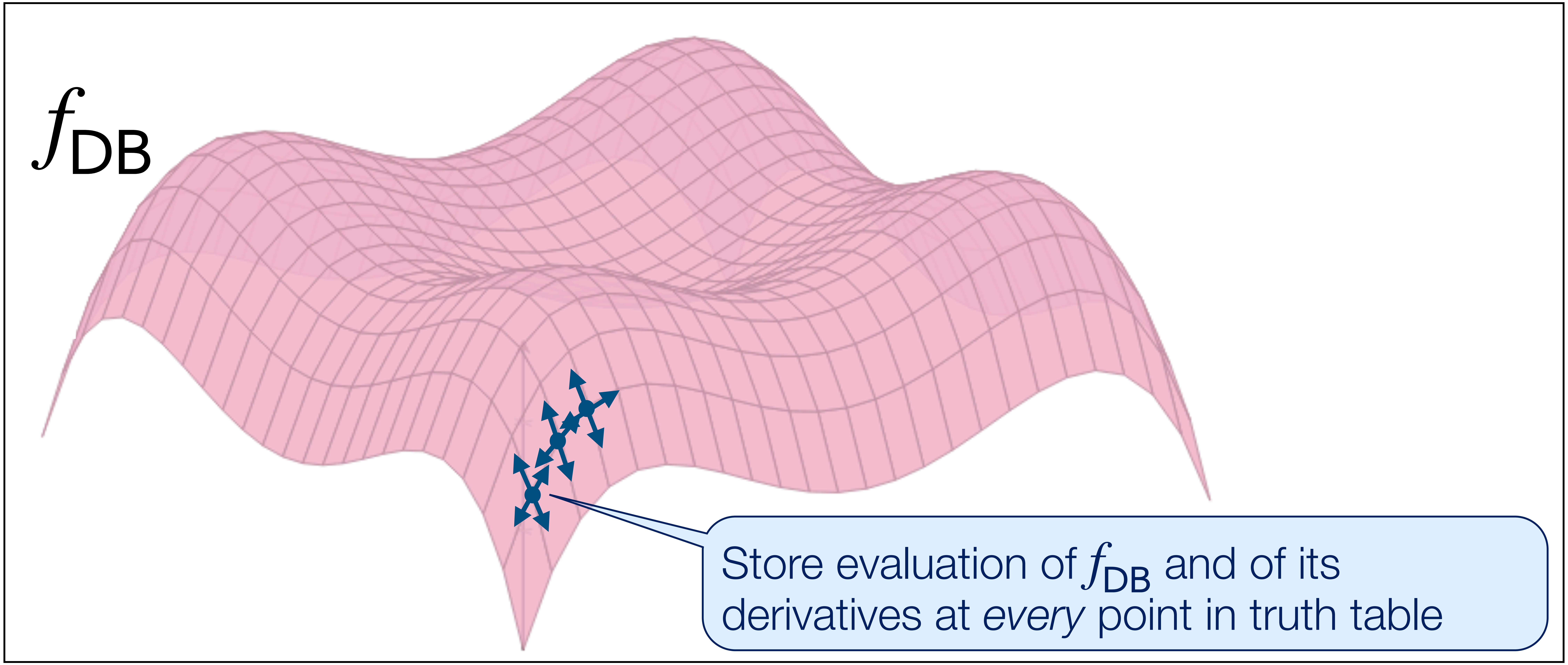

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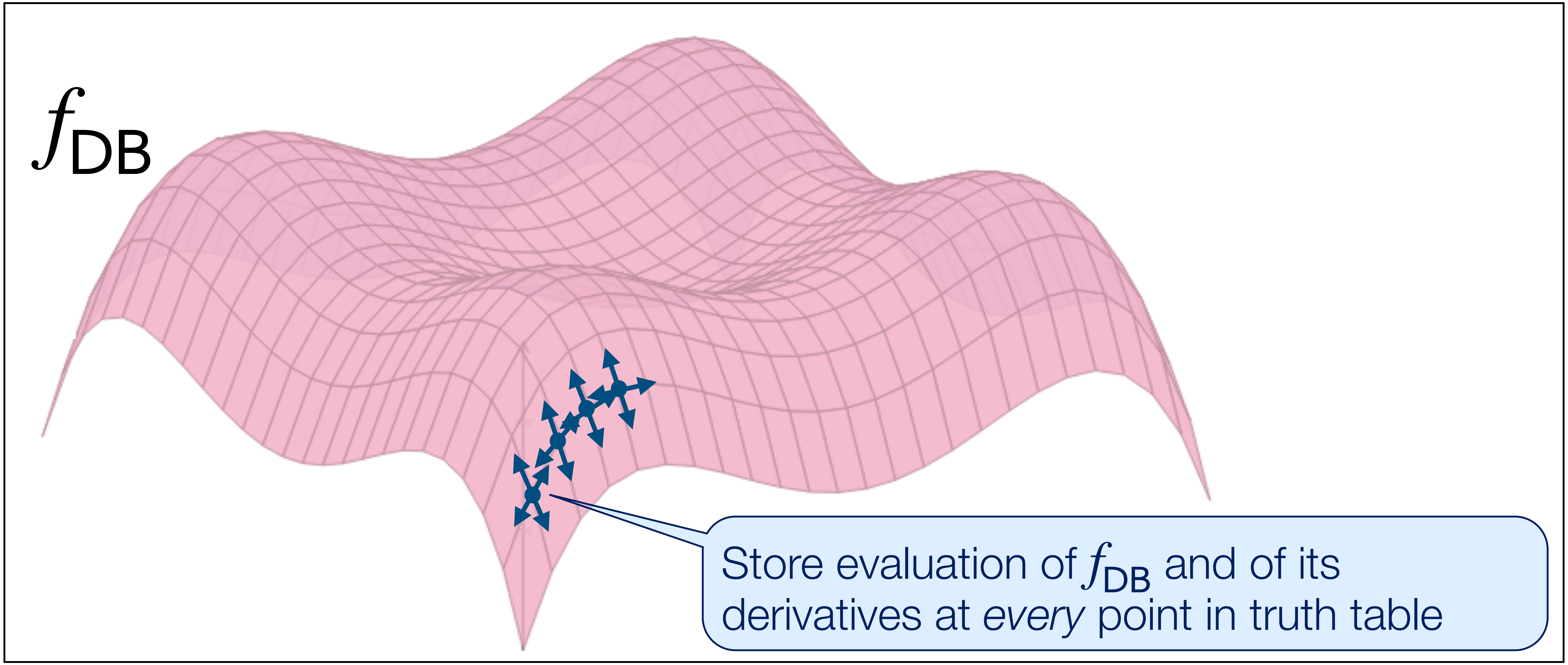

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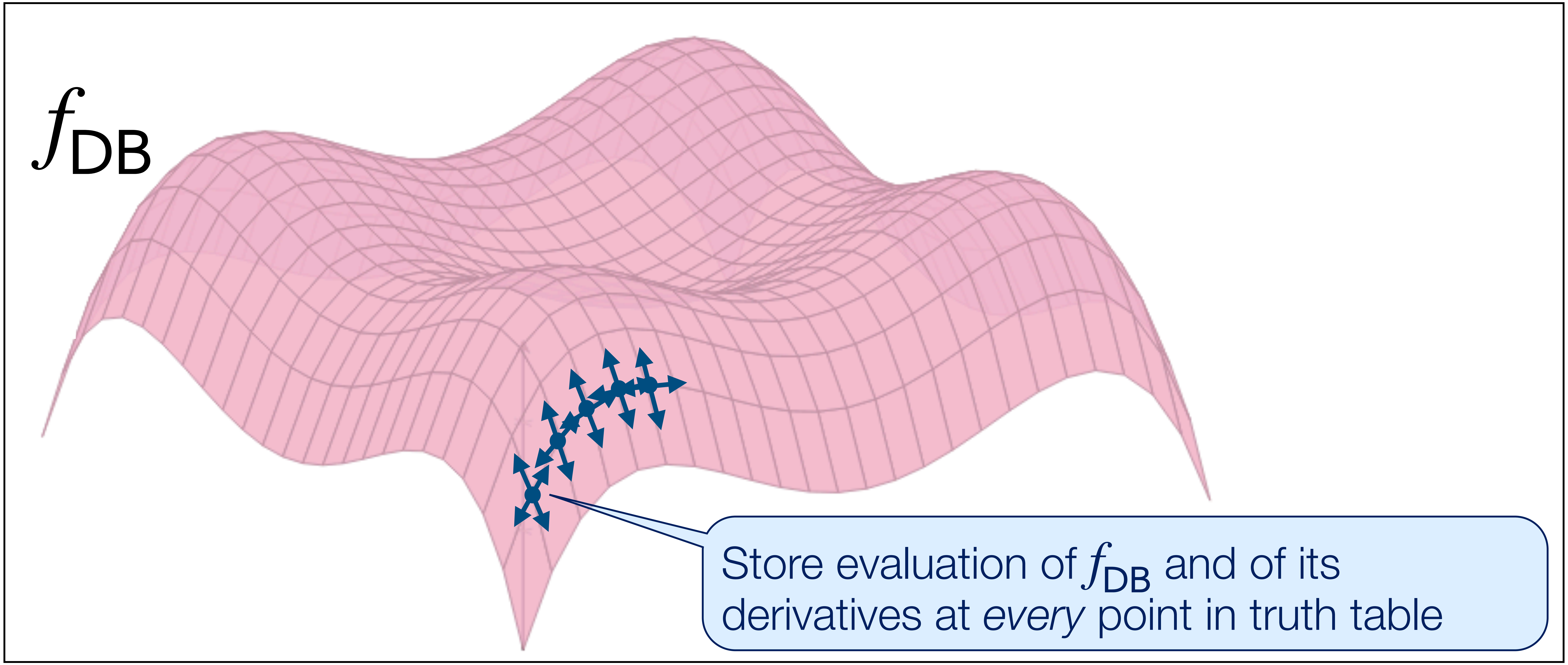

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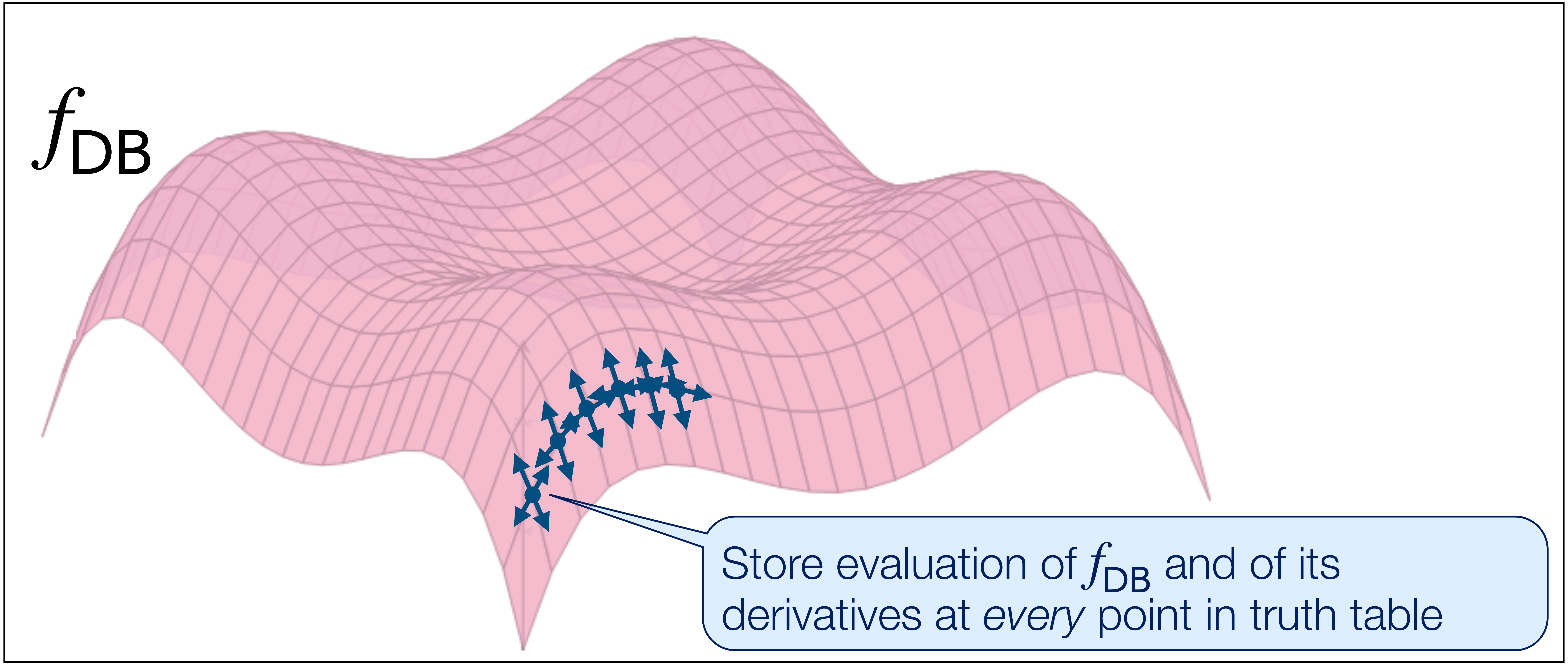

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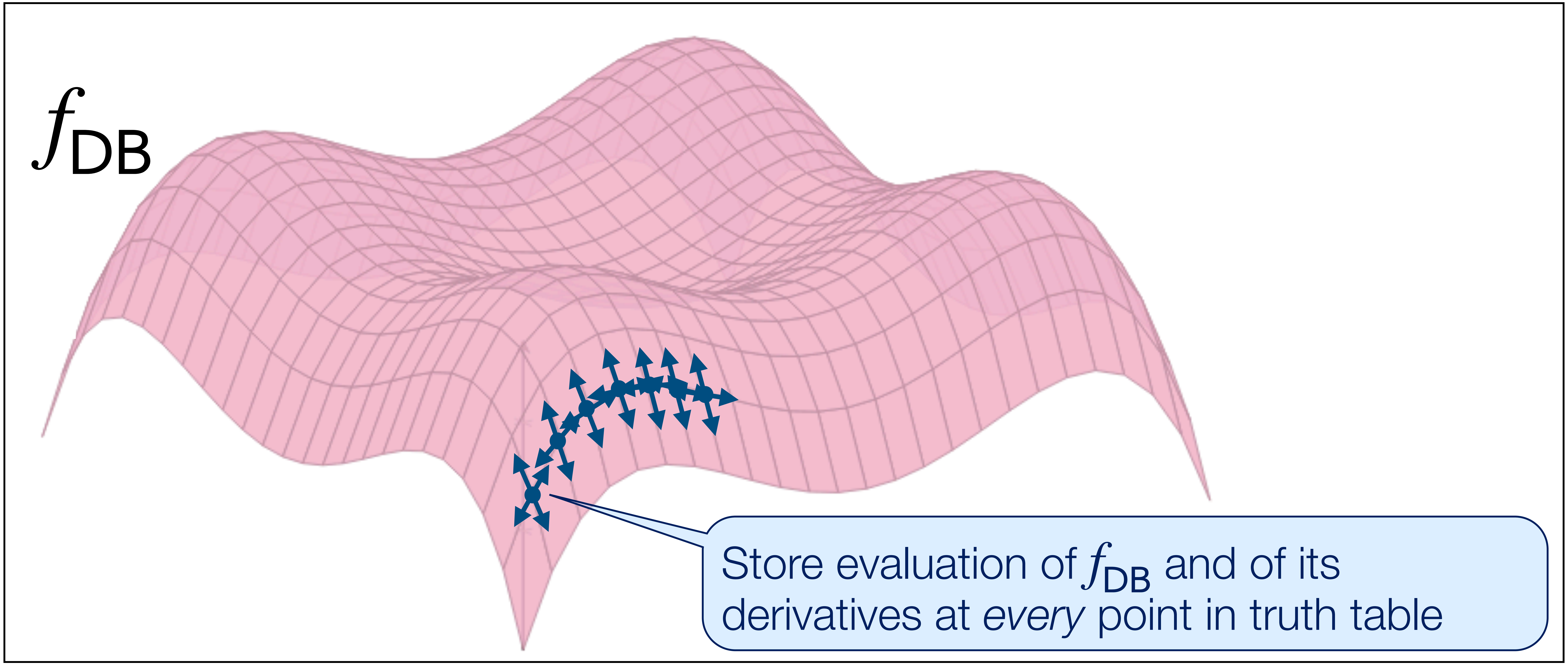

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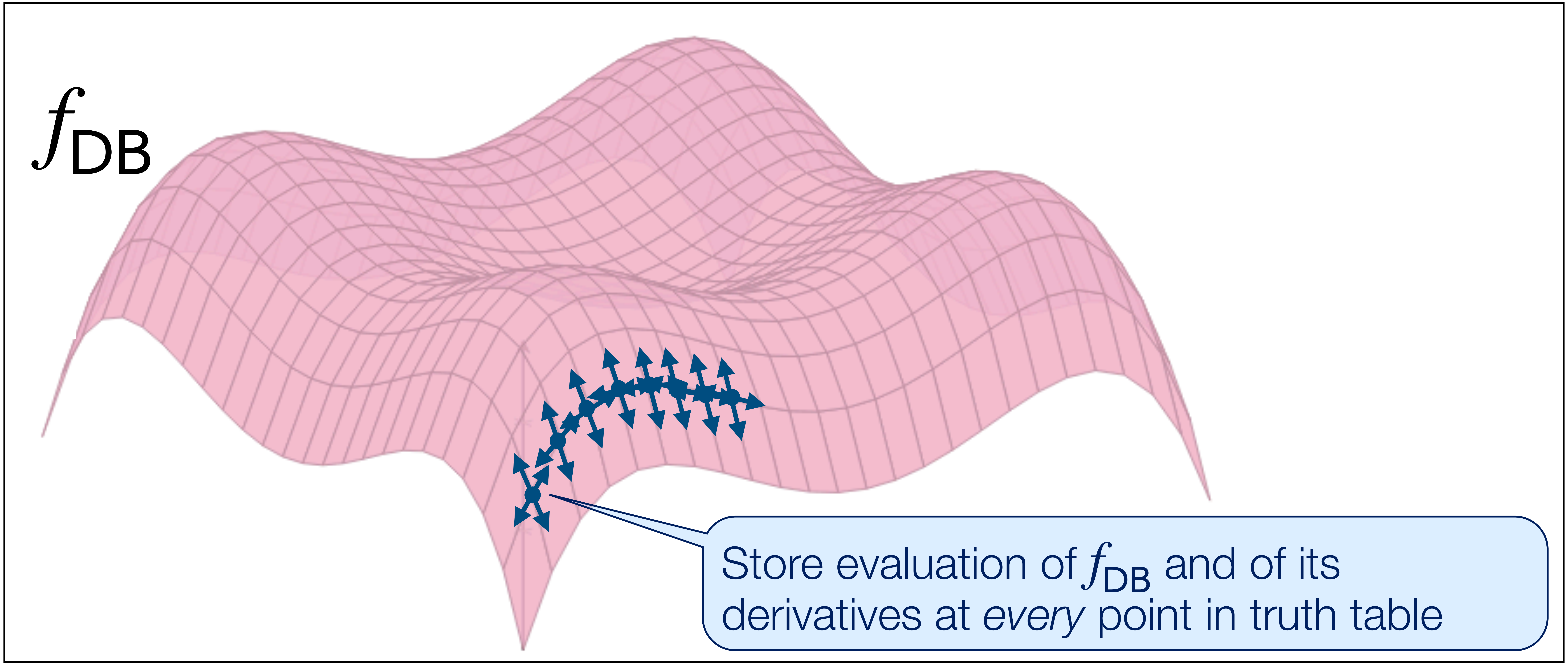

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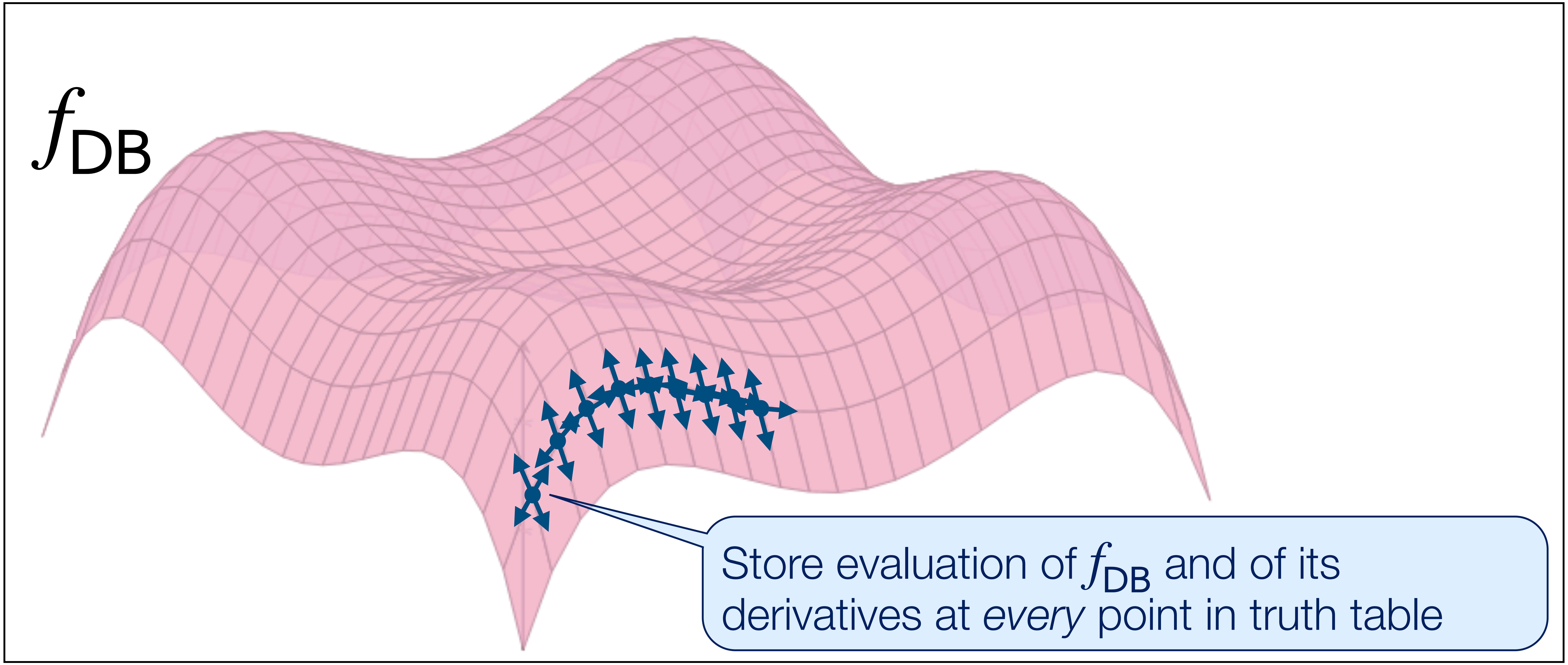

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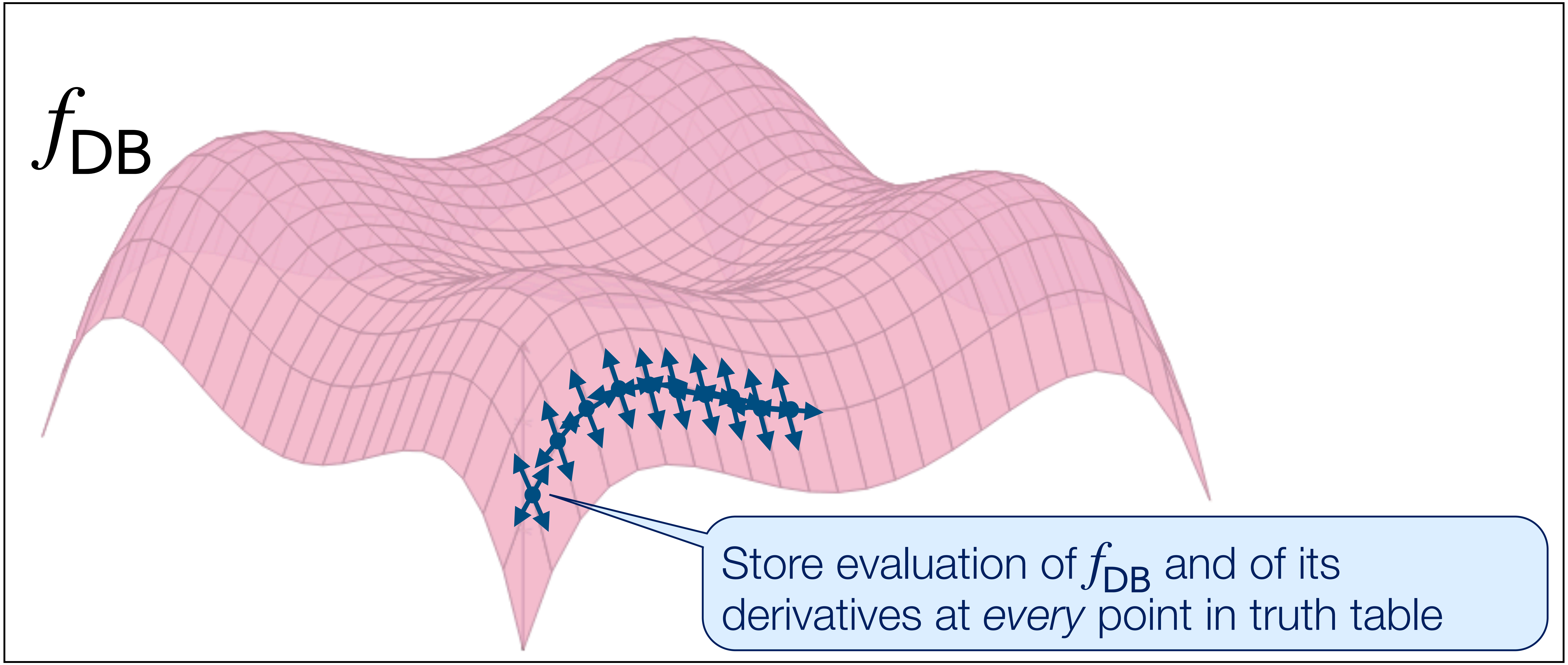

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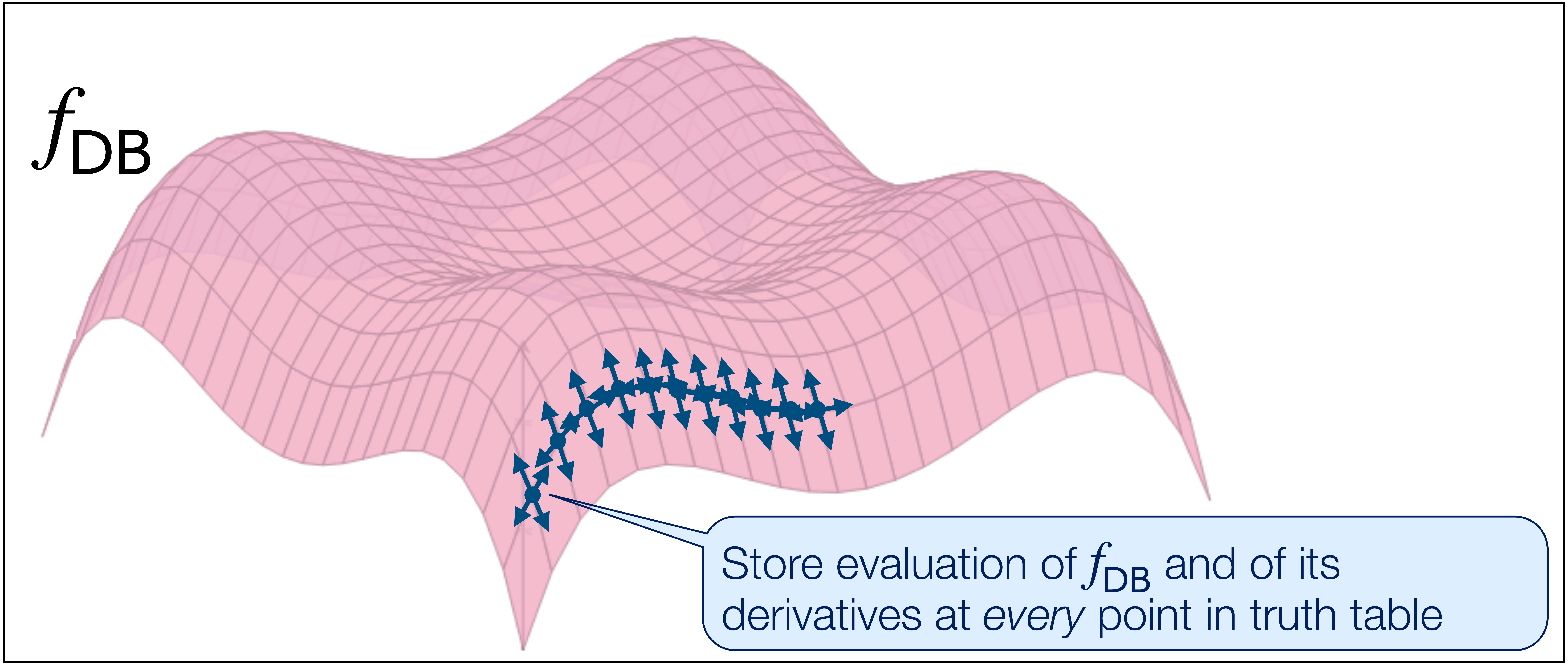

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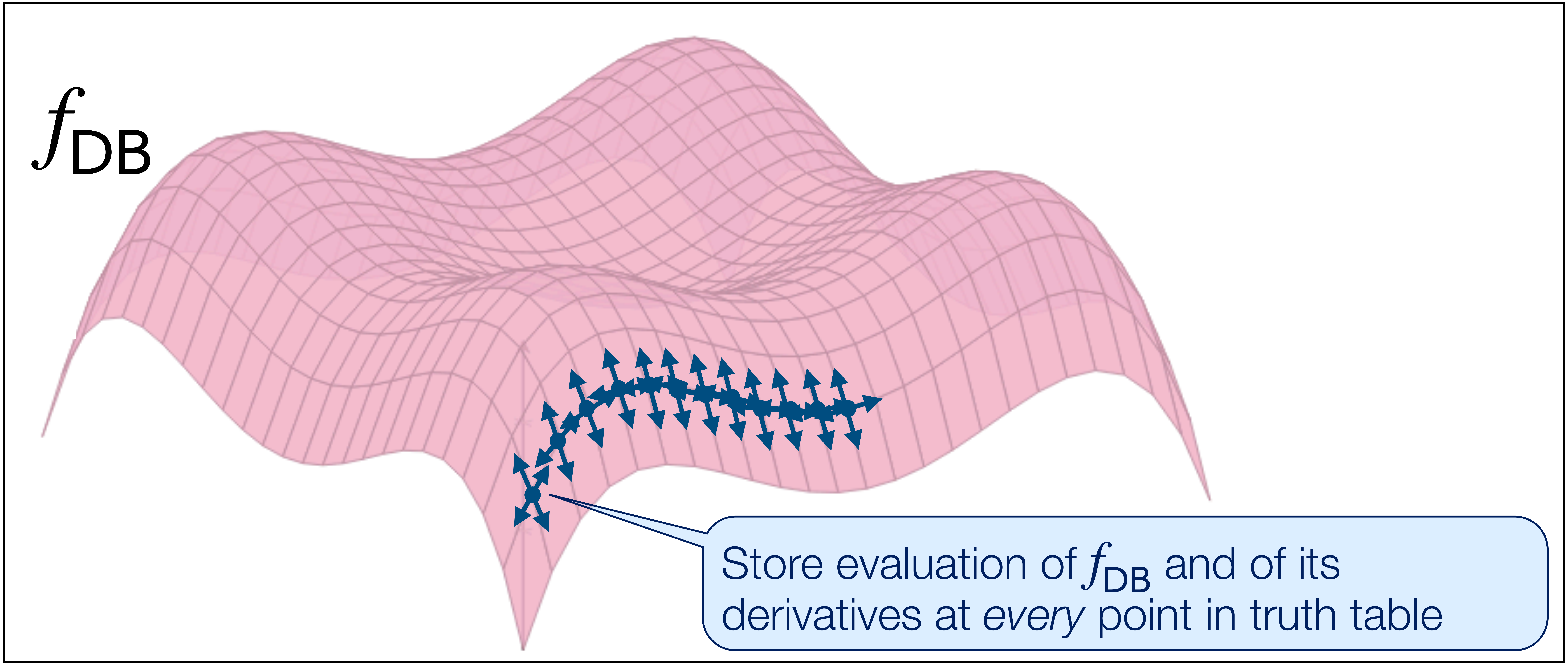

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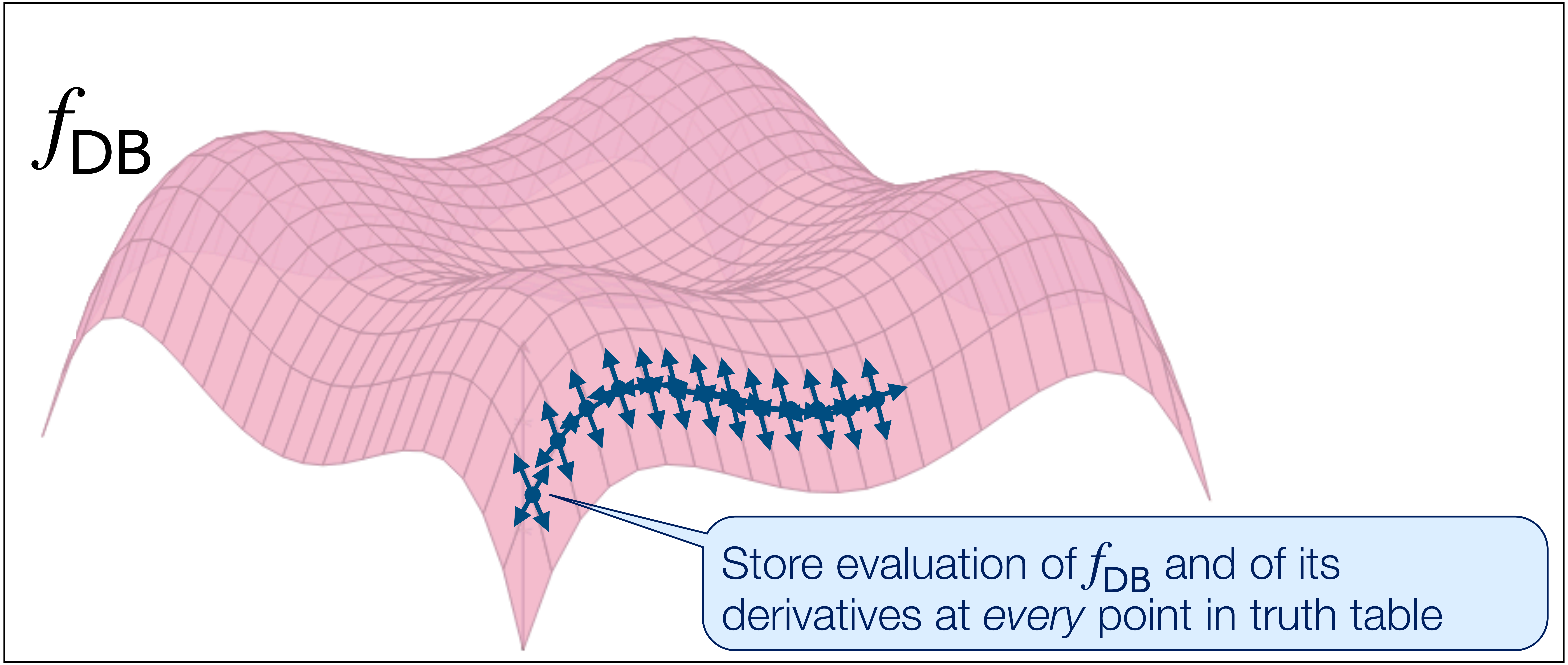

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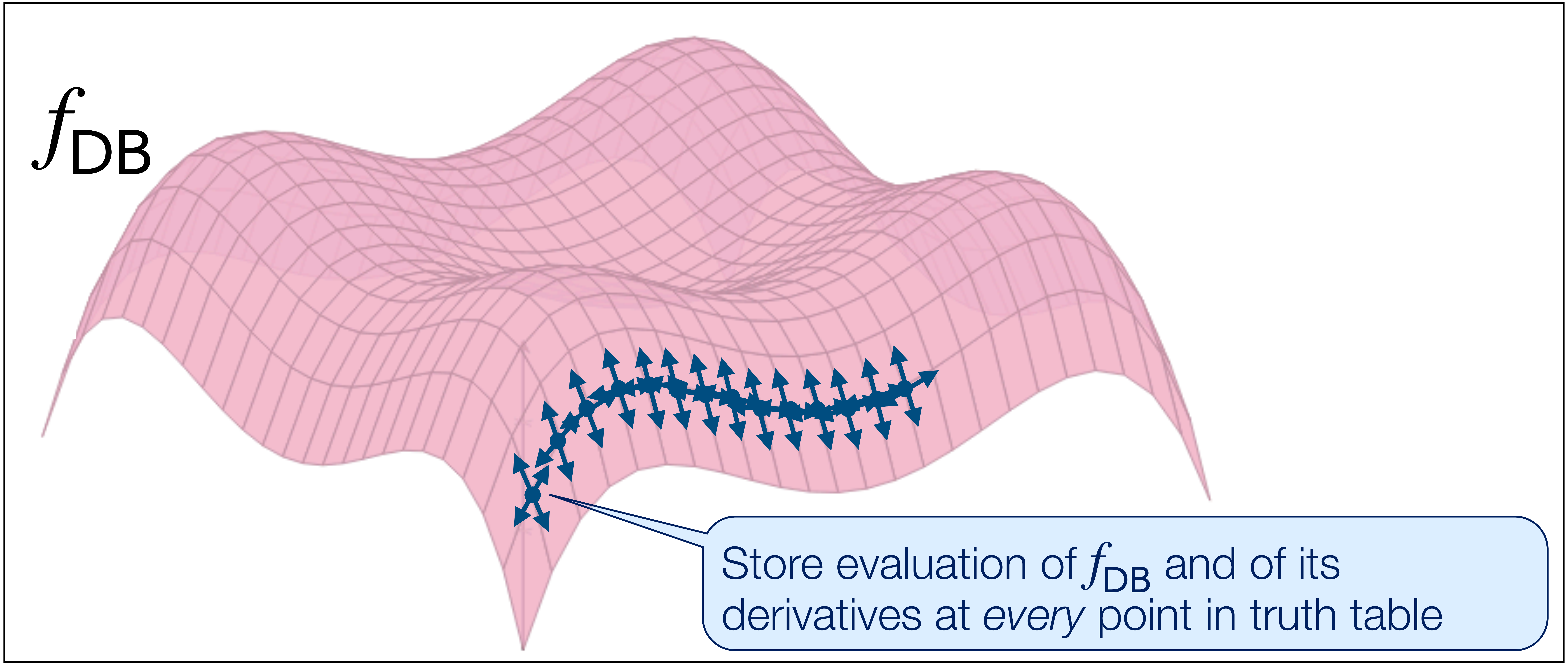

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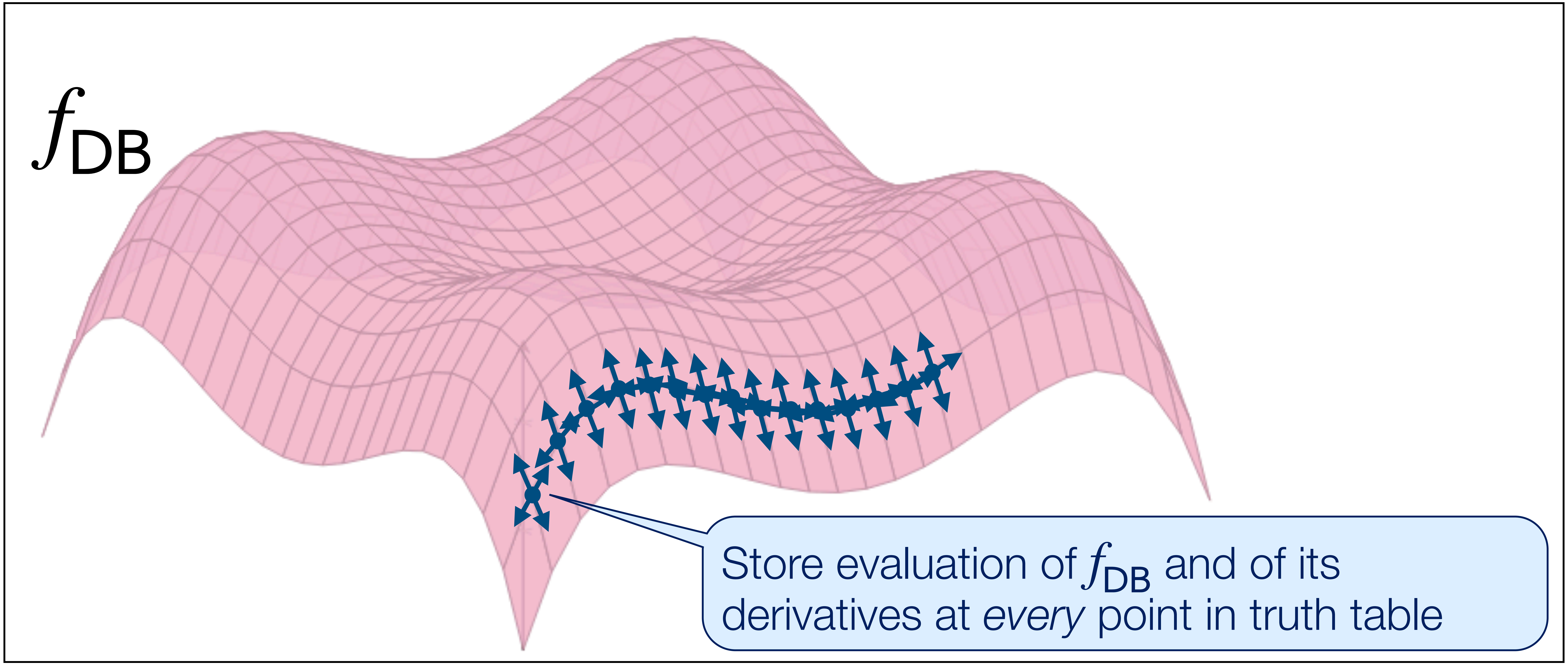

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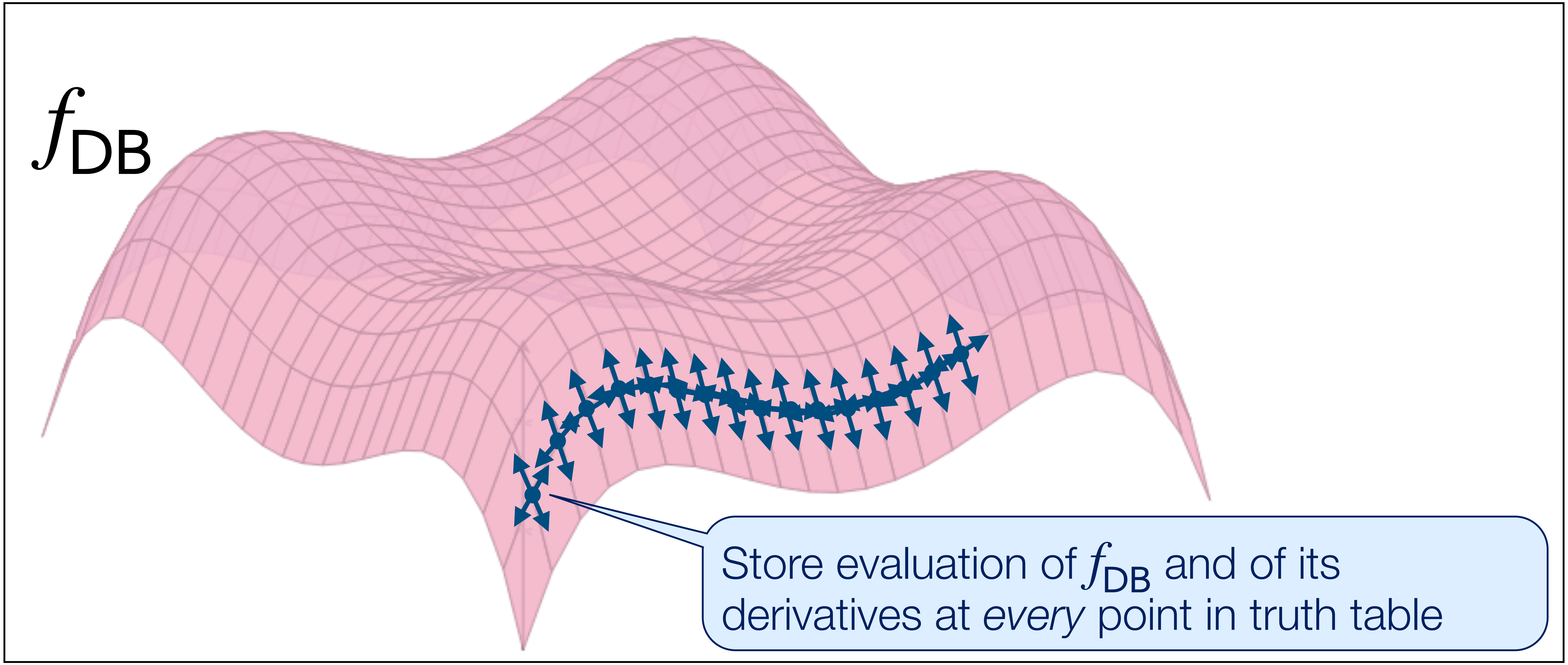

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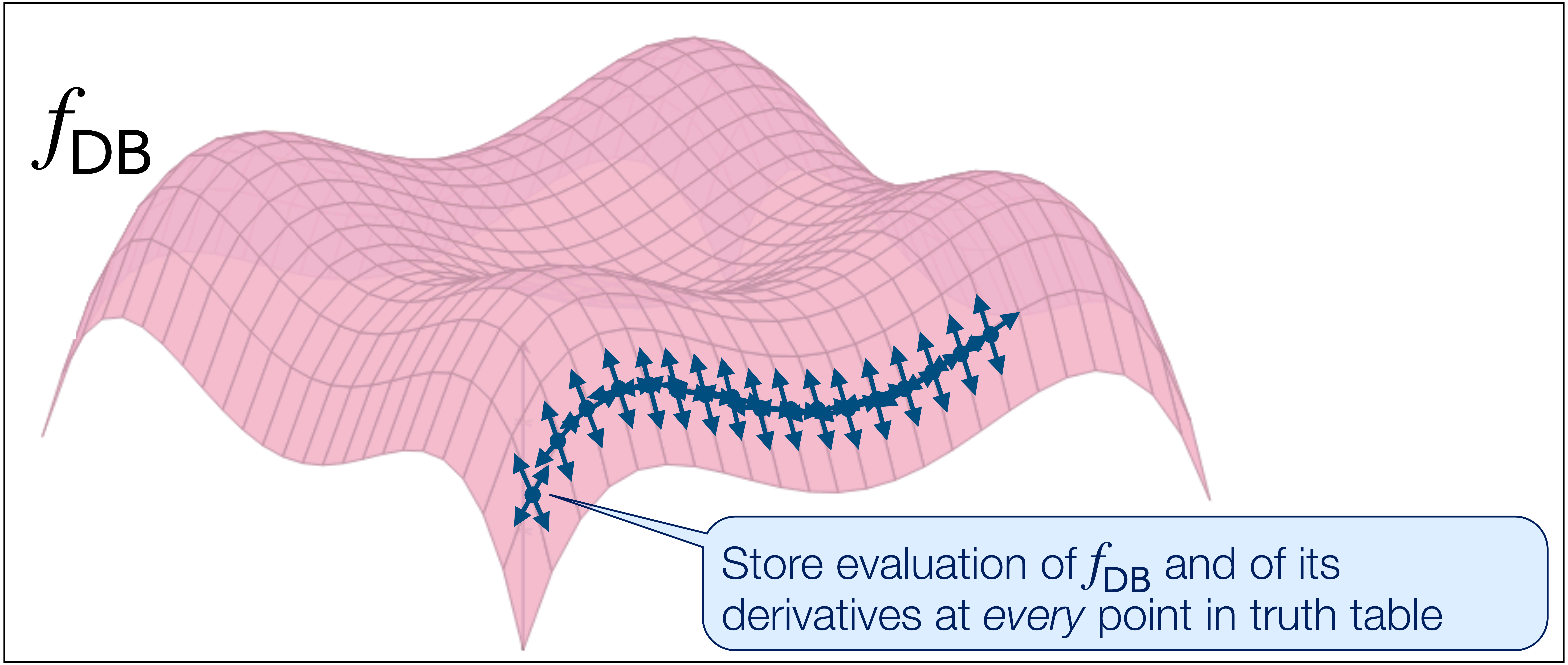

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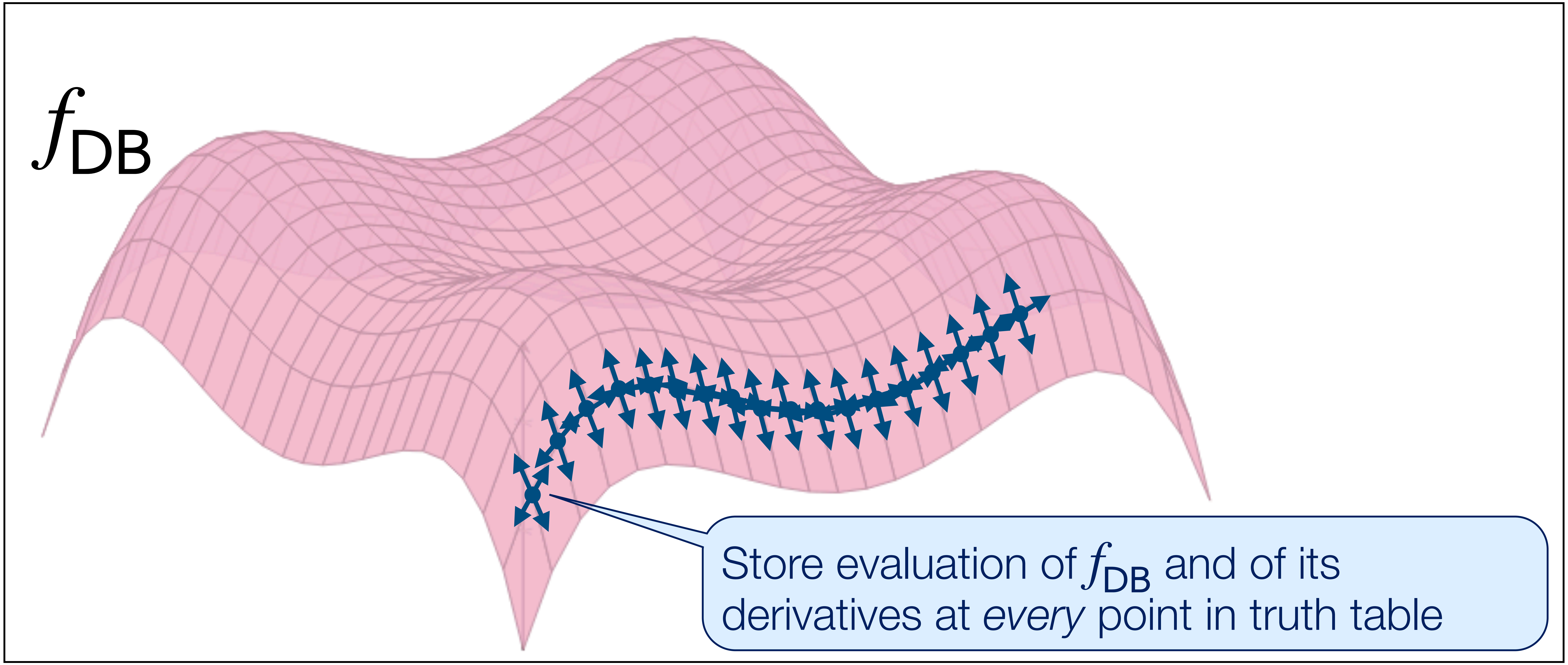

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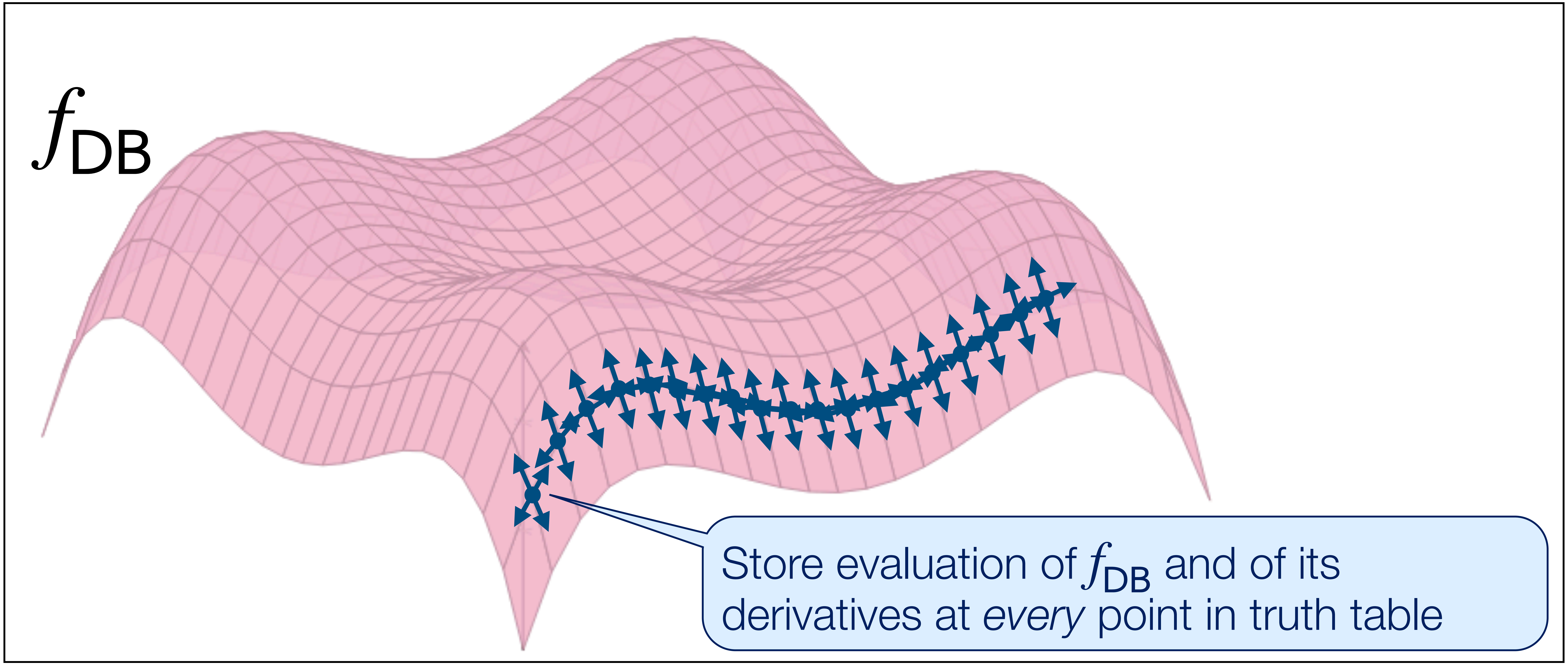

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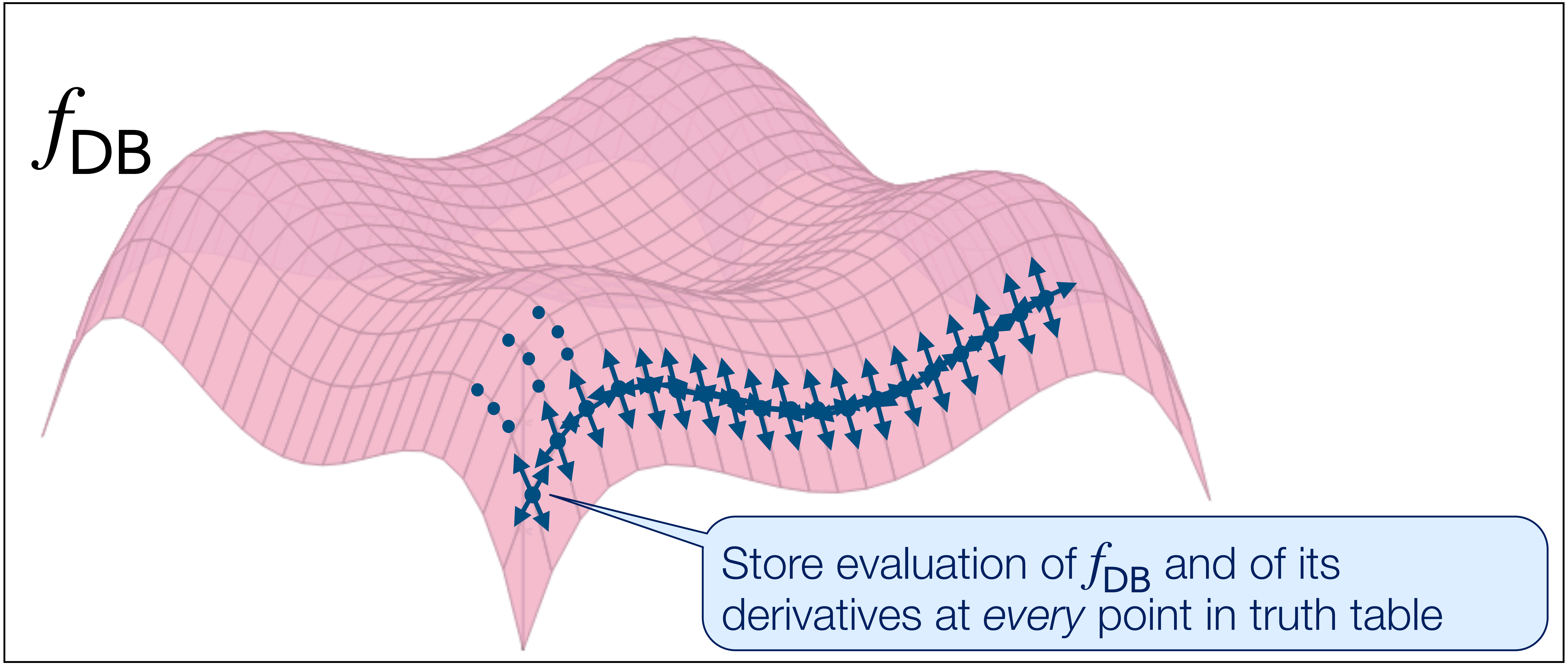

Store evaluation of f_{DB} and of its derivatives at *every* point in truth table

Fact 0. For a linear function $f : \mathbb{F} \to \mathbb{F}$, we have

$$f'(x) = f(x + 1) - f(x) \,.$$

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Fact 1. Since f_{DB} is multilinear, for any evaluation point $\mathbf{x} \in \mathbb{F}_2^m$,

$$\nabla f_{\mathsf{DB}}(\mathbf{x}) = \begin{bmatrix} \vdots \\ f_{\mathsf{DB}}(\mathbf{x} + \mathbf{u}_i) - f_{\mathsf{DB}}(\mathbf{x}) \\ \vdots \end{bmatrix} \in \mathbb{F}_2^m$$

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In other words: anyone can deduce $\nabla f_{\mathsf{DB}}(\mathbf{x})$ from the evaluations of f_{DB} in a Hamming ball of radius 1 around point $\mathbf{x}$.

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Idea: Save storage by using evaluations in Hamming balls instead of derivatives.

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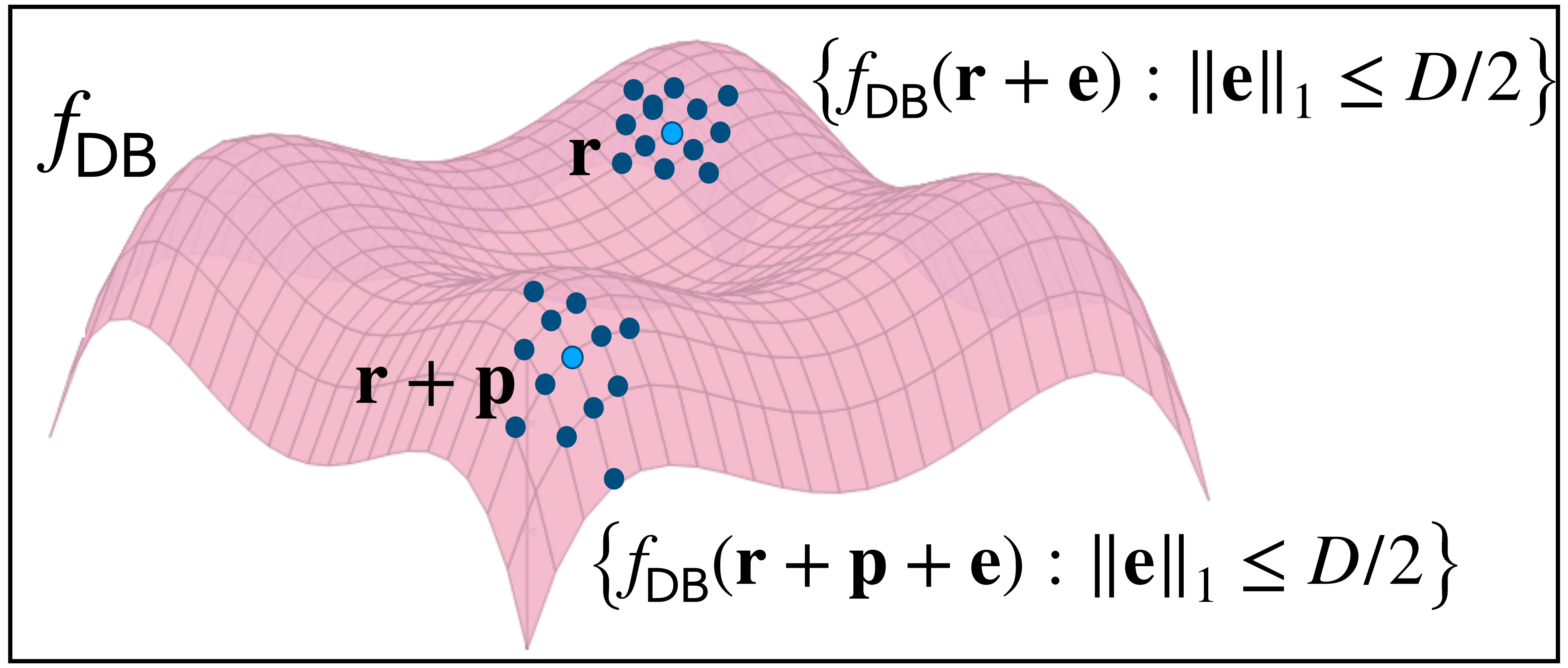
On query points $\mathbf{r}$ and $\mathbf{r} + \mathbf{p}$, the servers send back:


f_{DB}
$\mathbf{r}$
$\{f_{\mathsf{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\|_1 \leq D/2\}$
$\mathbf{r}+\mathbf{p}$
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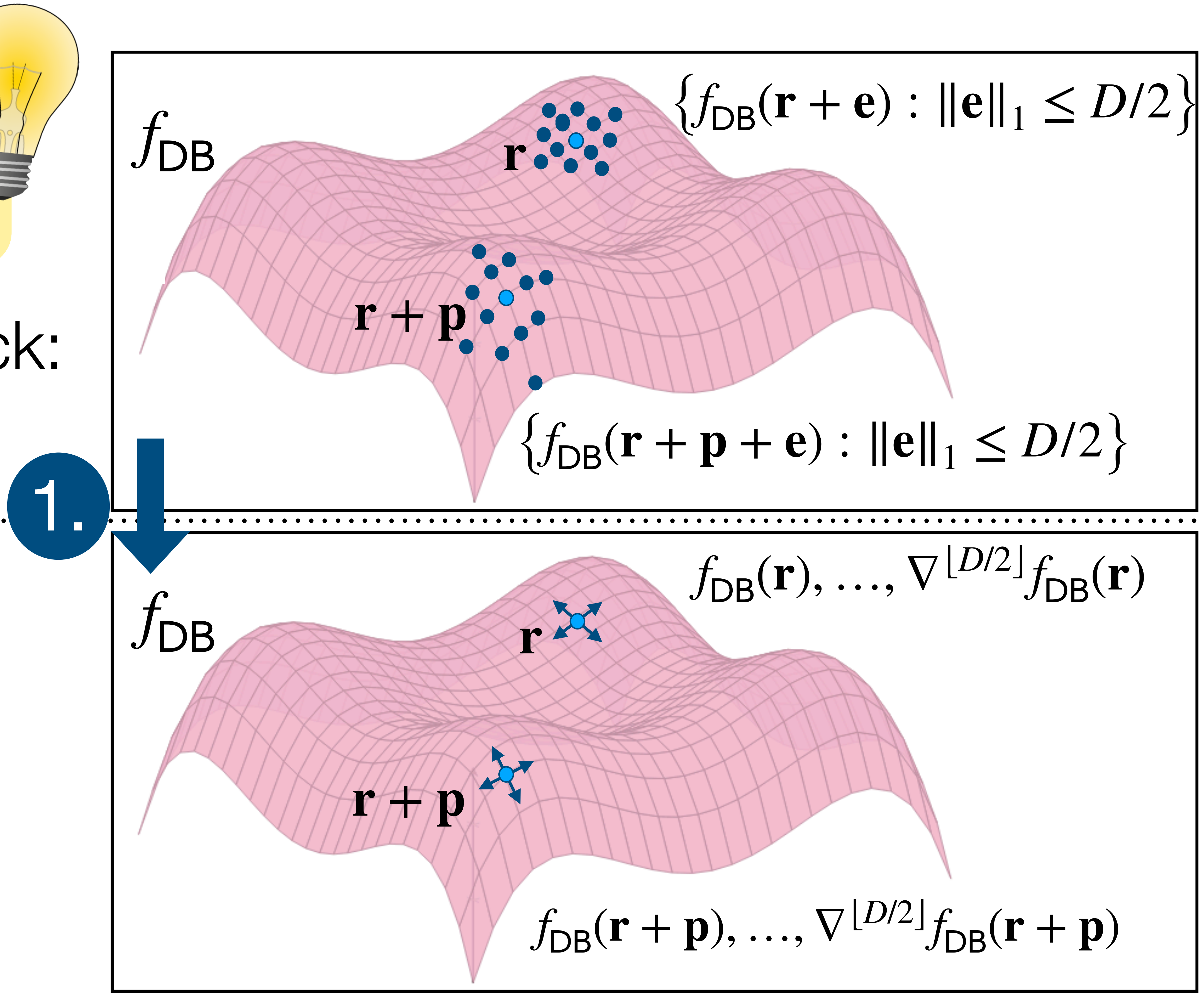
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1.
f_{DB}
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$\mathbf{r}$
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From these replies, the user computes:

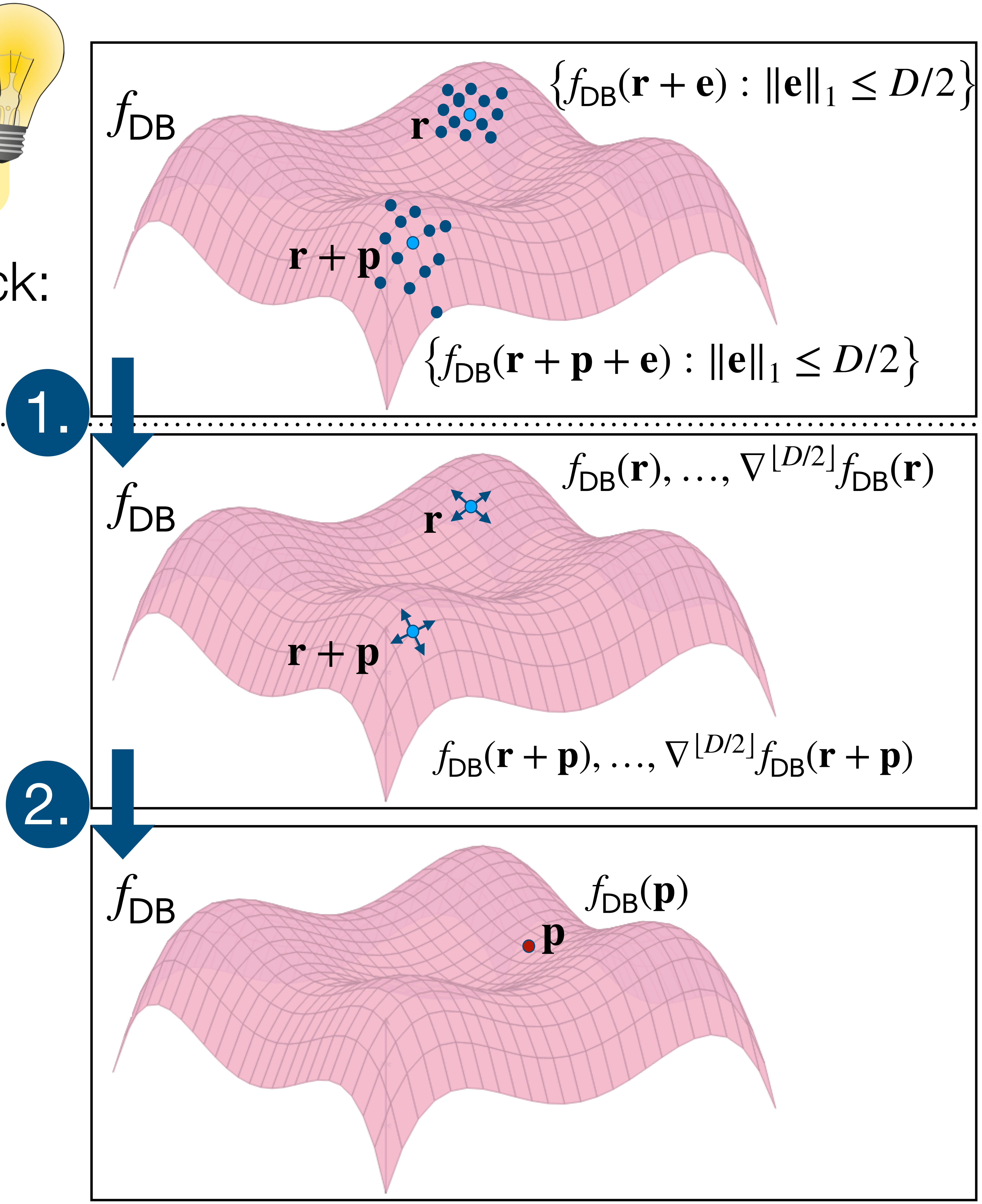
1. Finite differences

Idea: Save storage by using evaluations in Hamming balls instead of derivatives.

On query points $\mathbf{r}$ and $\mathbf{r}+\mathbf{p}$, the servers send back:

From these replies, the user computes:

1. Finite differences
2. Chain rule and Hermite interpolation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$\mathbf{r}$
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2.
f_{DB}
$f_{\mathrm{DB}}(\mathbf{p})$
$\mathbf{p}$

New PIR with preprocessing

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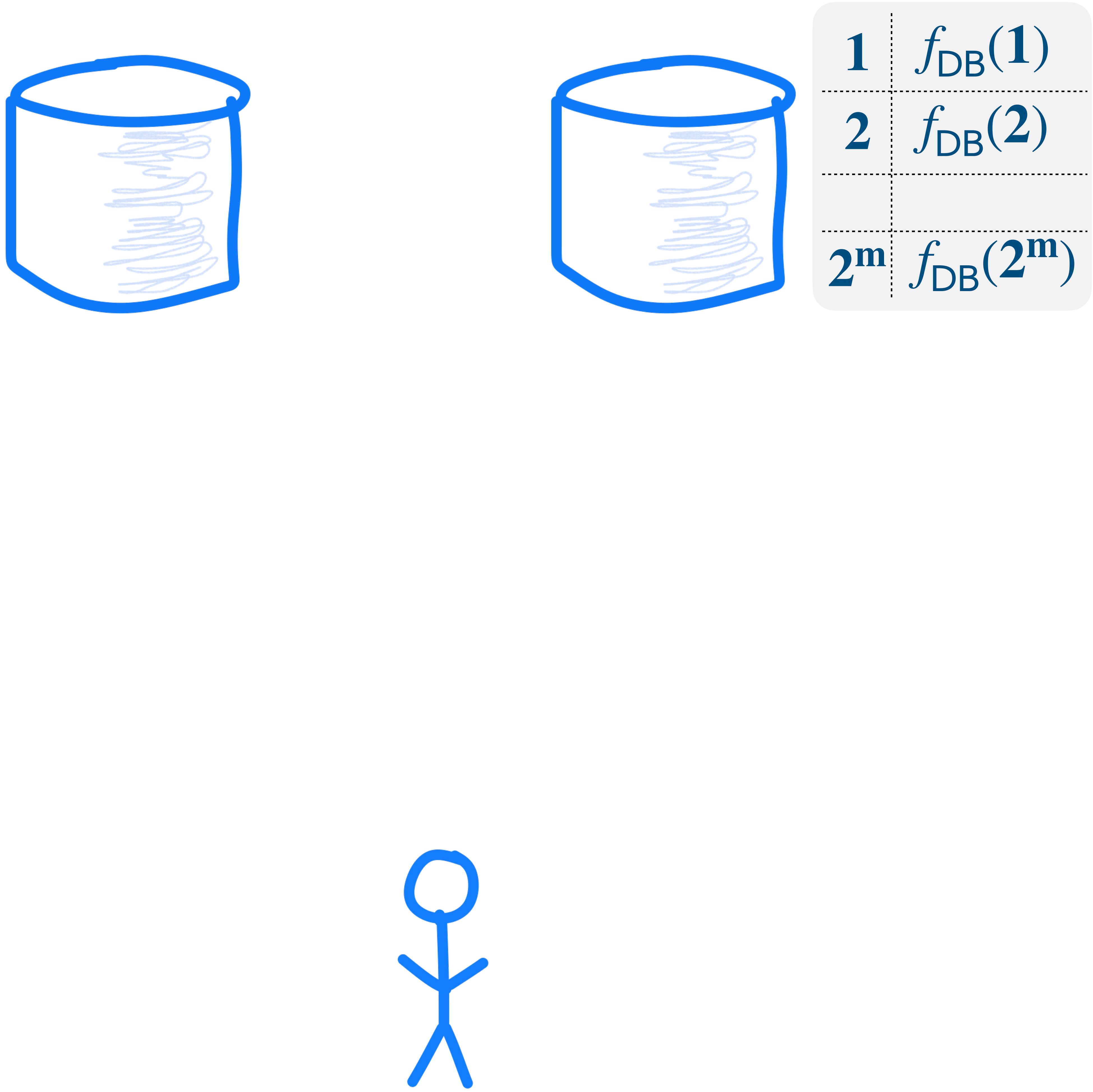

$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1})$
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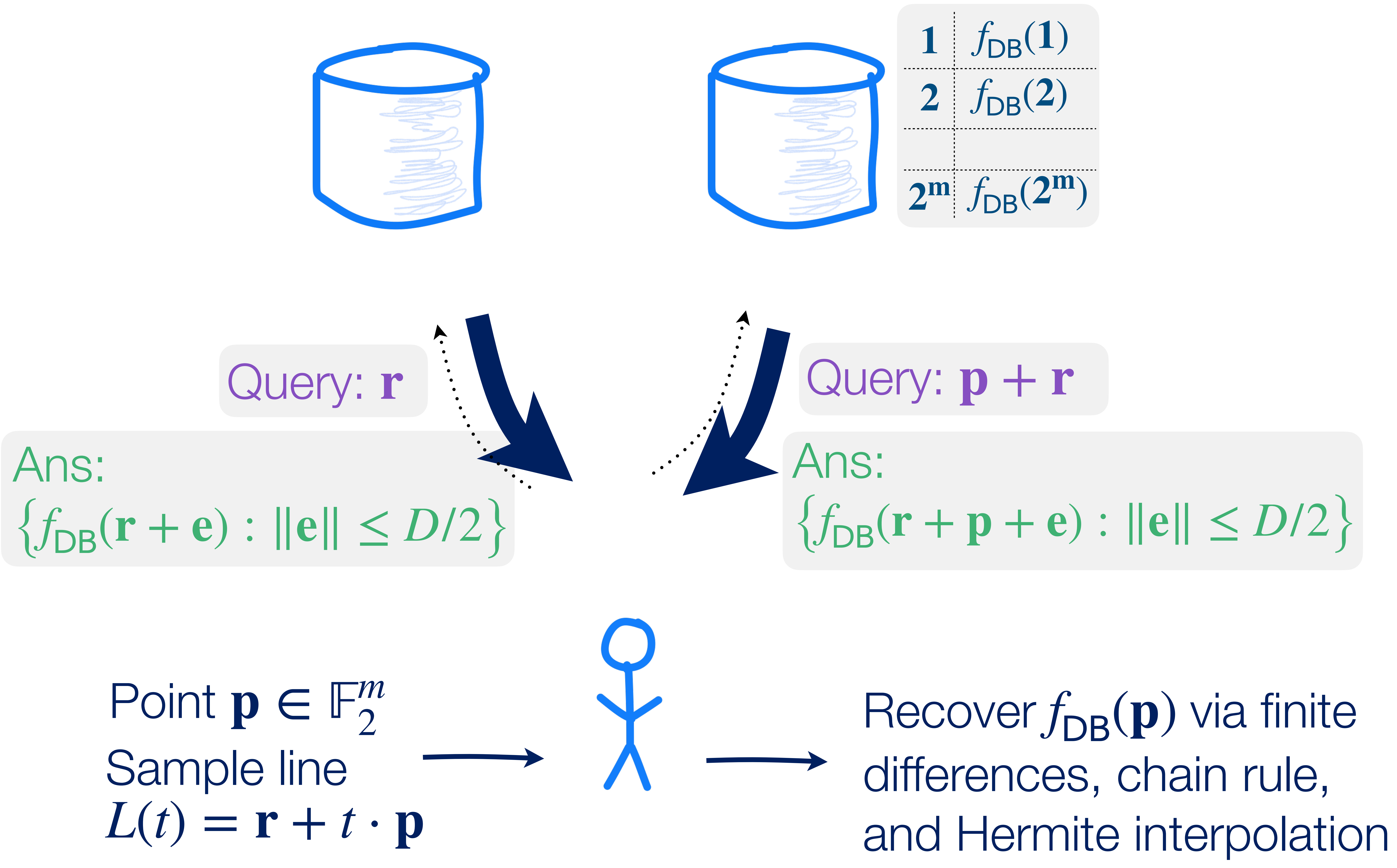
New PIR with preprocessing



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Query: $\mathbf{r}$

Ans: $\{f_{\mathsf{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\| \le D/2\}$

Query: $\mathbf{p}+\mathbf{r}$

Ans: $\{f_{\mathsf{DB}}(\mathbf{r}+\mathbf{p}+\mathbf{e}) : \|\mathbf{e}\| \le D/2\}$

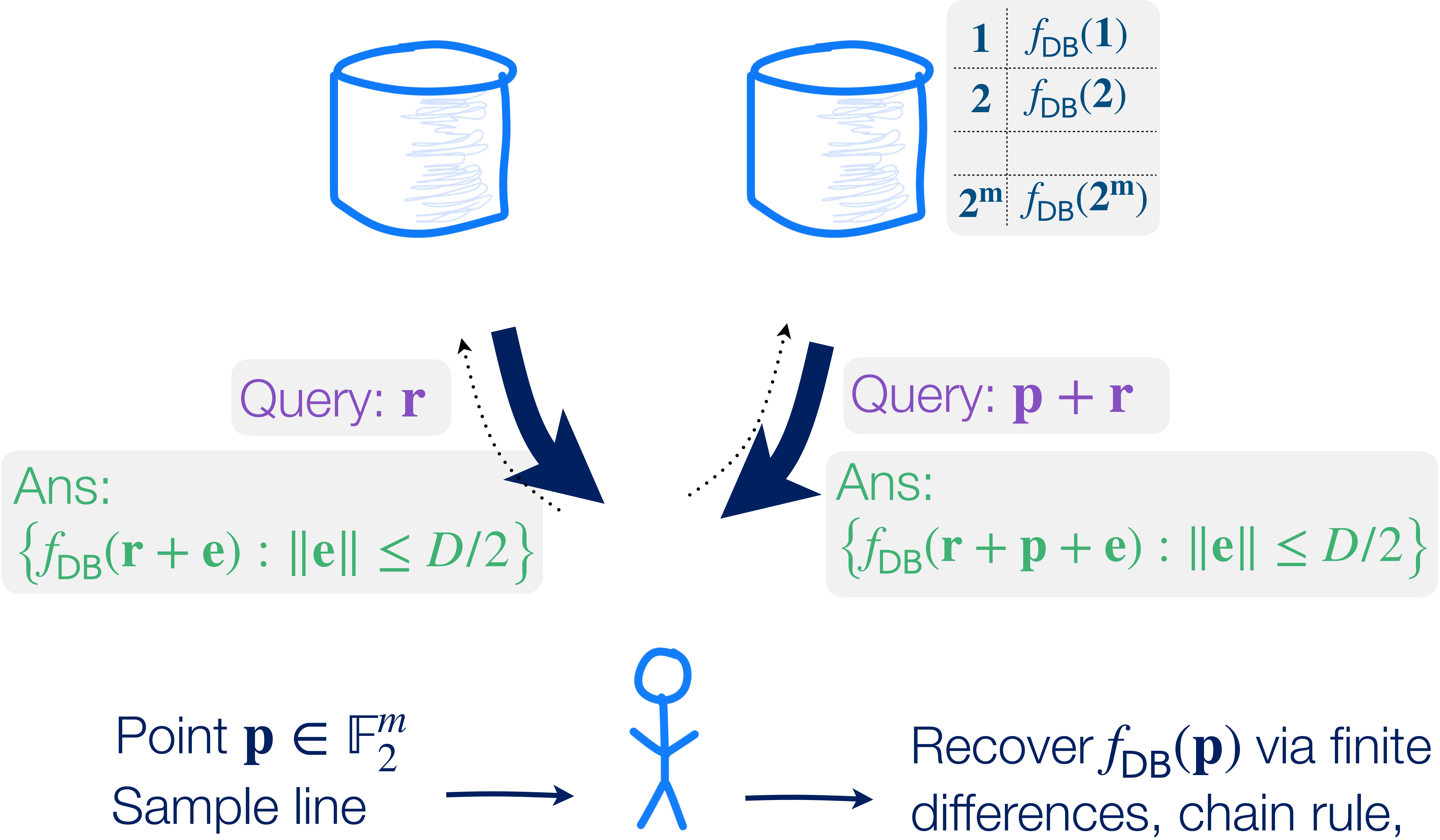
Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line

$L(t) = \mathbf{r} + t \cdot \mathbf{p}$

Recover $f_{\mathsf{DB}}(\mathbf{p})$ via finite differences, chain rule, and Hermite interpolation

New PIR with preprocessing



$\mathbf{1}$	$f_{\text{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\text{DB}}(\mathbf{2})$
$\mathbf{2^m}$	$f_{\text{DB}}(\mathbf{2^m})$

Query: $\mathbf{r}$

Ans: $\{f_{\text{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\| \leq D/2\}$

Query: $\mathbf{p}+\mathbf{r}$

Ans: $\{f_{\text{DB}}(\mathbf{r}+\mathbf{p}+\mathbf{e}) : \|\mathbf{e}\| \leq D/2\}$

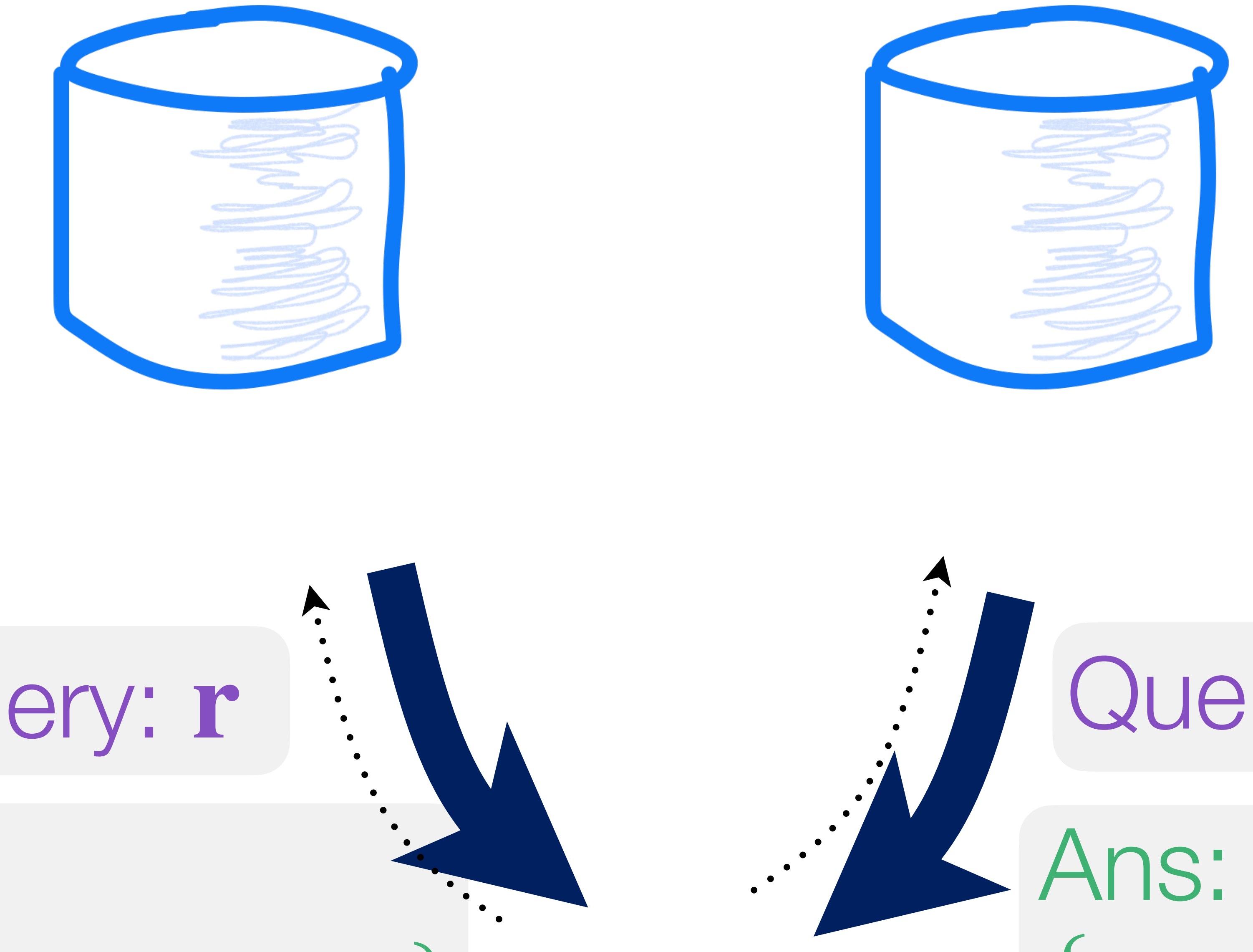
Point $\mathbf{p} \in \mathbb{F}_2^m$
Sample line
$L(t) = \mathbf{r} + t \cdot \mathbf{p}$

Recover $f_{\text{DB}}(\mathbf{p})$ via finite differences, chain rule, and Hermite interpolation

With 2 servers, gives preprocessing PIR with

- ➡ Same comm. as [BIM00]:
  $O(\log n)$ upload
  $n^{0.82}$ download
- ➡ Same time as [BIM00]:
  $O(n^{0.82})$ work
- ➡ Quasilinear space:
  $2^m = n^{1+o(1)}$ bits

New PIR with preprocessing



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Query: $\mathbf{r}$

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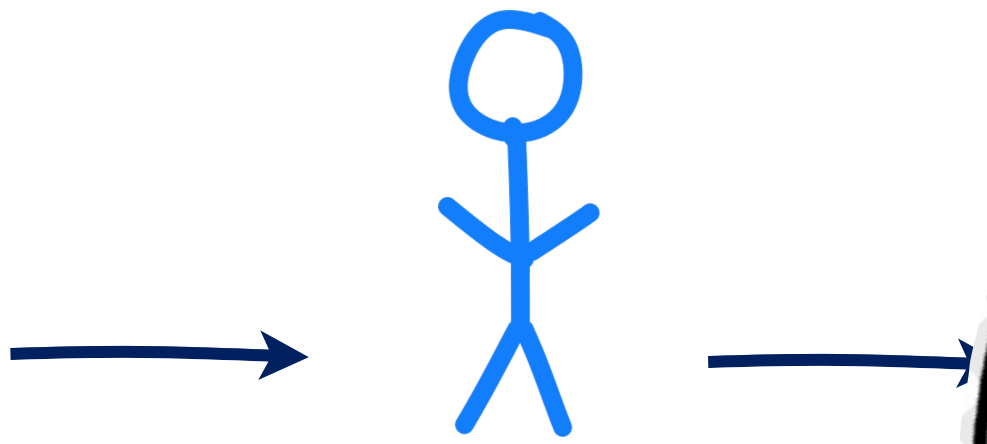

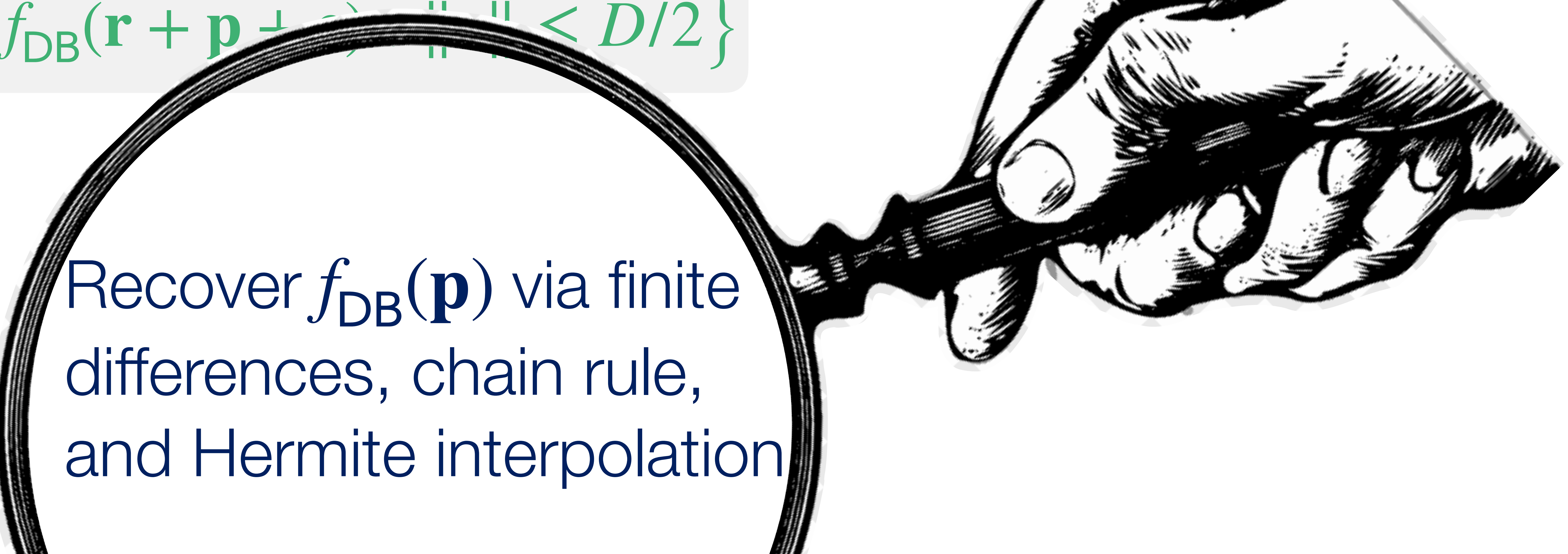

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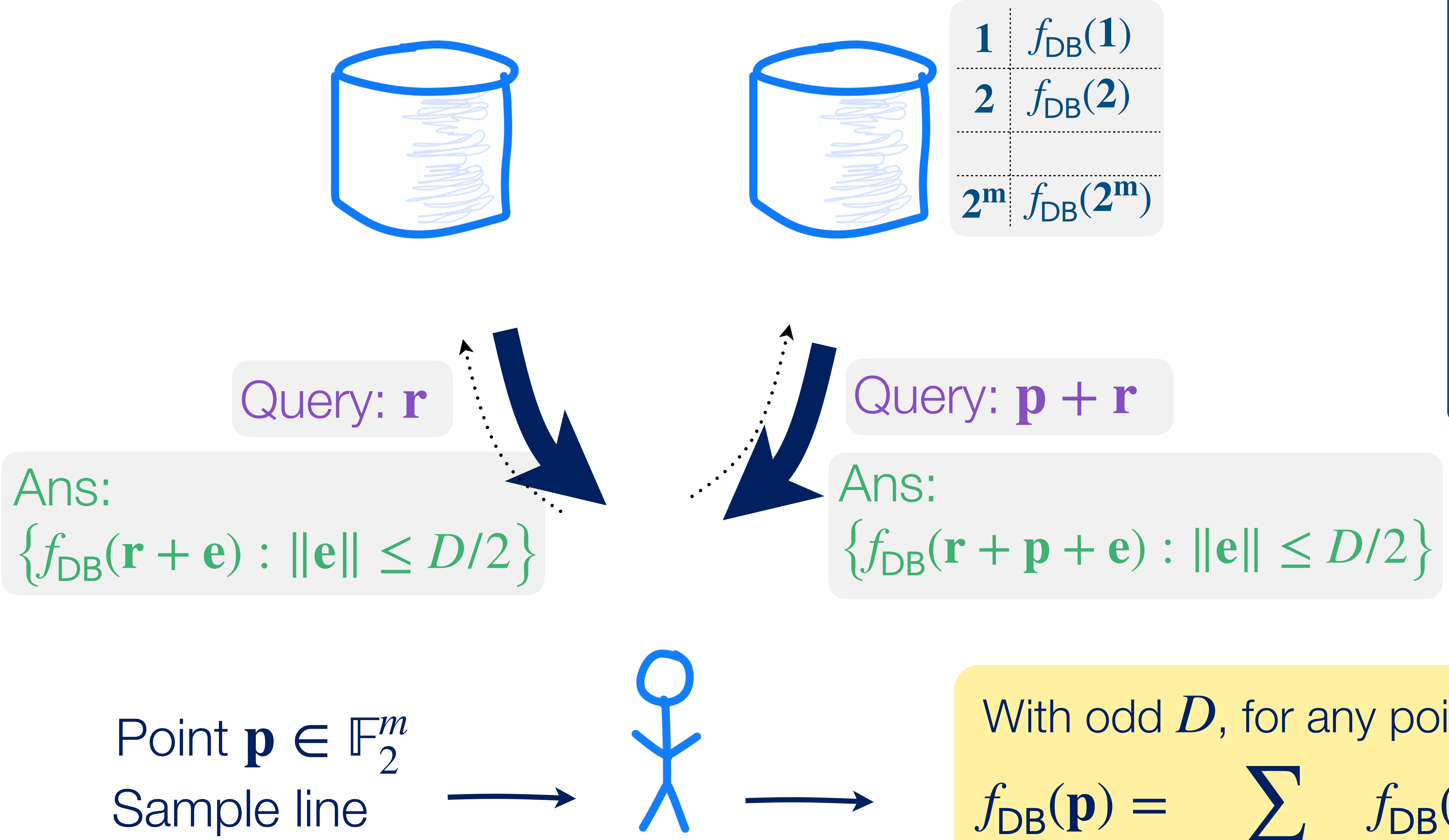

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$\mathbf{2^m}$	$f_{\mathsf{DB}}(\mathbf{2^m})$

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Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line

$L(t) = \mathbf{r} + t \cdot \mathbf{p}$

With odd D, for any point $\mathbf{p}$ with $\|\mathbf{p}\| = D$:

$$f_{\mathsf{DB}}(\mathbf{p}) = \sum_{\substack{\|\mathbf{e}\| \leq \lfloor D/2 \rfloor \\ \mathbf{e} \leq \mathbf{p}}} f_{\mathsf{DB}}(\mathbf{r}+\mathbf{e}) + f_{\mathsf{DB}}(\mathbf{p}+\mathbf{r}+\mathbf{e})$$

With 2 servers, gives preprocessing PIR with

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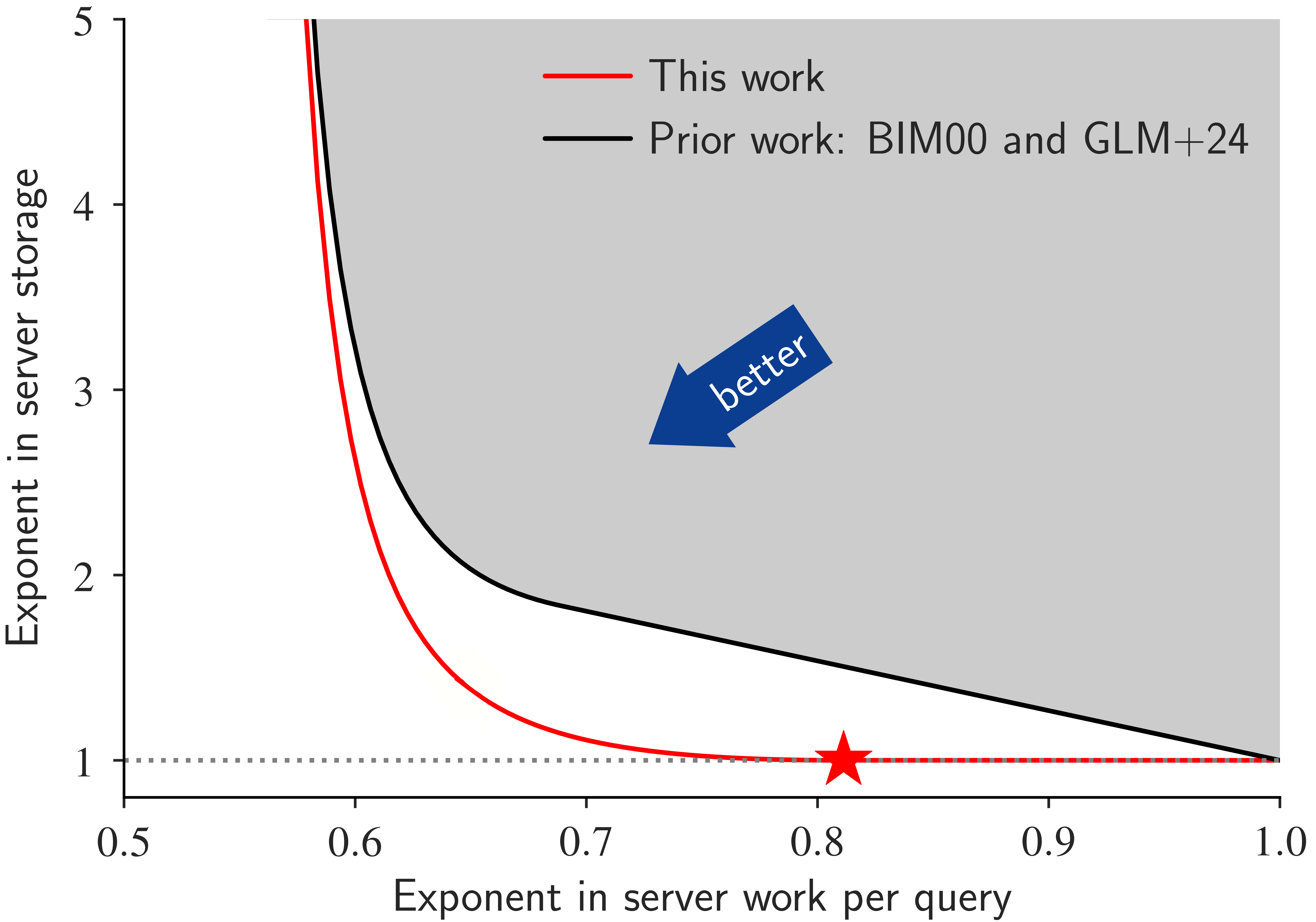
$2^m = n^{1+o(1)}$ bits

Theorem. On any database of $n > 10^6$ bits, there exists information-theoretic, two-server PIR with preprocessing with:

- $1.5 \cdot \sqrt{\log n} \cdot n$ bits of server storage,
- $12 \cdot n^{0.82}$ server RAM lookups per query, and
- $12 \cdot n^{0.82}$ bits of communication per query.

✓


This work
Prior work: BIM00 and GLM+24
better
5
4
3
2
1
0.5
0.6
0.7
0.8
0.9
1.0
Exponent in server storage
Exponent in server work per query

This talk

1. **Background:** PIR with preprocessing
2. **[HR26] New two-server PIR:** sublinear time, quasilinear space
3. **Evaluation:** what does this mean for practice?
4. **Bonuses** 😁
    - Reducing communication using crypto
    - **[HPR26]** Connecting multi-server PIR to complexity theory

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Concrete Evaluation: Storage

Compared to prior PIR with preprocessing [BIM00,GLM+25]

DB size (GB) with 1-byte records	Communication (MB)	Our storage (TB)	BIM00 storage (TB)
2	0.7	1	7.6×10^5
11	4.4	1	4.4×10^6
37	22.2	1	4.9×10^6
82	95.5	1	1.3×10^6

Concrete Evaluation: Space-Time Tradeoff

Compared to fastest two-server, linear-time PIR with $\sqrt{n}$ communication

DB size (GB) with 1-byte records	**Storage blowup**	**Communication blowup**	**Memory accesses saved**	**Throughput improvement**
2	512x	14x	2,926x	10.2x
11	93x	37x	2,560x	9.0x
37	28x	101x	1,707x	5.5x
82	12x	298x	879x	1.8x

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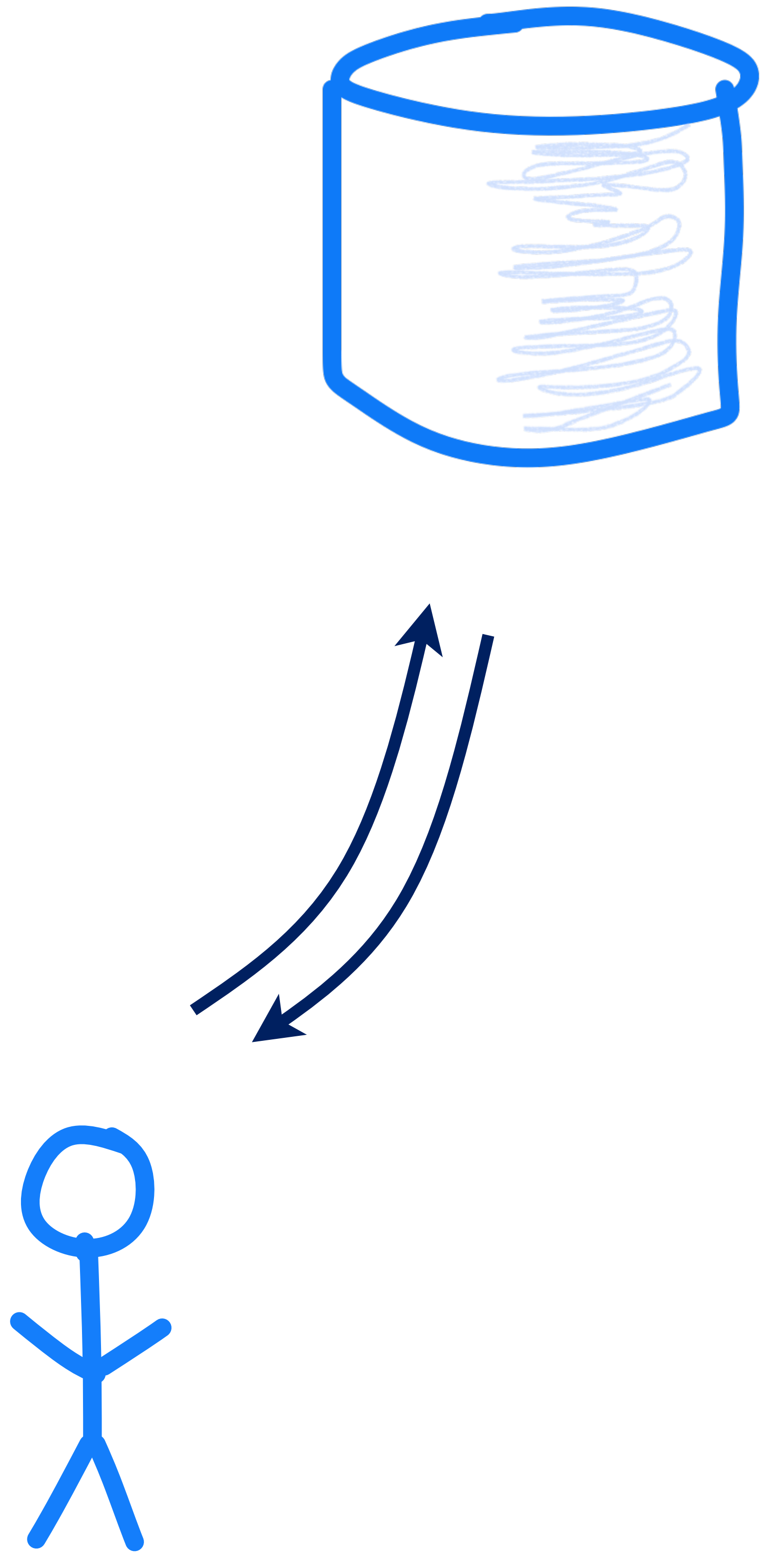
Homomorphic Encryption for Single-Server PIR

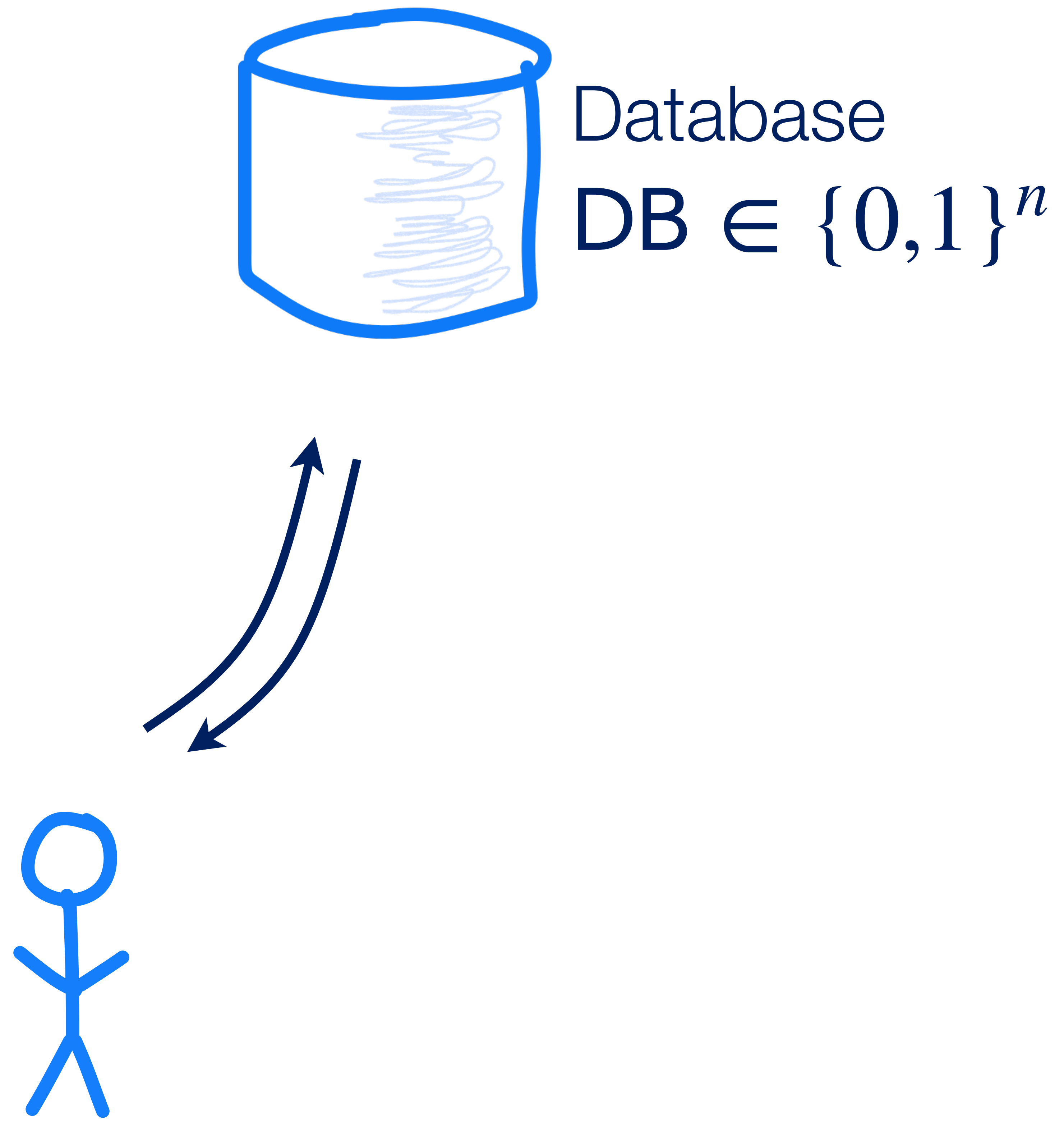


Database

DB $\in \{0,1\}^n$

Homomorphic Encryption for Single-Server PIR

- PIR: computing $\mathsf{DB}[i]$ without revealing i



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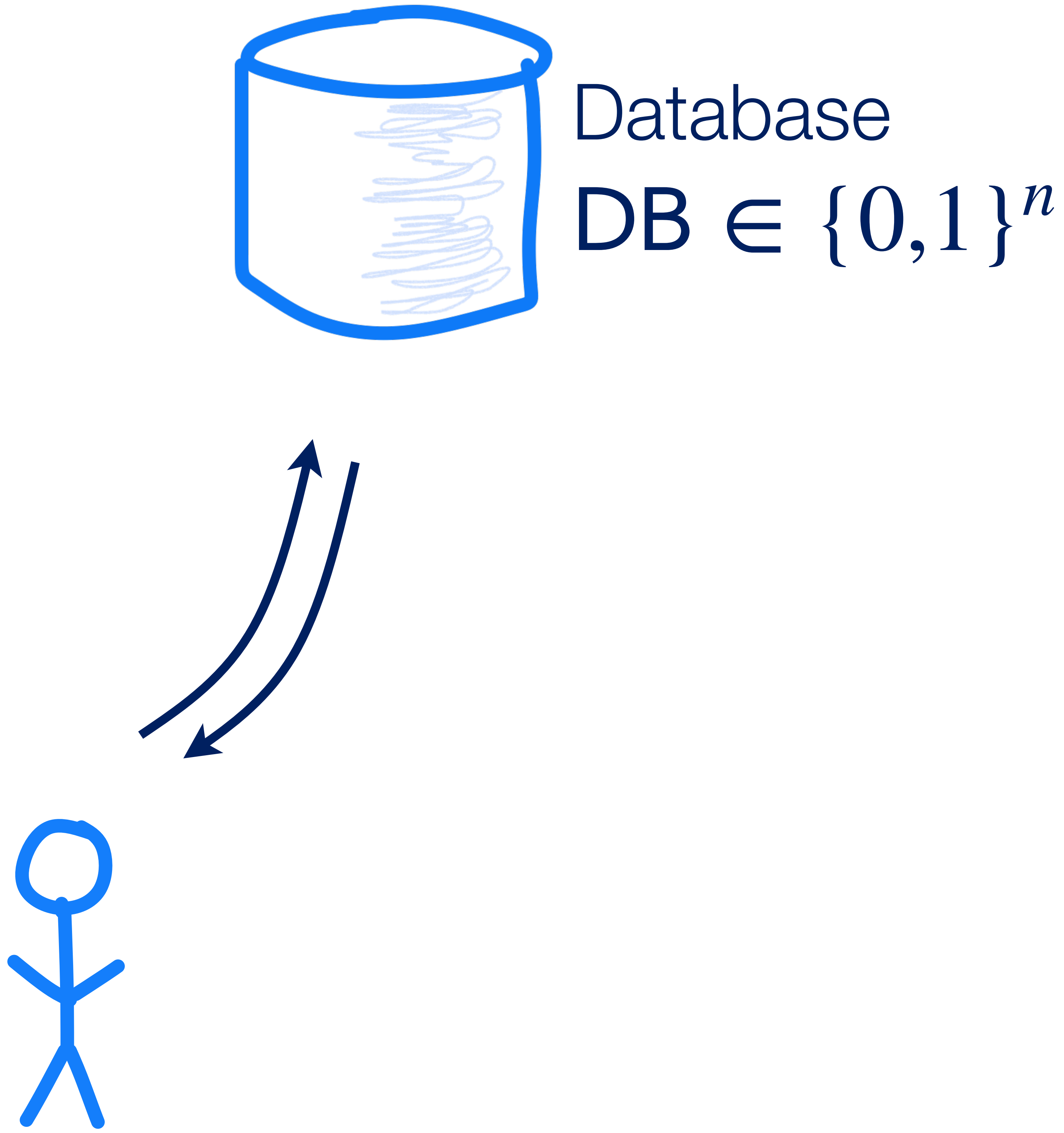

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- PIR: computing $\mathsf{DB}[i]$ without revealing i
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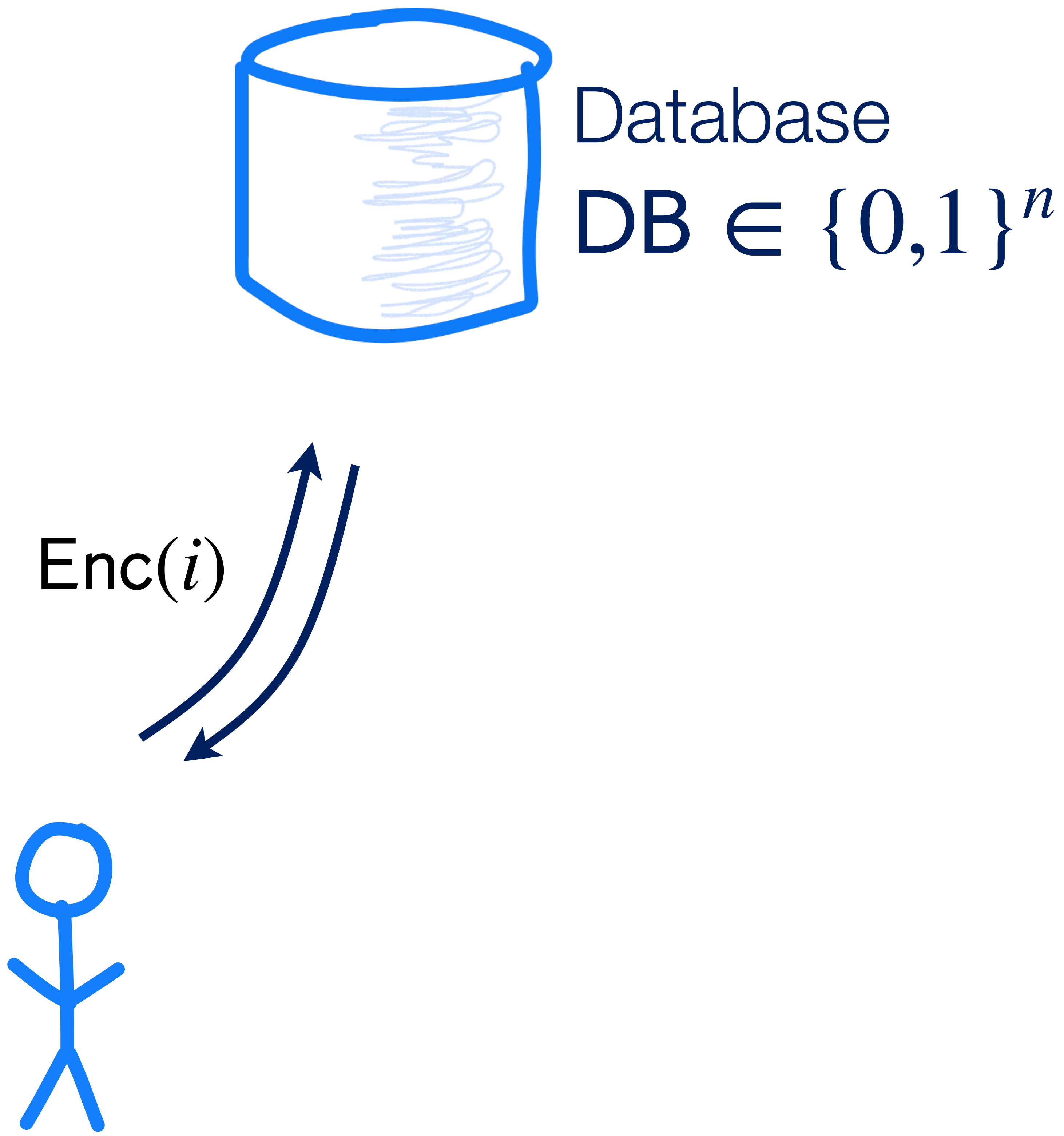
  - Query: $\mathsf{ct} \leftarrow \mathsf{FHE.Enc}(i)$



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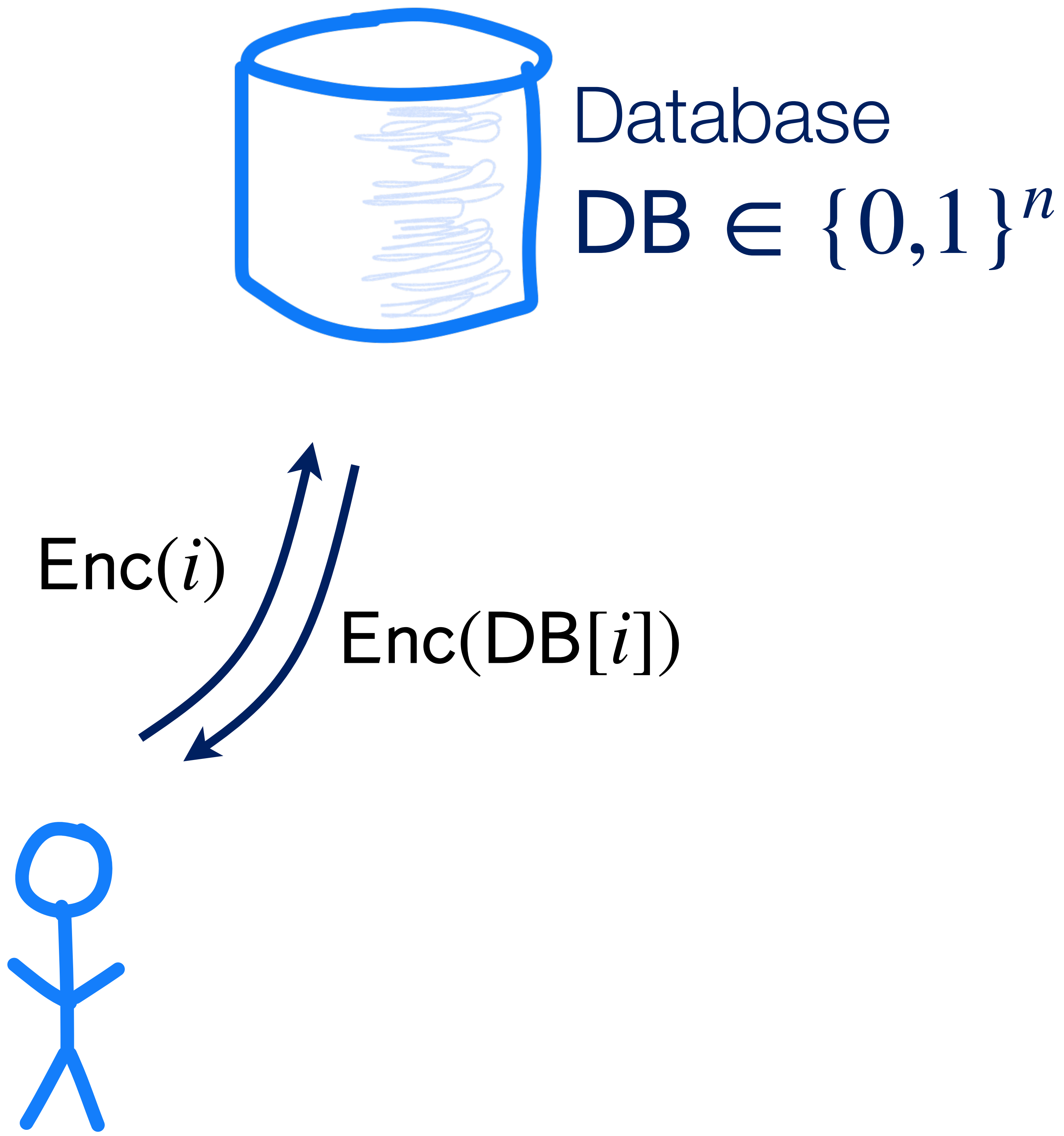

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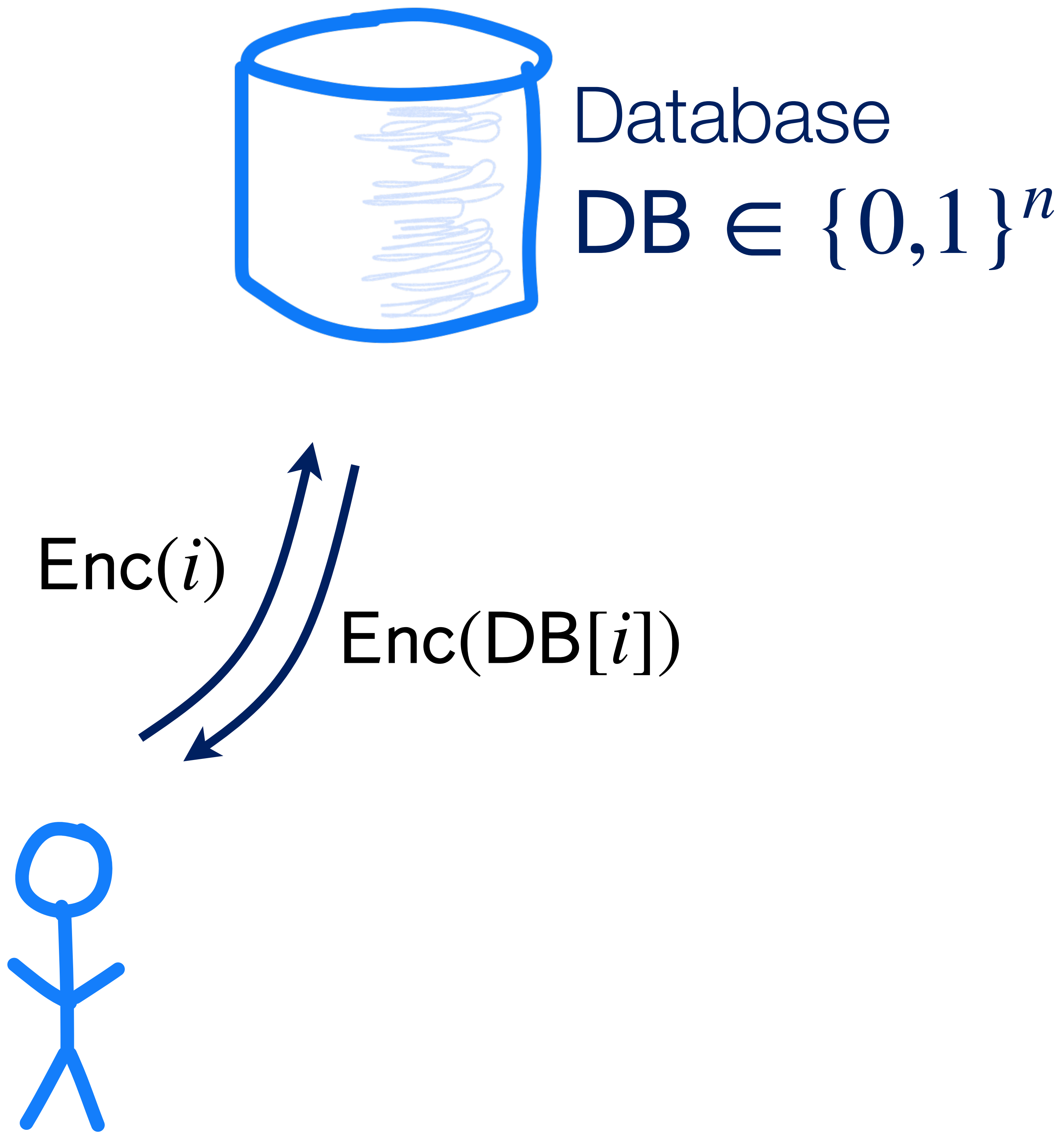

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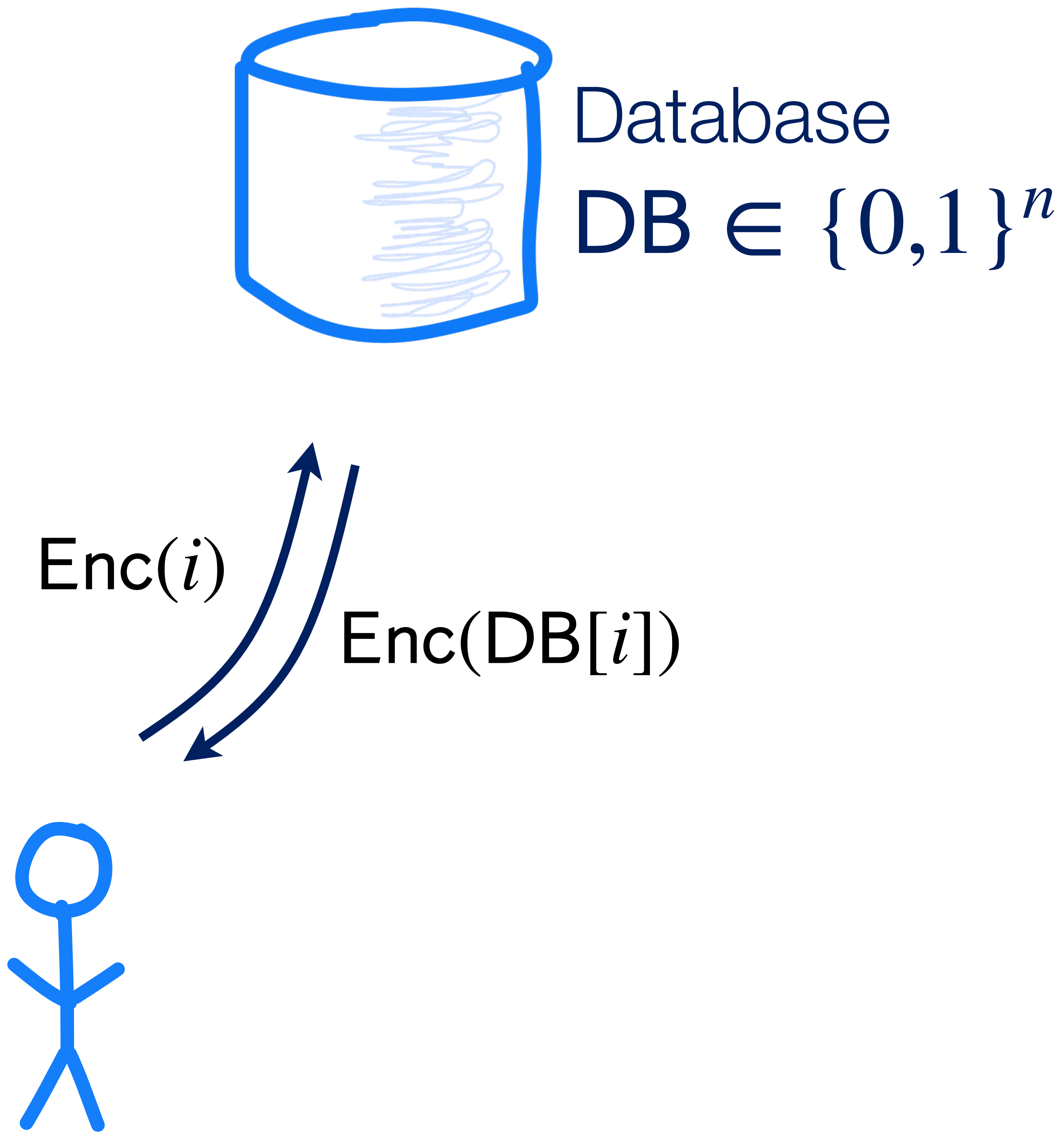
- But $\mathsf{DB}[\,\cdot\,]$ is a size $O(n)$ circuit!



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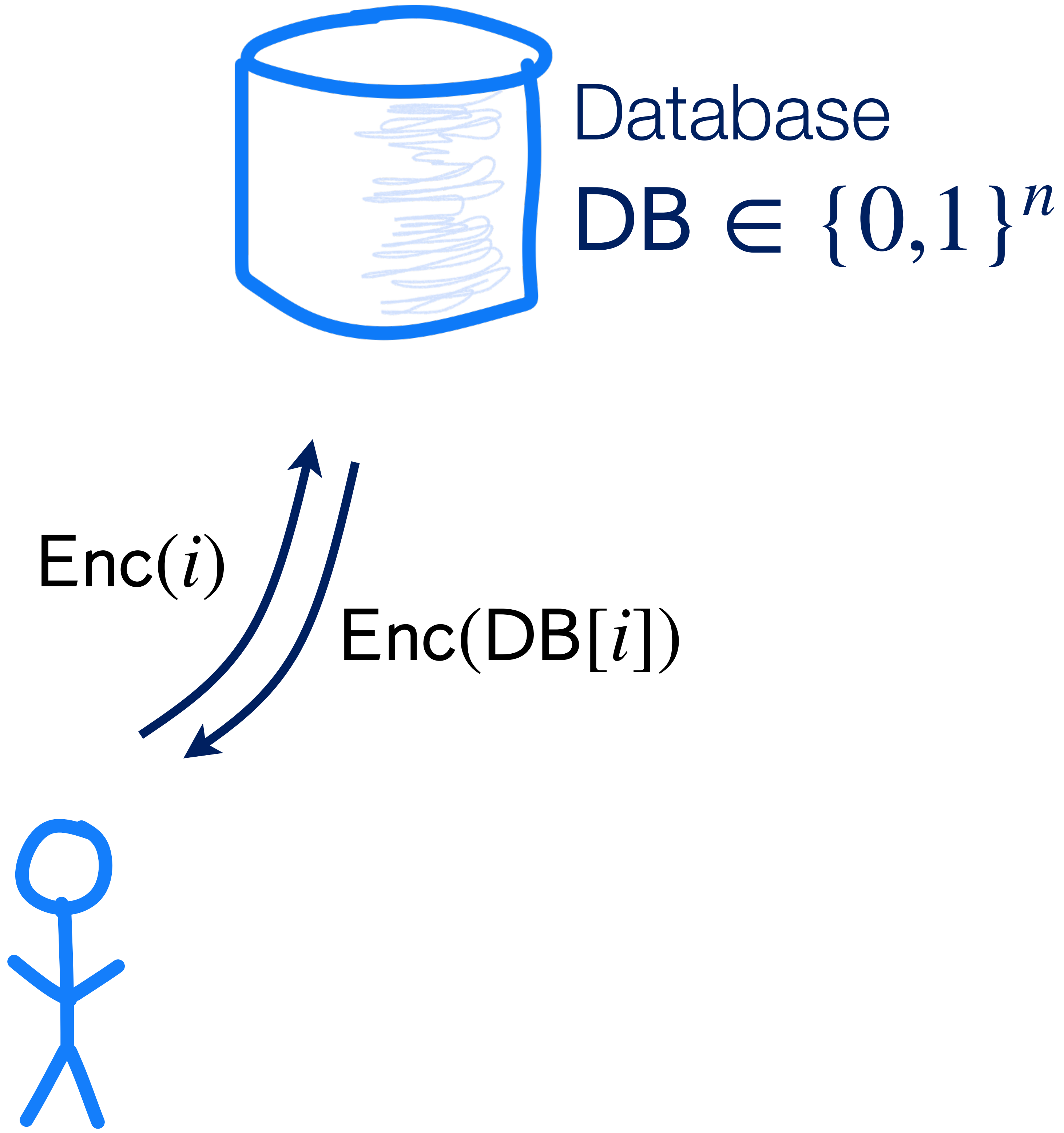
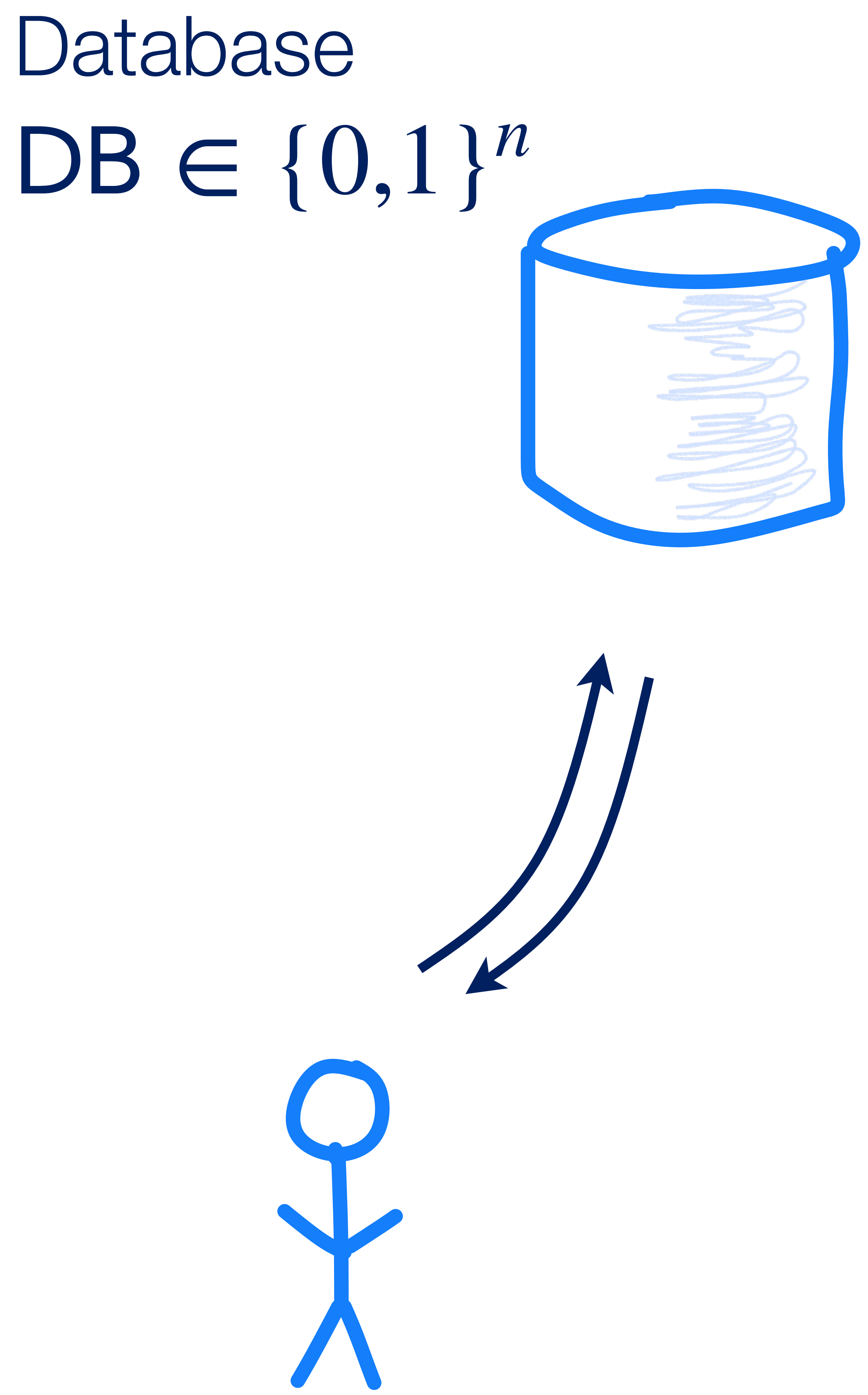
  - Server time per query: $O(n)$



Homomorphic Encryption for 2-Server PIR

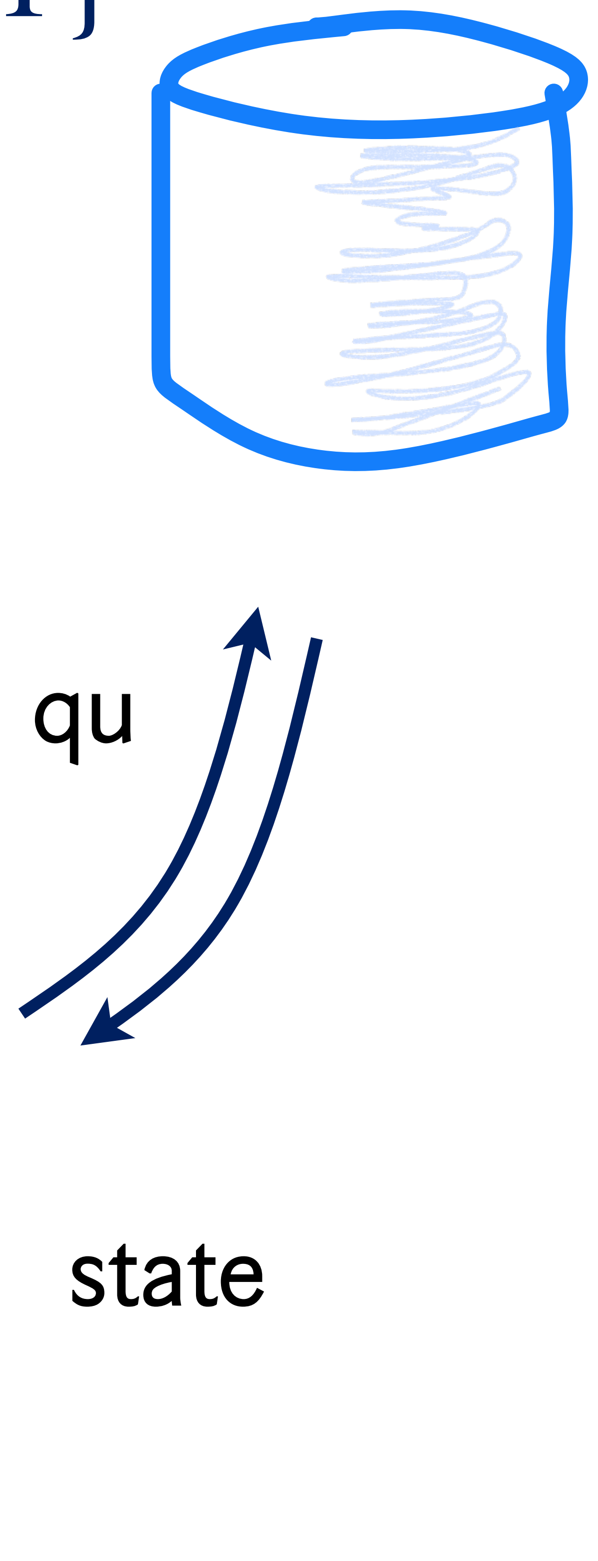


Homomorphic Encryption for 2-Server PIR

- Three phases:
    - Query: $\mathsf{state}, \mathsf{qu}_1 \leftarrow \mathsf{Query}(i)$

Database

$\mathsf{DB} \in \{0,1\}^n$



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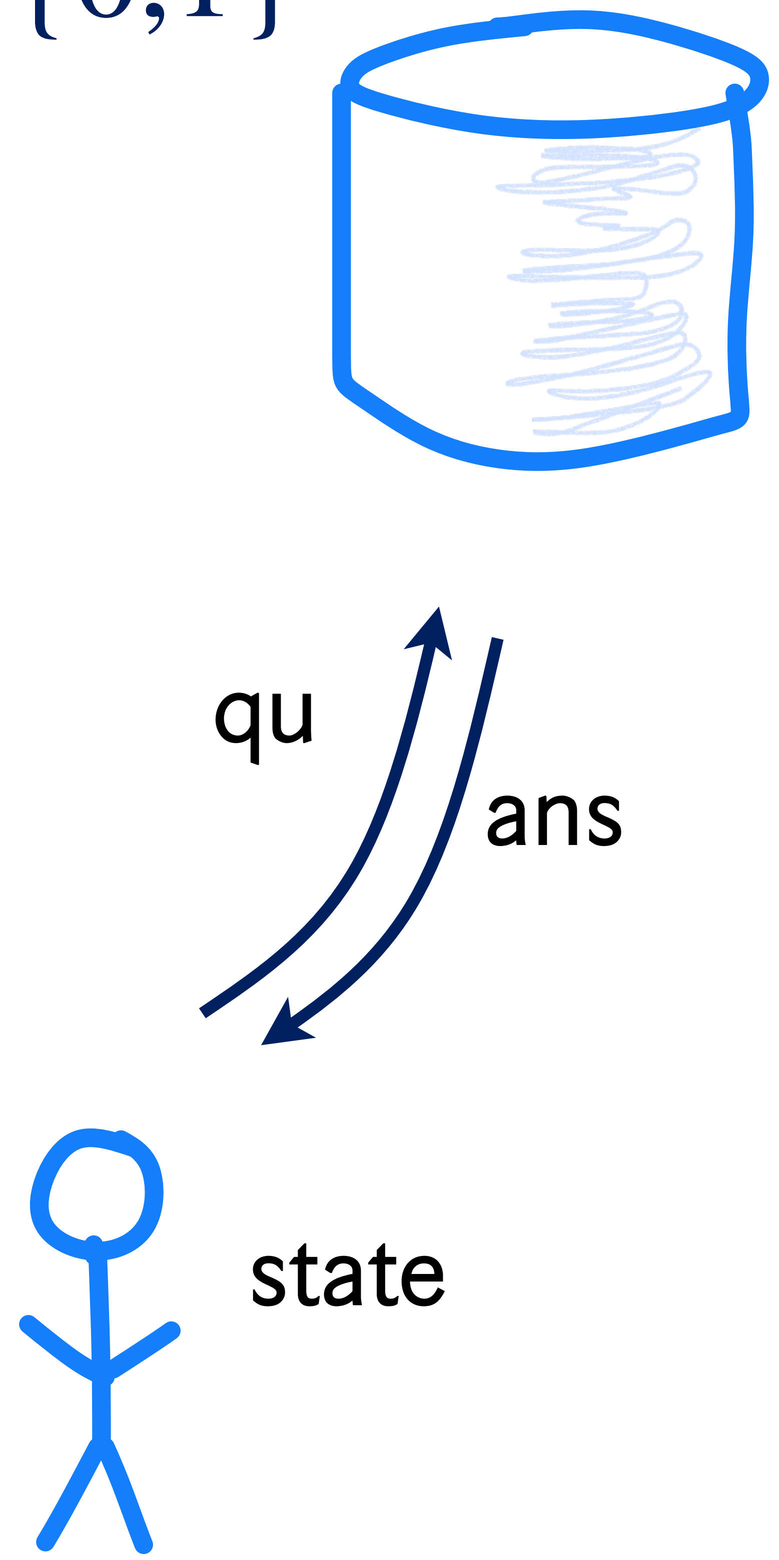

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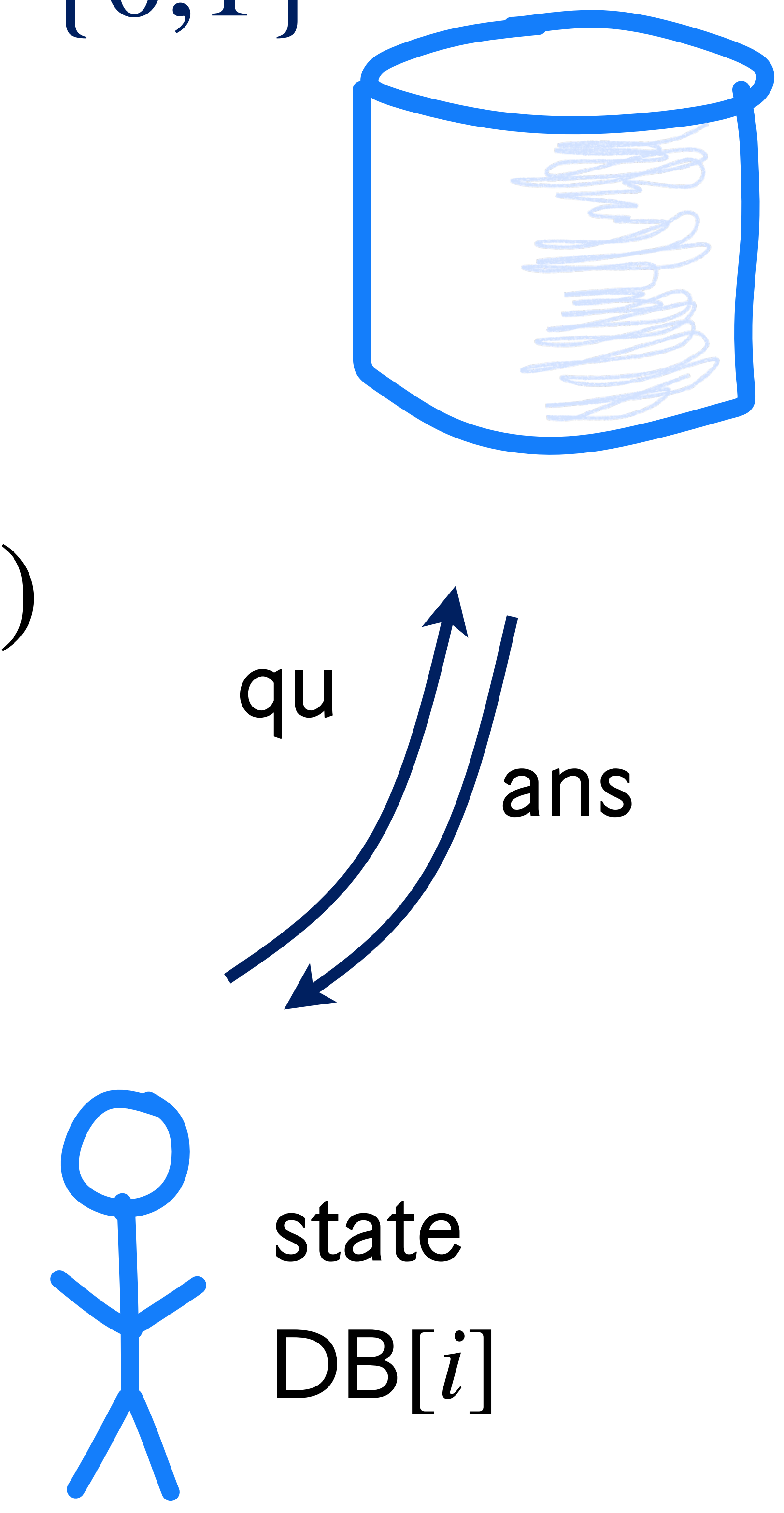

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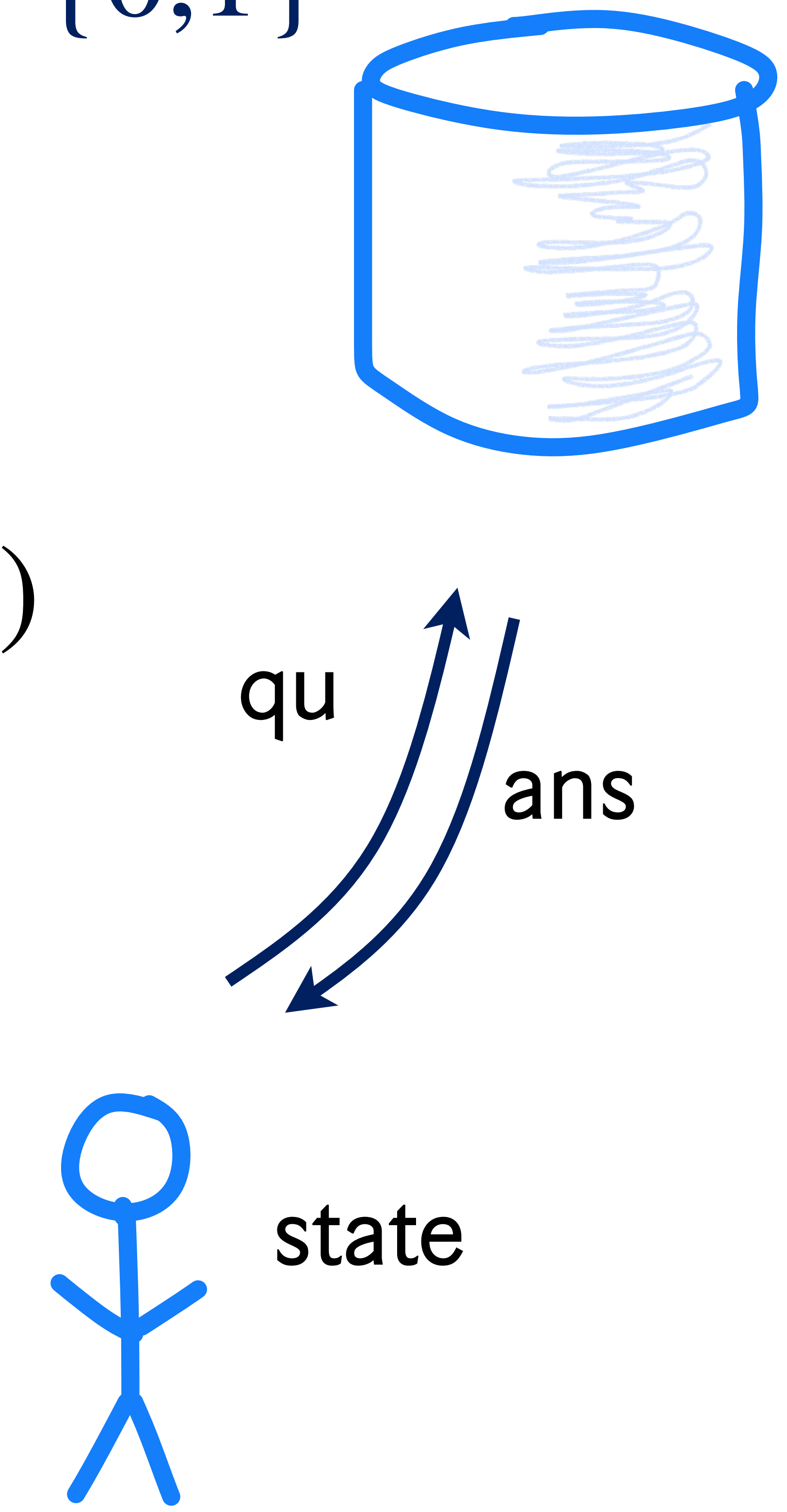

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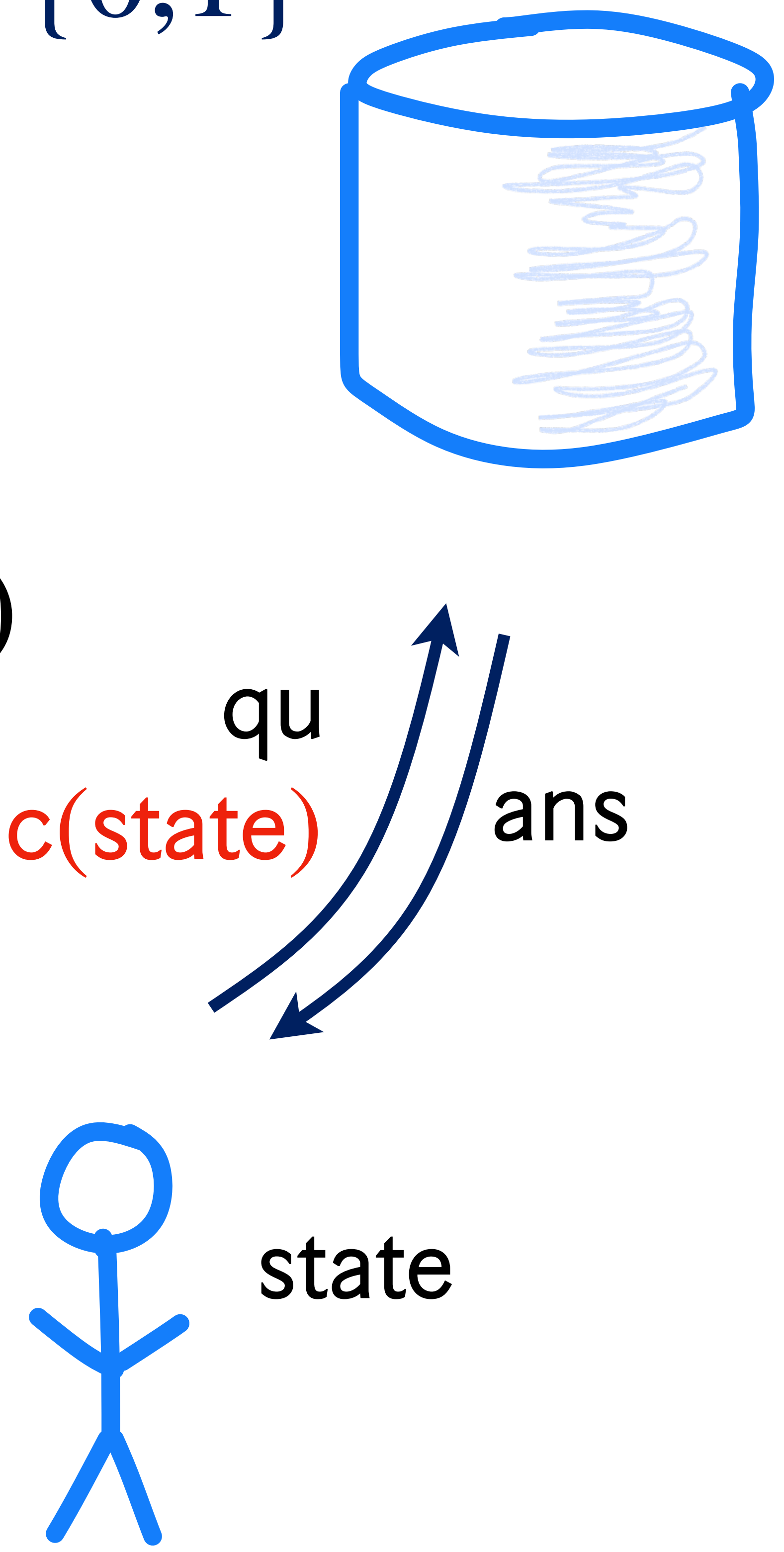

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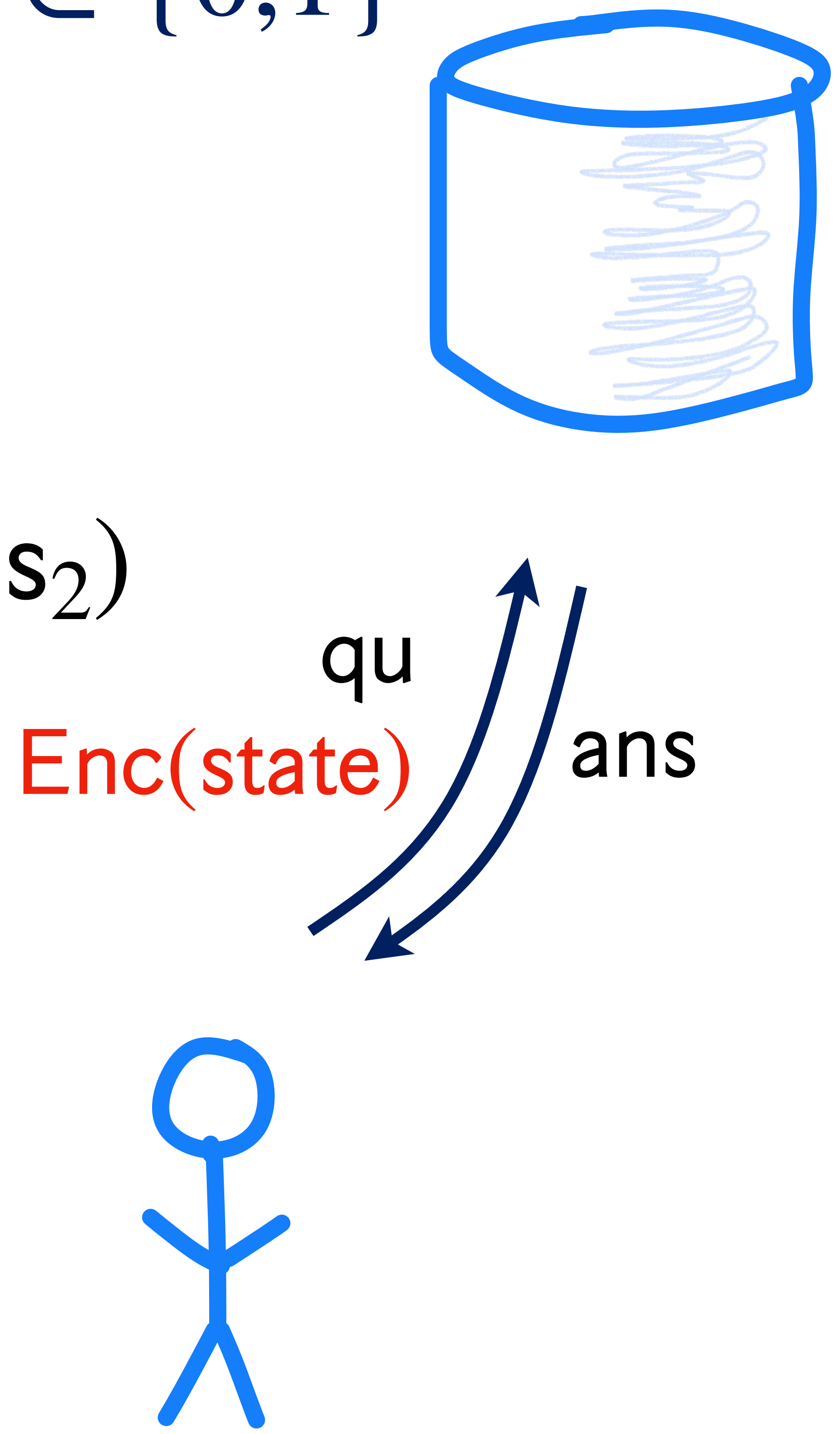

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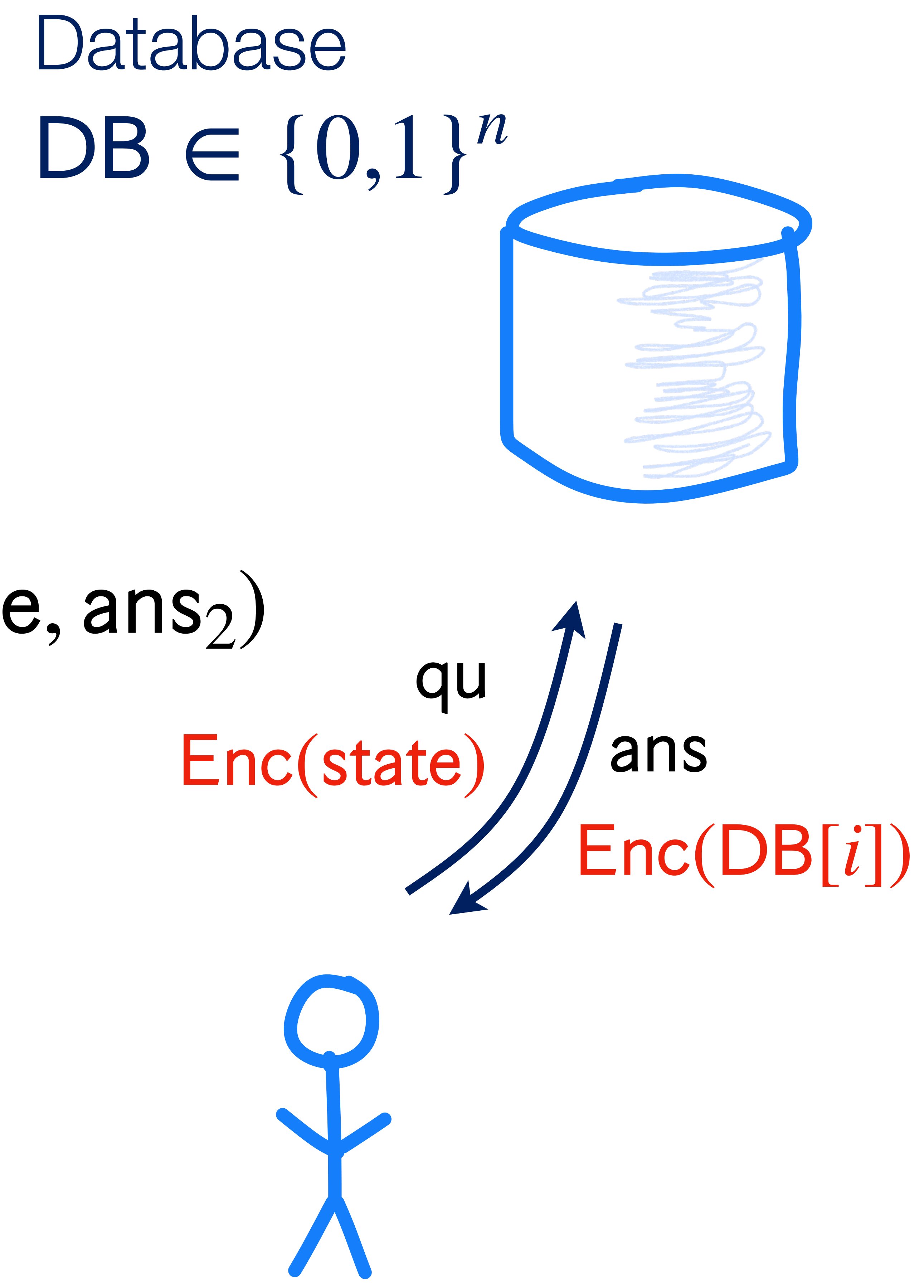

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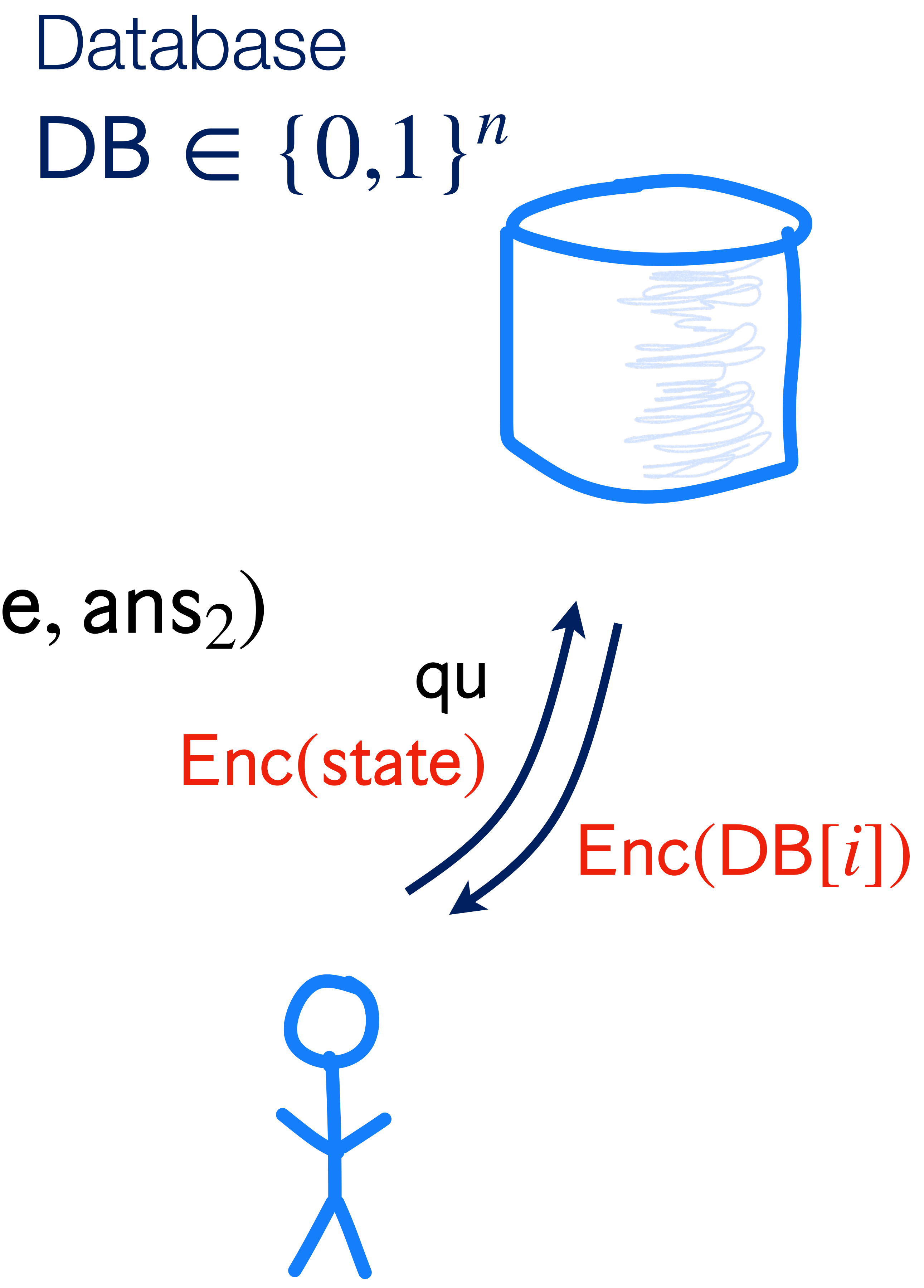

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Database

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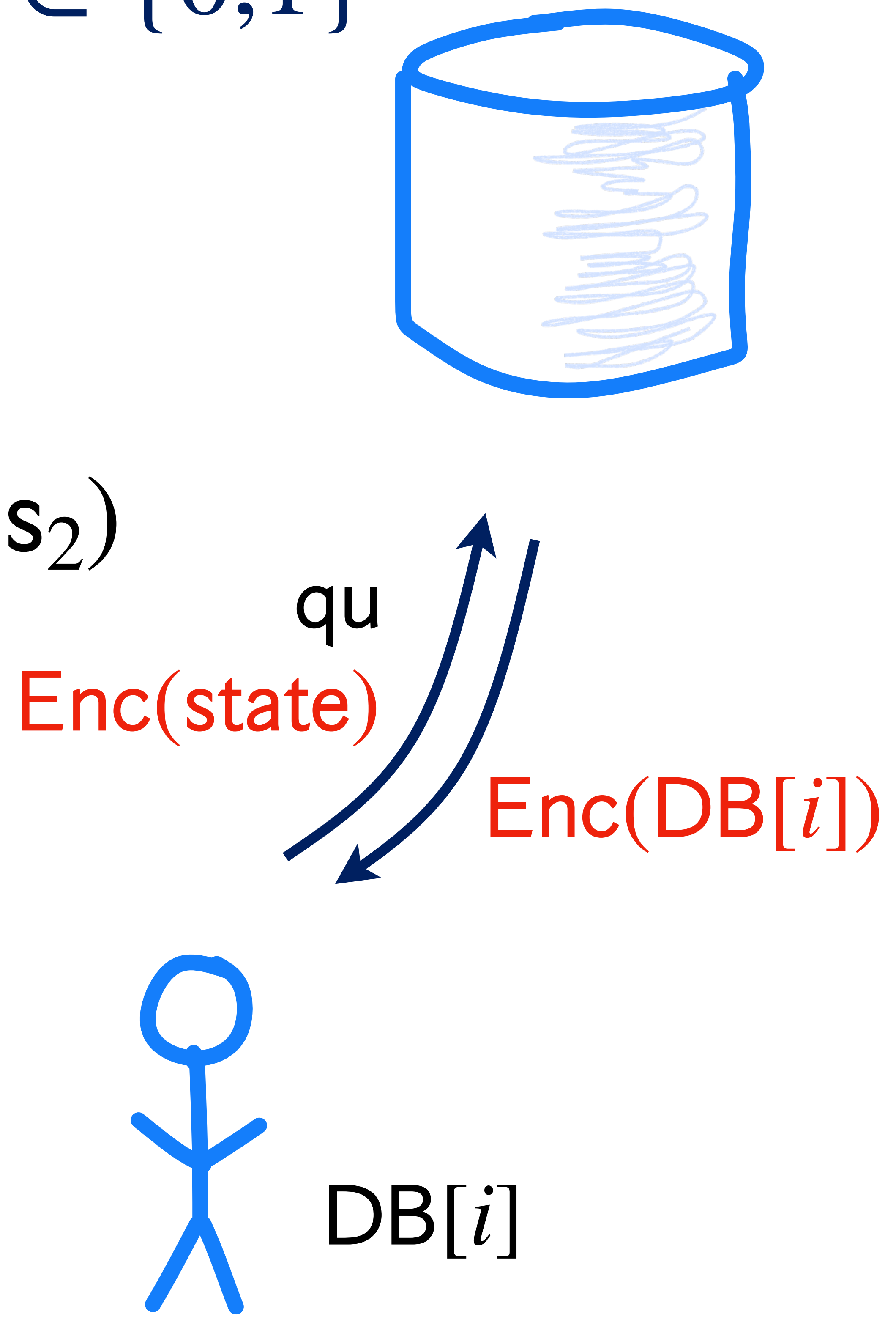

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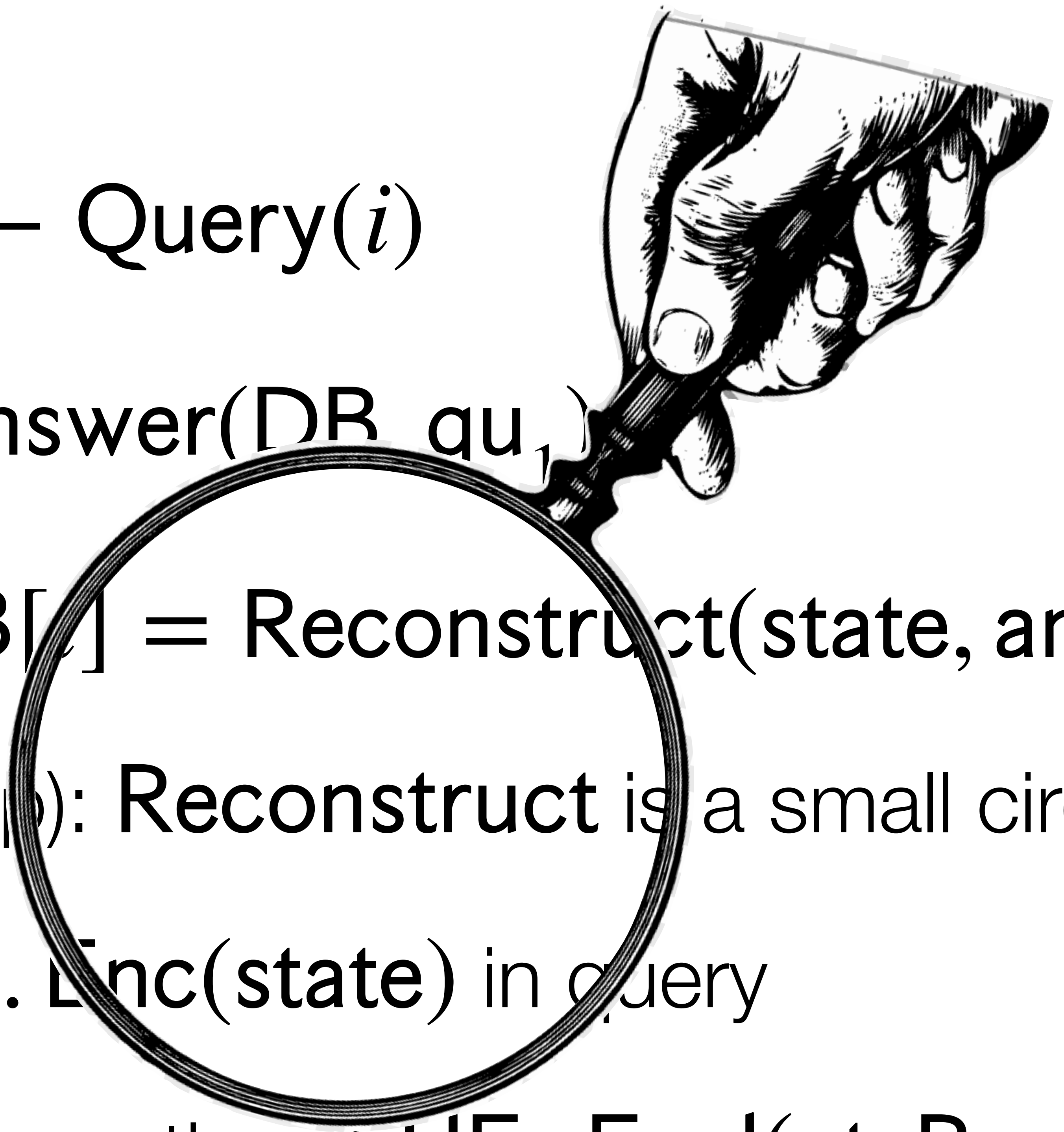

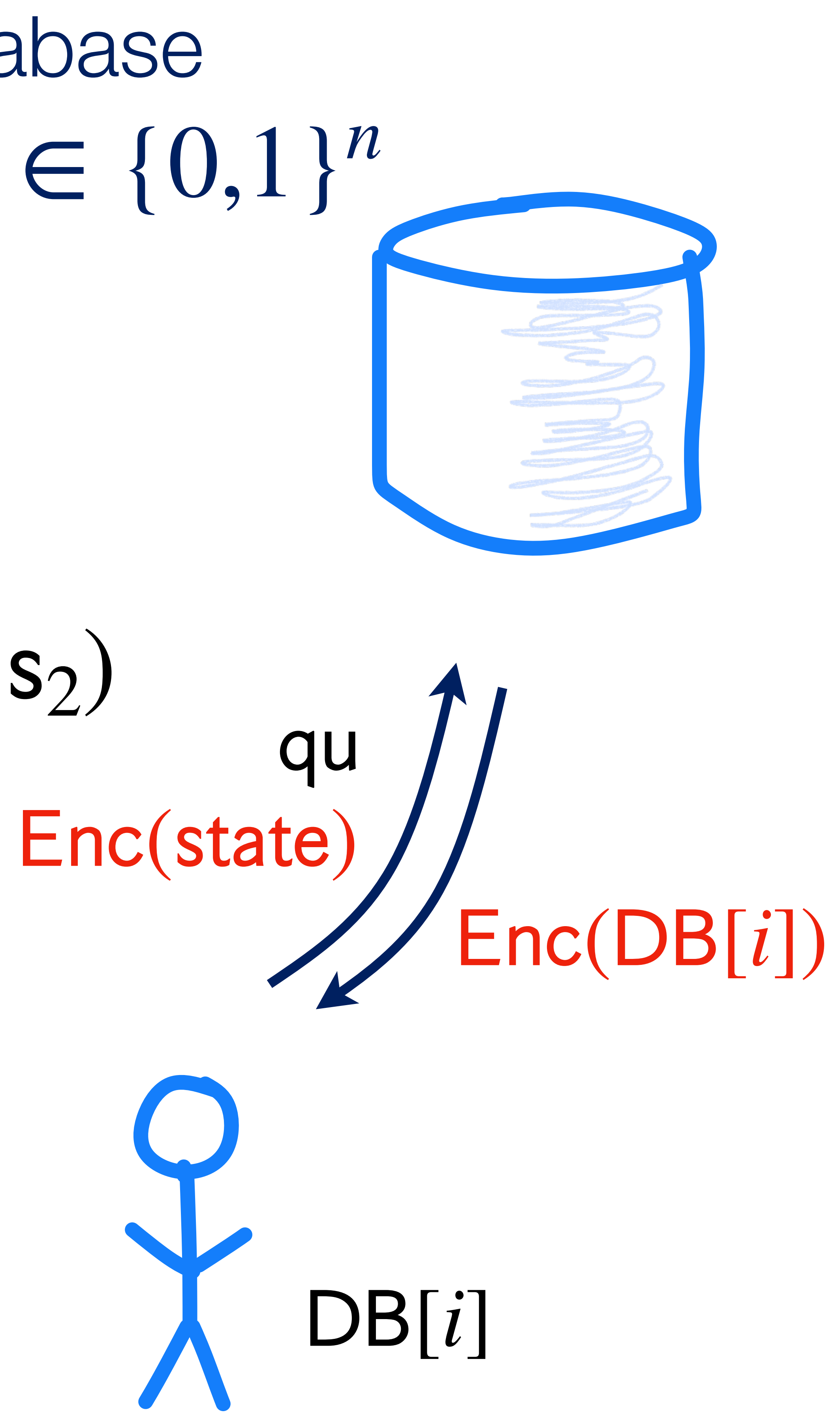

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Our PIR Reconstruction

Cheatsheet

$m \approx \log n$

$D \approx m/2$

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- $f_{\mathsf{DB}} : \mathbb{F}_2^m \to \mathbb{F}_2$ of degree D

Our PIR Reconstruction

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- $f_{\mathsf{DB}} : \mathbb{F}_2^m \to \mathbb{F}_2$ of degree D
- Queries: $\mathbf{r}, \mathbf{r} + \mathbf{p}$
  - $\mathsf{state} = \mathbf{p}$

Our PIR Reconstruction

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- Answers: $\left\{f(\mathbf{x} + \mathbf{e}) : \mathbf{x} \in \{\mathbf{r}, \mathbf{r} + \mathbf{p}\}, \|\mathbf{e}\| \leq D/2\right\}$

Our PIR Reconstruction

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- Reconstruction:

$$\mathsf{DB}_i = f_{\mathsf{DB}}(\mathbf{p}) = \sum_{\substack{\| \mathbf{e} \| \ \le \lfloor D/2 \rfloor \\ \mathbf{e} \le \mathbf{p}}} \left(\underbrace{f(\mathbf{r} + \mathbf{e})}_{\text{from server 1's answer}} + \underbrace{f(\mathbf{p} + \mathbf{r} + \mathbf{e})}_{\text{from server 2's answer}} \right)$$

Our PIR Reconstruction

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- For each $\mathbf{e}$, $\mathbf{1}[\mathbf{e} \le \mathbf{p}] = \mathbf{p}^{\mathbf{e}} = \prod_{i \in [m]} p_i^{e_i}$, which is degree $\le D/2$ in $\mathbf{p}$

Abstract Setup

Cheatsheet

$m \approx \log n$

$D \approx m/2$

- Two polynomials $g_1, g_2 : \mathbb{F}_2^m \to \mathbb{F}_2$ of degree $D/2$ for servers 1 and 2 respectively

$$g_1(\mathbf{x}) = \sum_{\|\mathbf{e}\| \le \lfloor D/2 \rfloor} \left(\underbrace{f(\mathbf{r} + \mathbf{e})}_{\text{from server 1's answer}} \right) \mathbf{x}^{\mathbf{e}}$$

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Cheatsheet

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- All we need to do is get server 1 to help the user evaluate $g_1(\mathbf{p})$ and likewise for server 2

Succinct PIR from FHE

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Theorem: with compact **fully homomorphic encryption***, we get 2-server PIR with server storage $n^{1+o(1)}$, time per query $O(n^{0.82})$ and communication $O(\log n)$.

✓

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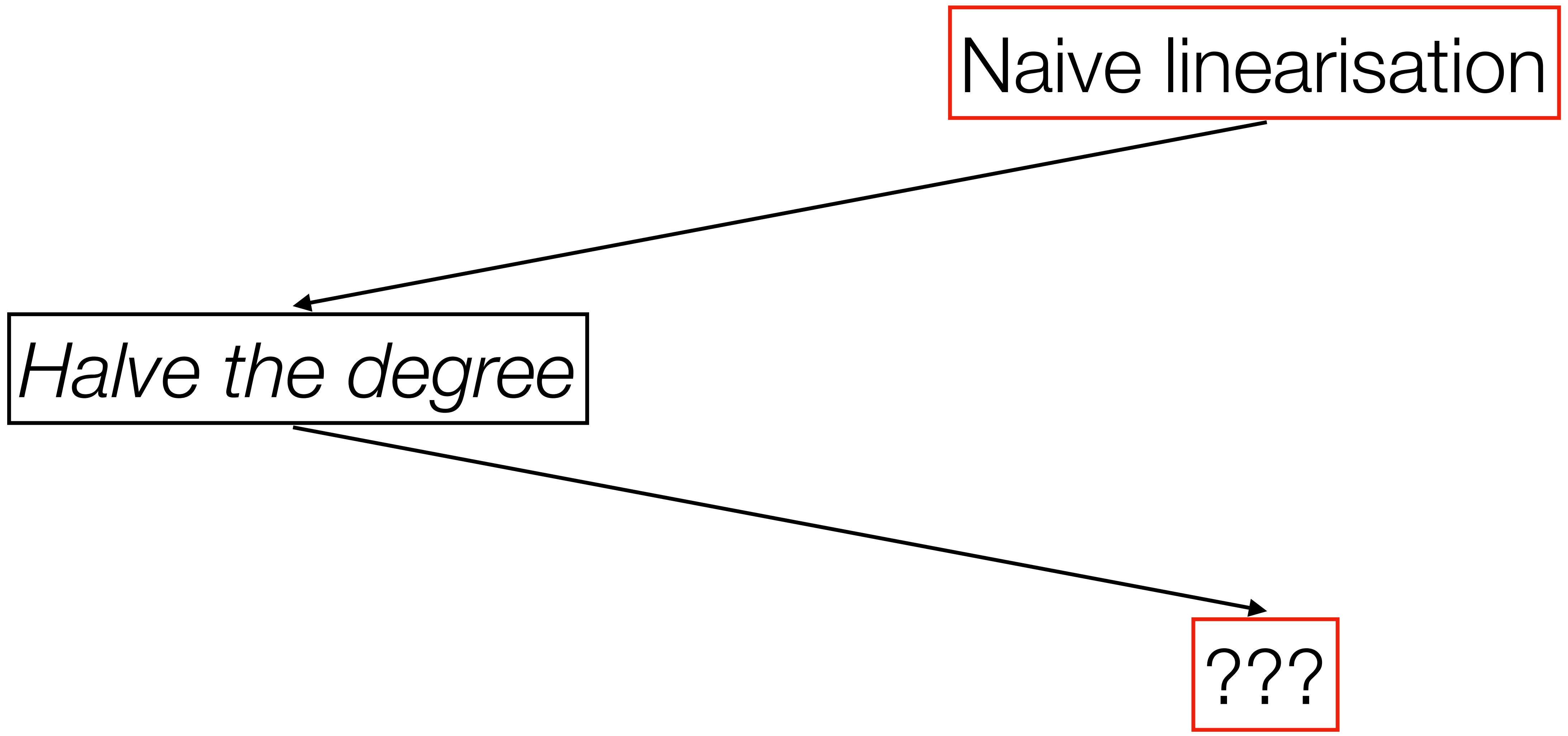

Naive linearisation

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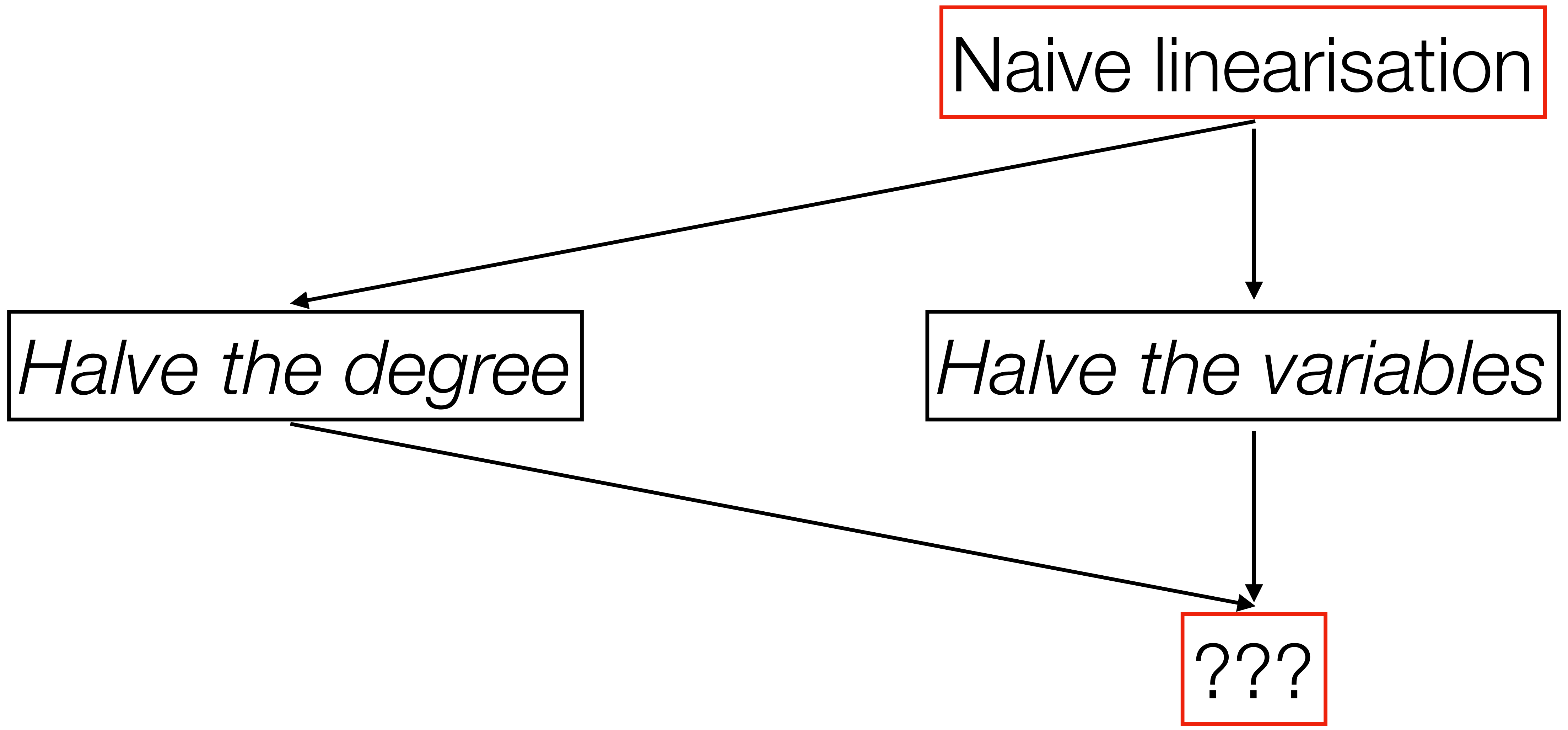

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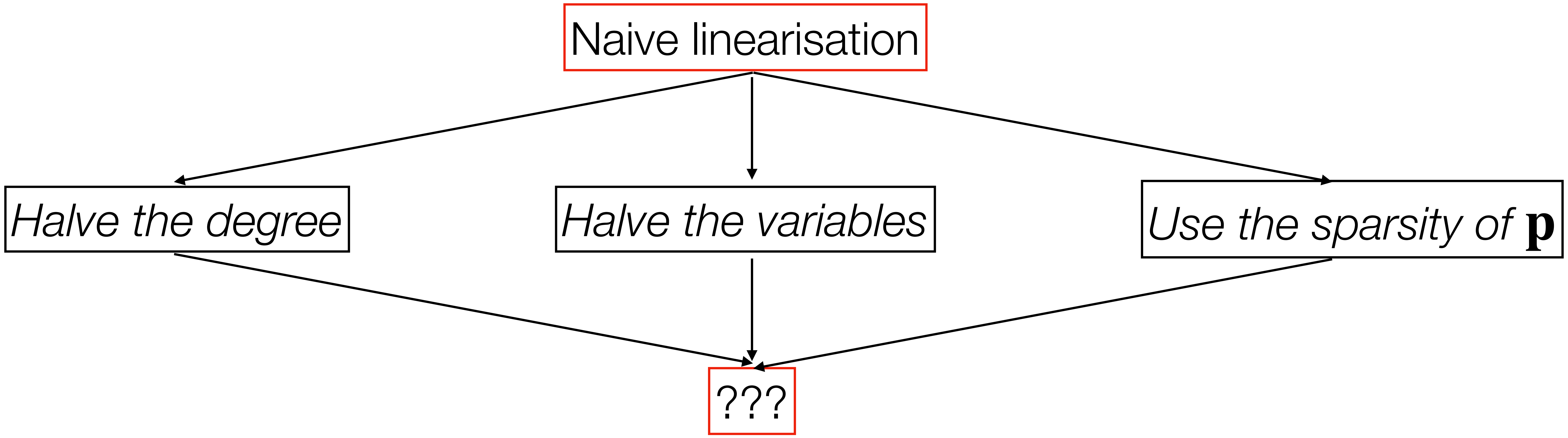

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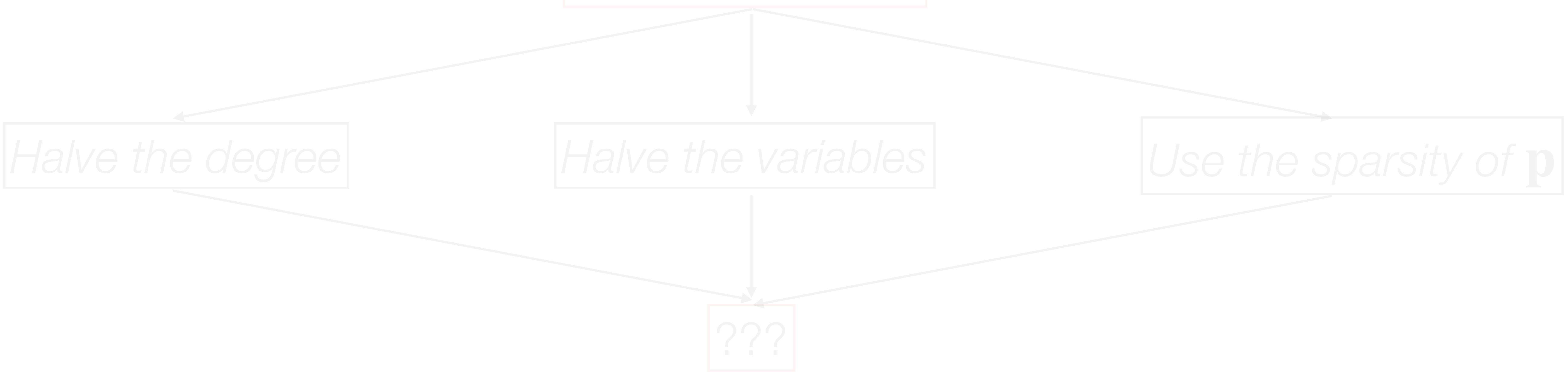
Natural target: reduce communication from $n^{0.82}$ to $\sqrt{n^{0.82}} = n^{0.41}$



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LHE → Computing Deg 2 Polynomials

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- **Claim** [KO97]: assume LHE. Then if the user has $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{\ell}$ and the server has $\mathbf{A} \in \mathbb{F}^{\ell \times \ell}$, the user can learn $\mathbf{x}^{\top}\mathbf{A}\mathbf{y}$ without revealing $\mathbf{x}, \mathbf{y}$, with $\ell \cdot \mathsf{poly}(\lambda)$ communication.

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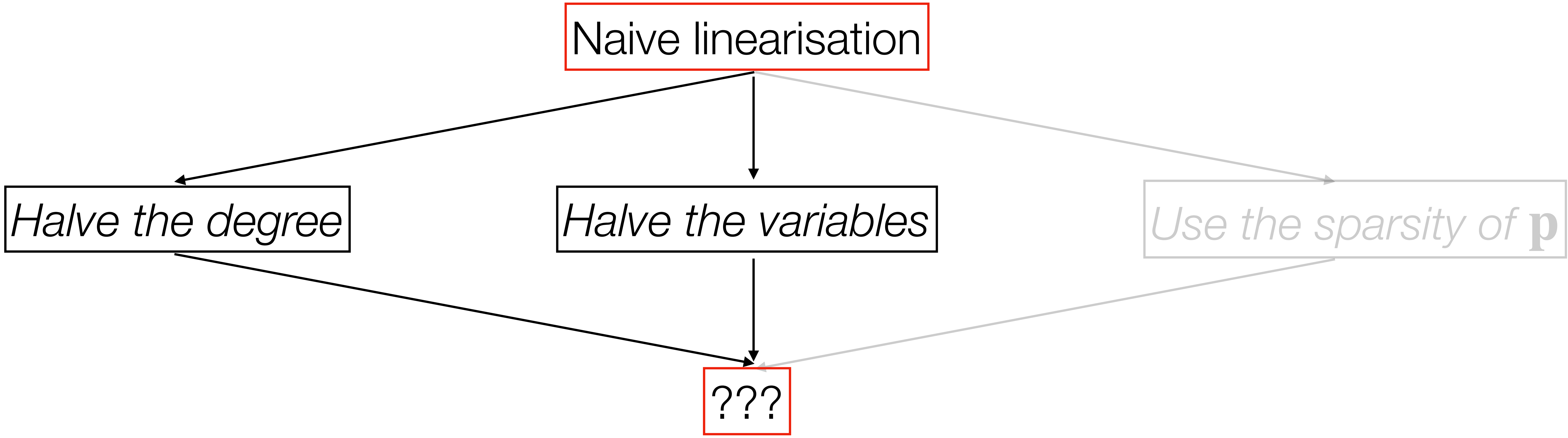
- **Proof:**
  - User sends $\mathsf{LHE}.\mathsf{Enc}(\mathbf{y})$
  - Server replies with $\mathsf{LHE}.\mathsf{Enc}(\mathbf{A}\mathbf{y})$
  - User decrypts and locally takes the inner product with $\mathbf{x}$

Rebalancing



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Halving the Degree/Variables

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$\mathbf{p} \in \mathbb{F}_2^m$

$D \approx m/2$

Evaluating $g(\mathbf{p})$

$\deg g \le D/2$

$\binom{m}{\alpha m} \approx 2^{mH(\alpha)} \approx n^{H(\alpha)}$

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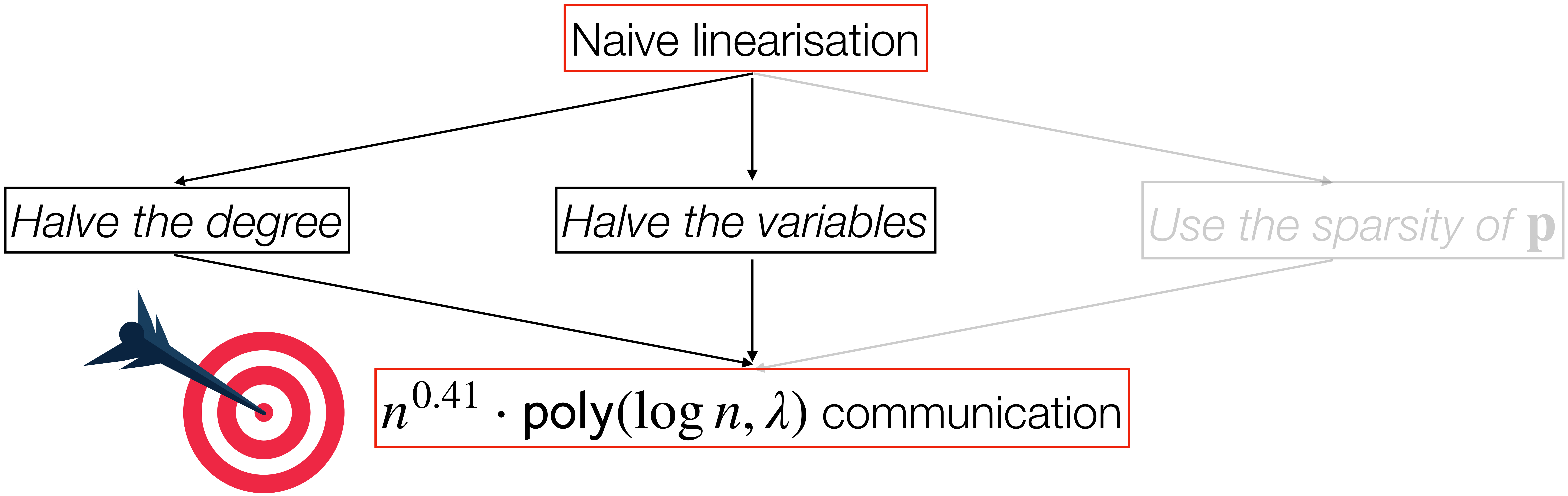
- Careful combination of these ideas: $\approx \binom{m/2}{D/4} \approx 2^{mH(1/4)/2} \approx n^{0.41}$



Naive linearisation

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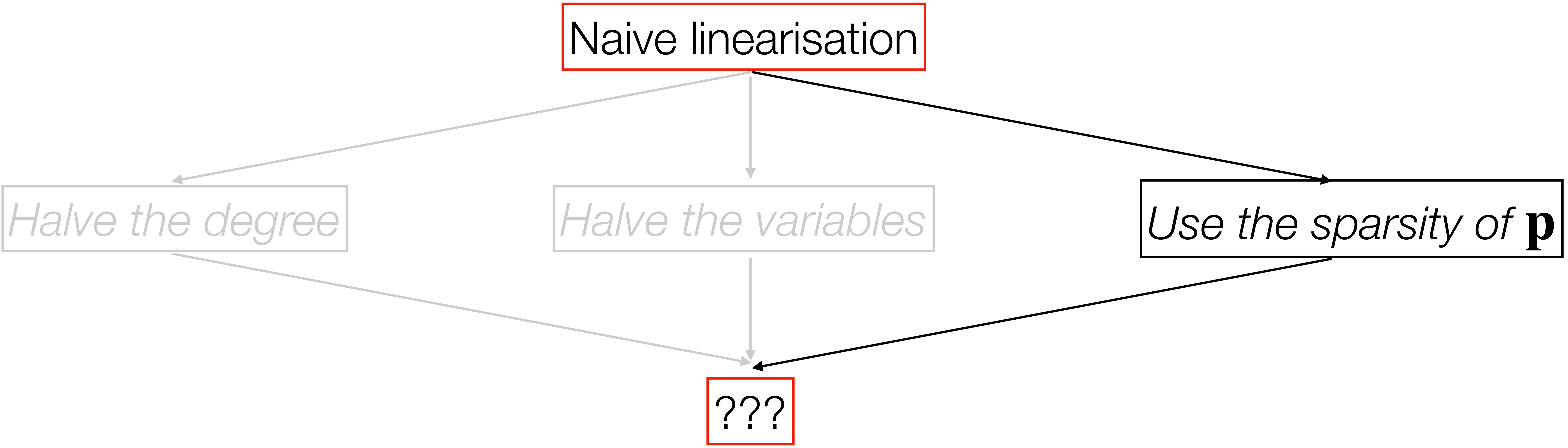
Halve the variables

Use the sparsity of $\mathbf{p}$

$n^{0.41} \cdot \mathsf{poly}(\log n, \lambda)$ communication



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Leveraging the Sparsity of $\mathbf{p}$ [BIM04]

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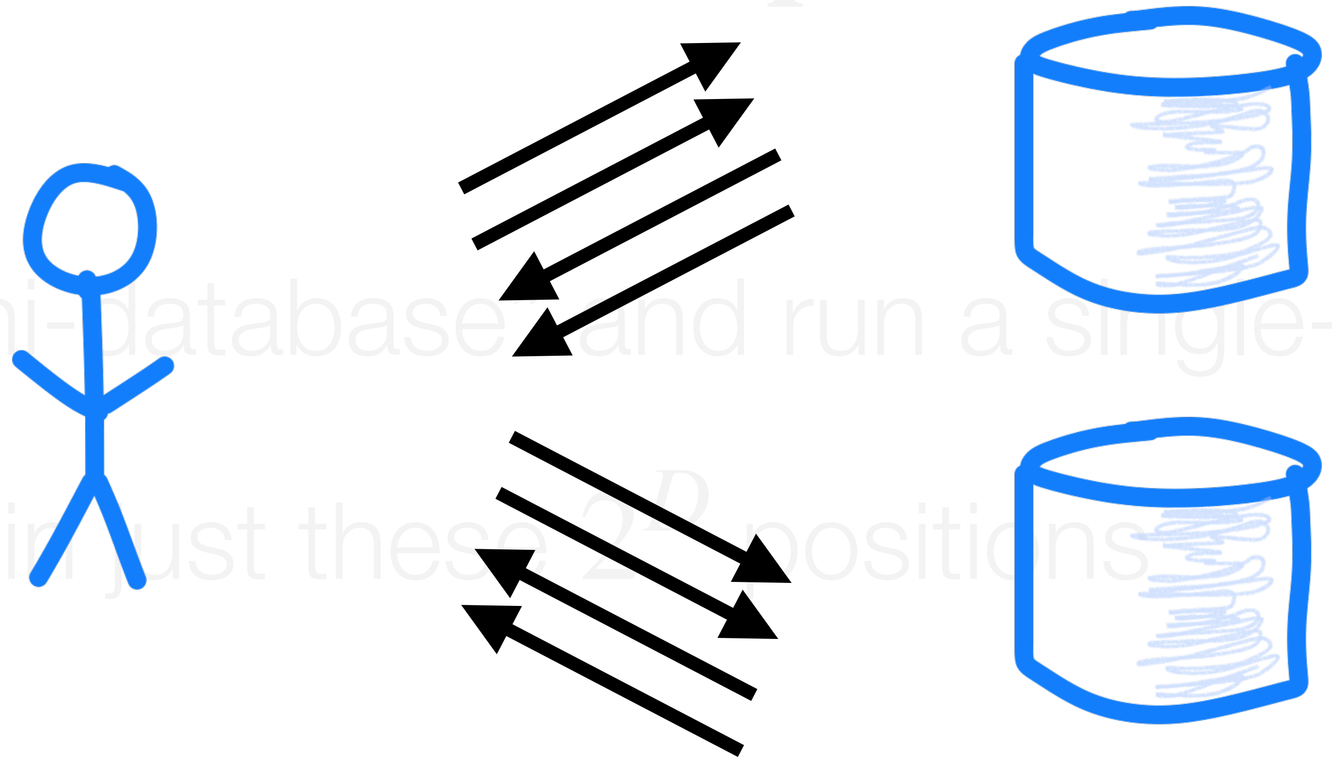
Batch PIR with many, non-adaptive queries



[IKOS’04,HHG’13,GKL’10,AS’16,H’16,ACLS’18,CHLR’18]

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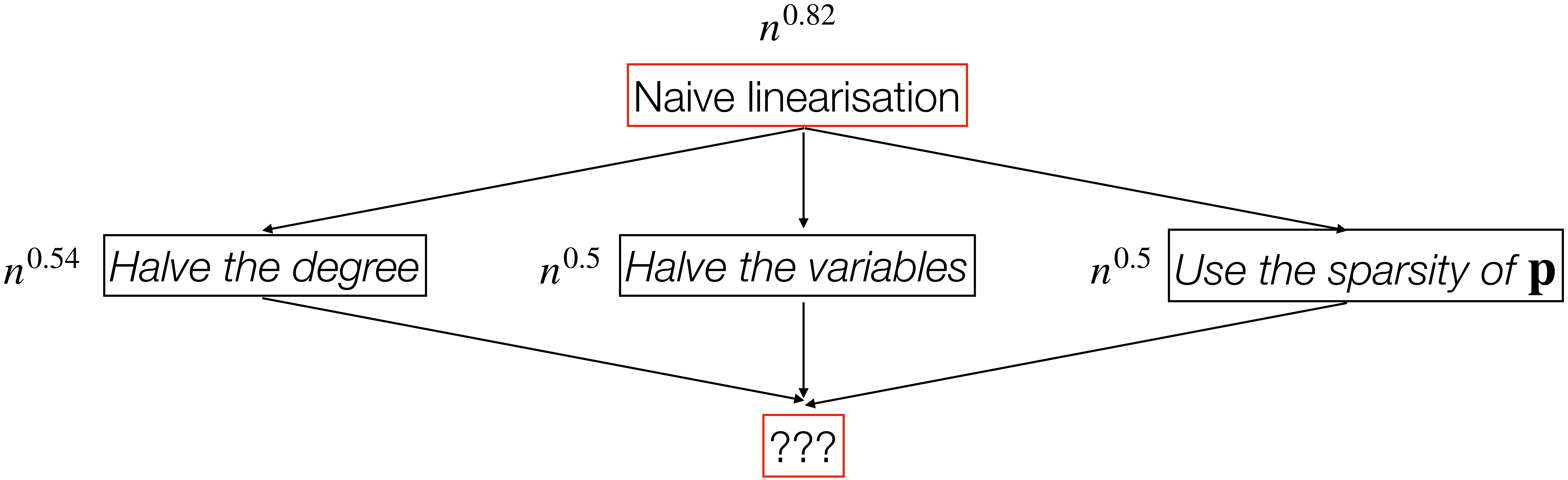
- Communication: $\approx 2^D \approx 2^{m/2} \approx n^{0.5}$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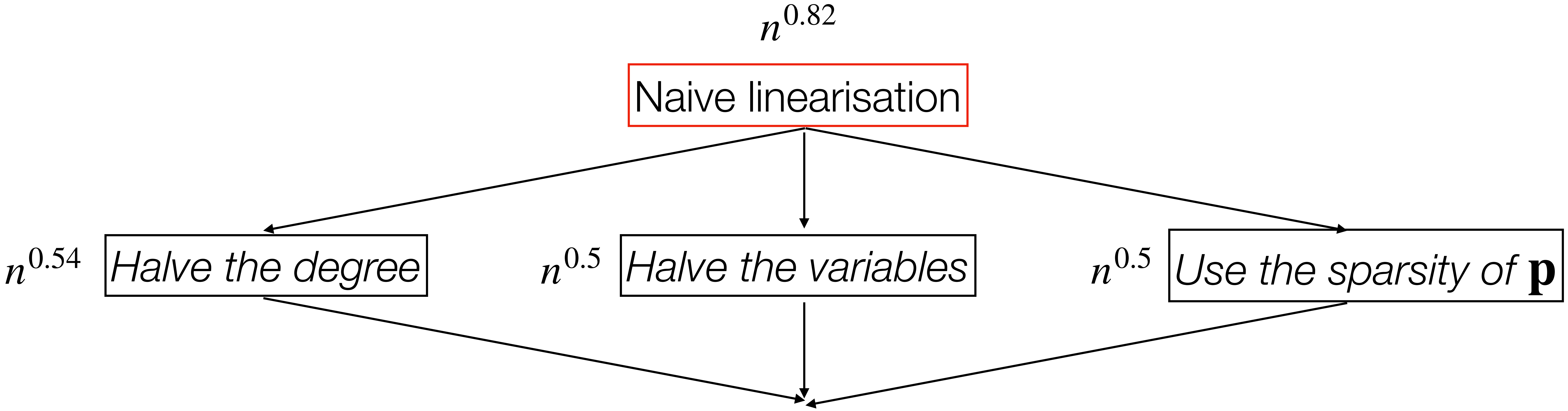

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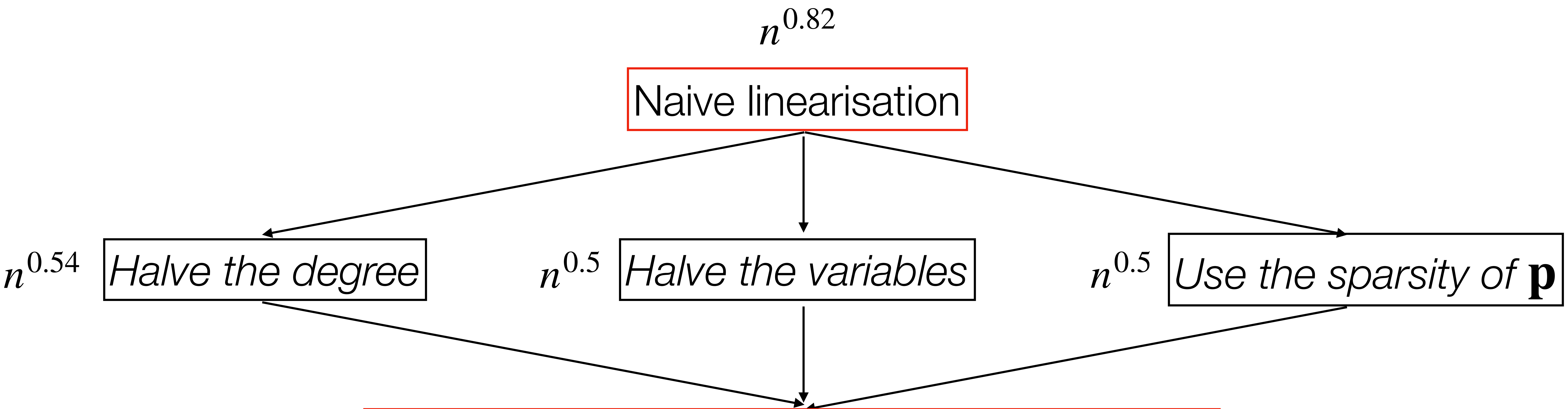

$n^{0.82}$ Naive linearisation

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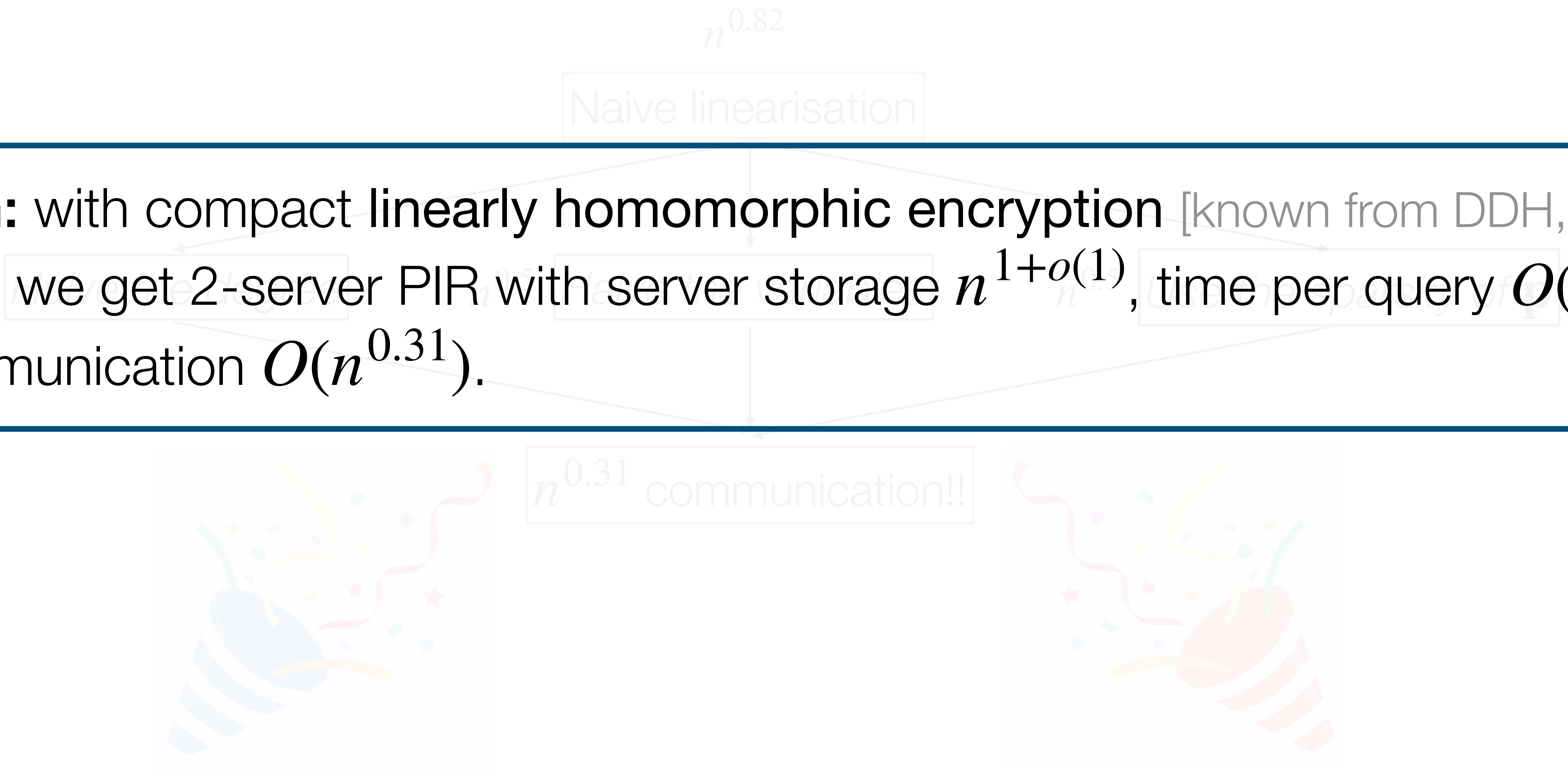
$n^{0.5}$ *Use the sparsity of* $\mathbf{p}$

$n^{0.31}$ communication!!

Even better than the $n^{0.41}$ we were aiming for!



Theorem: with compact **linearly homomorphic encryption** [known from DDH, DCR, QR, LWE], we get 2-server PIR with server storage $n^{1+o(1)}$, time per query $O(n^{0.82})$ and communication $O(n^{0.31})$.

✓

This talk

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2. **[HR26] New two-server PIR:** sublinear time, quasilinear space
3. **Evaluation:** what does this mean for practice?
4. **Bonuses** 😁
    - Reducing communication using crypto
    - **[HPR26]** Connecting multi-server PIR to complexity theory

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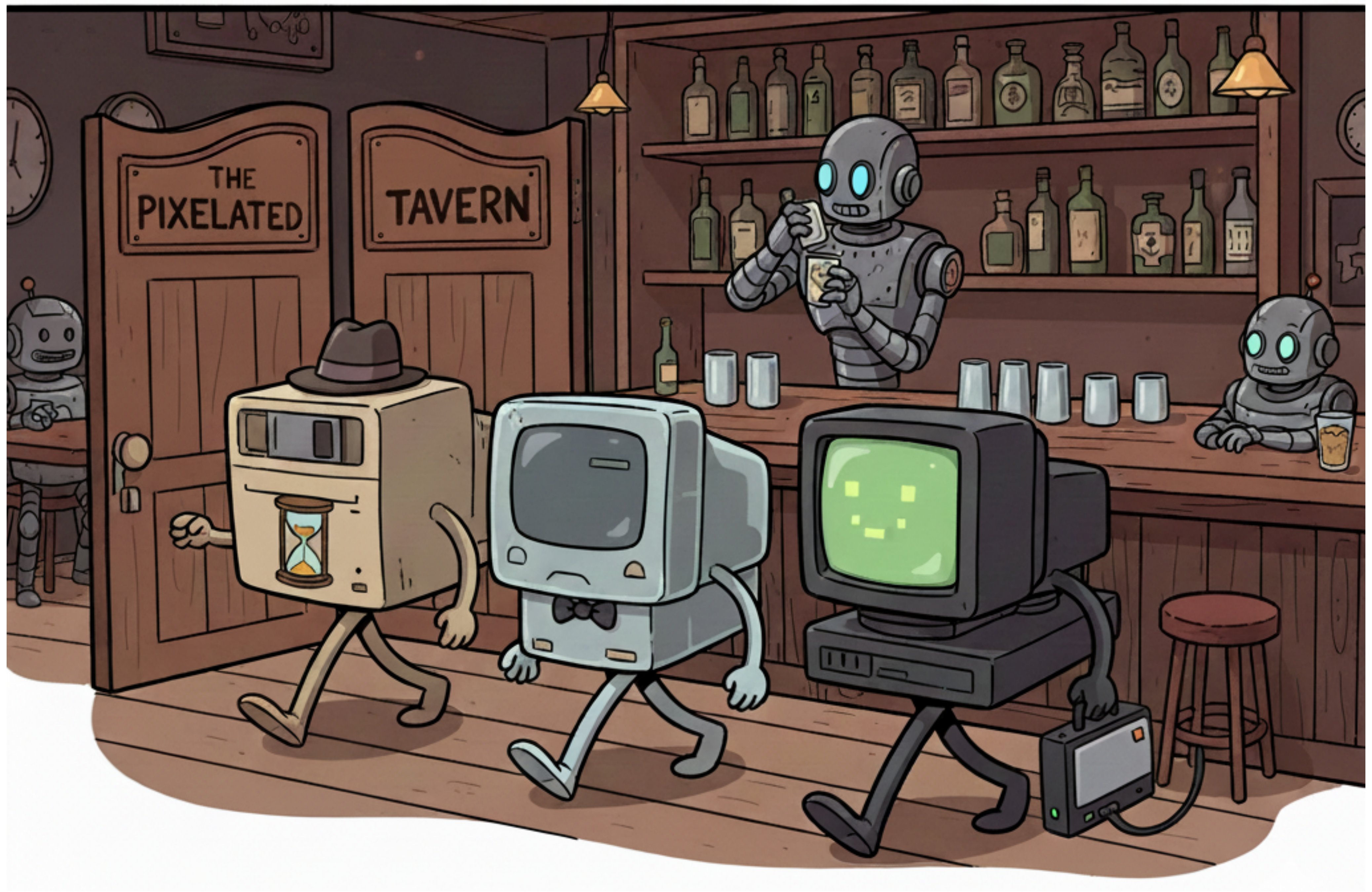
Three Turing Machines Walk Into a Bar



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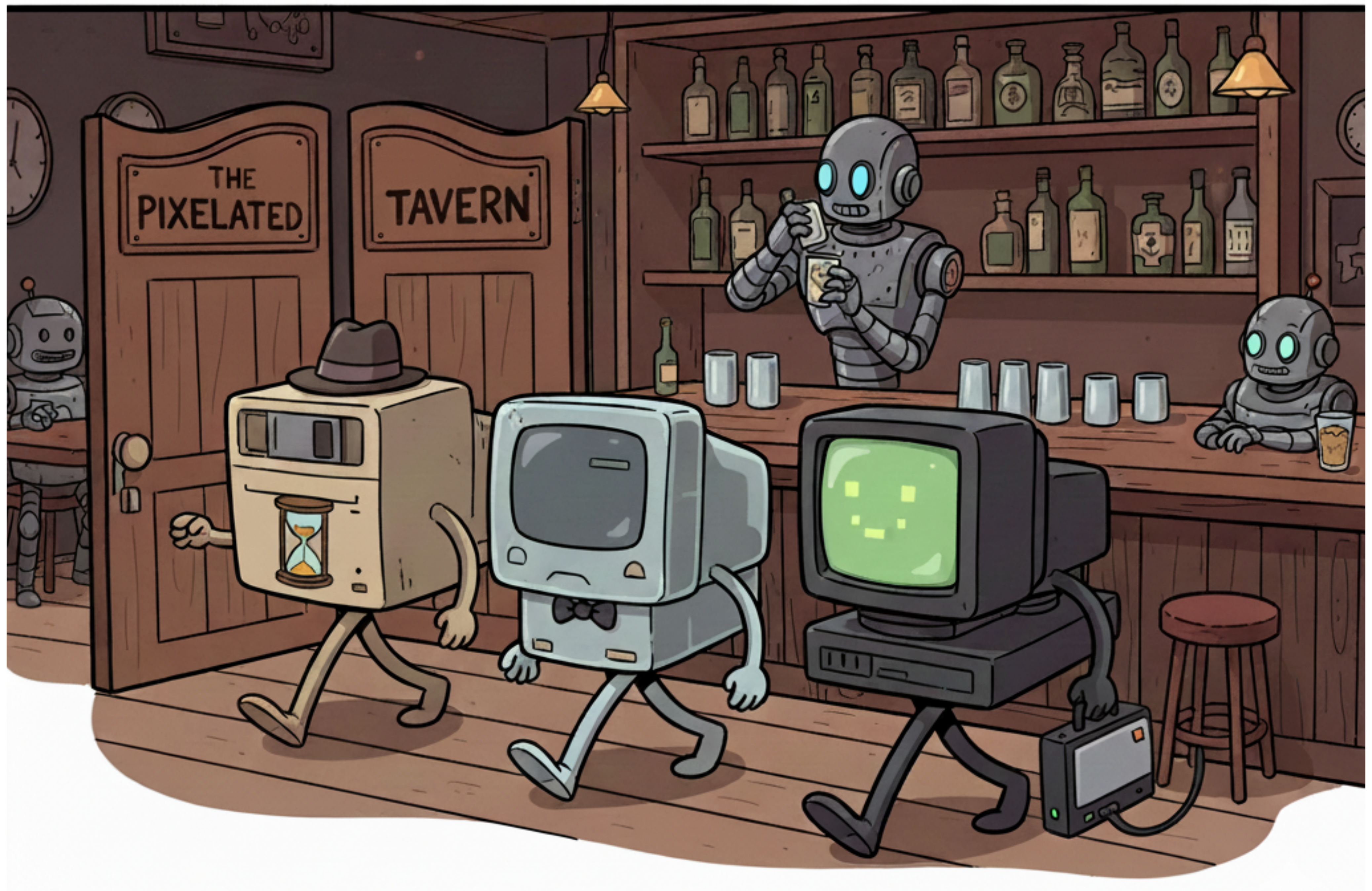

Three Turing Machines Walk Into a Bar

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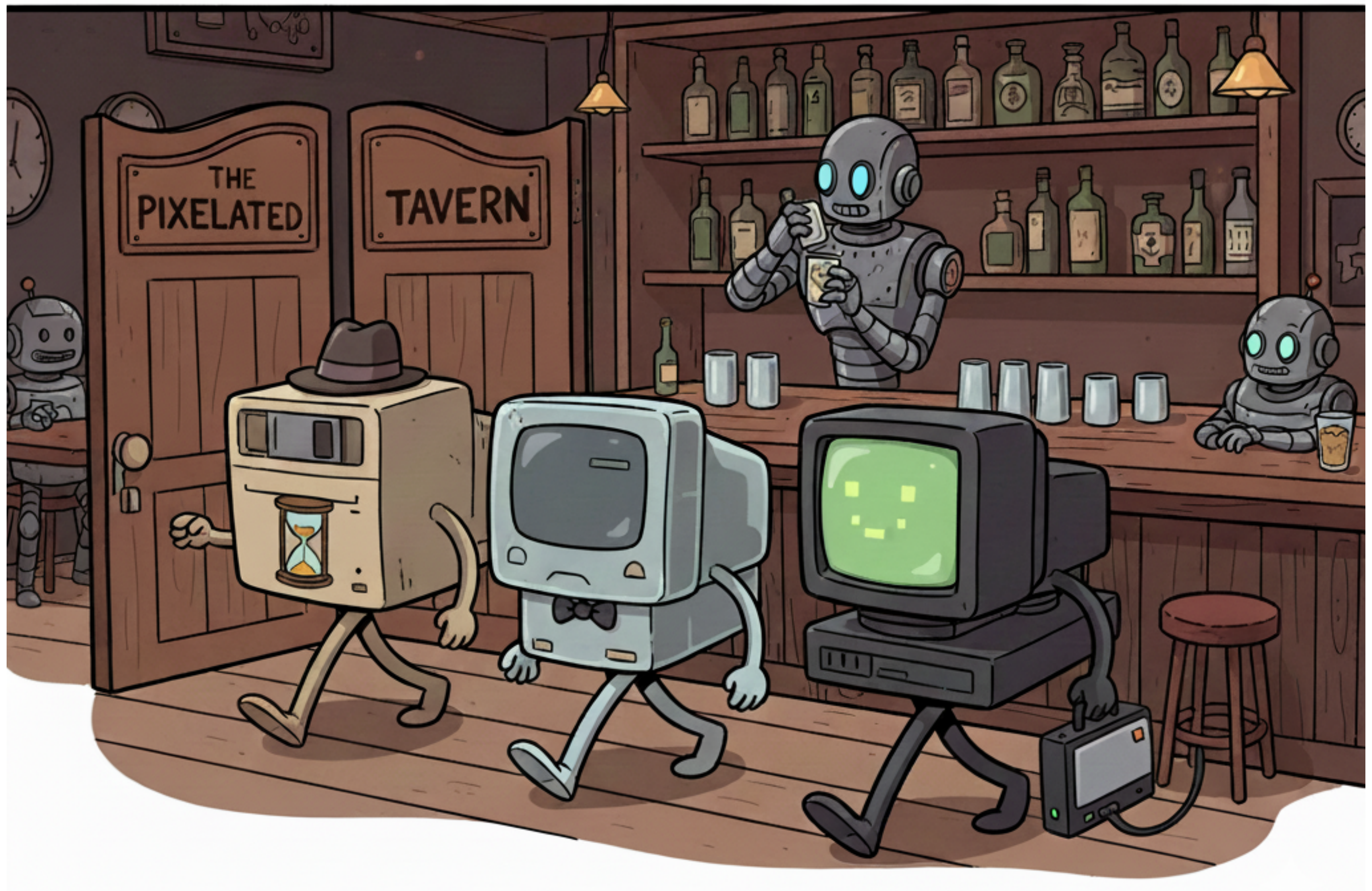

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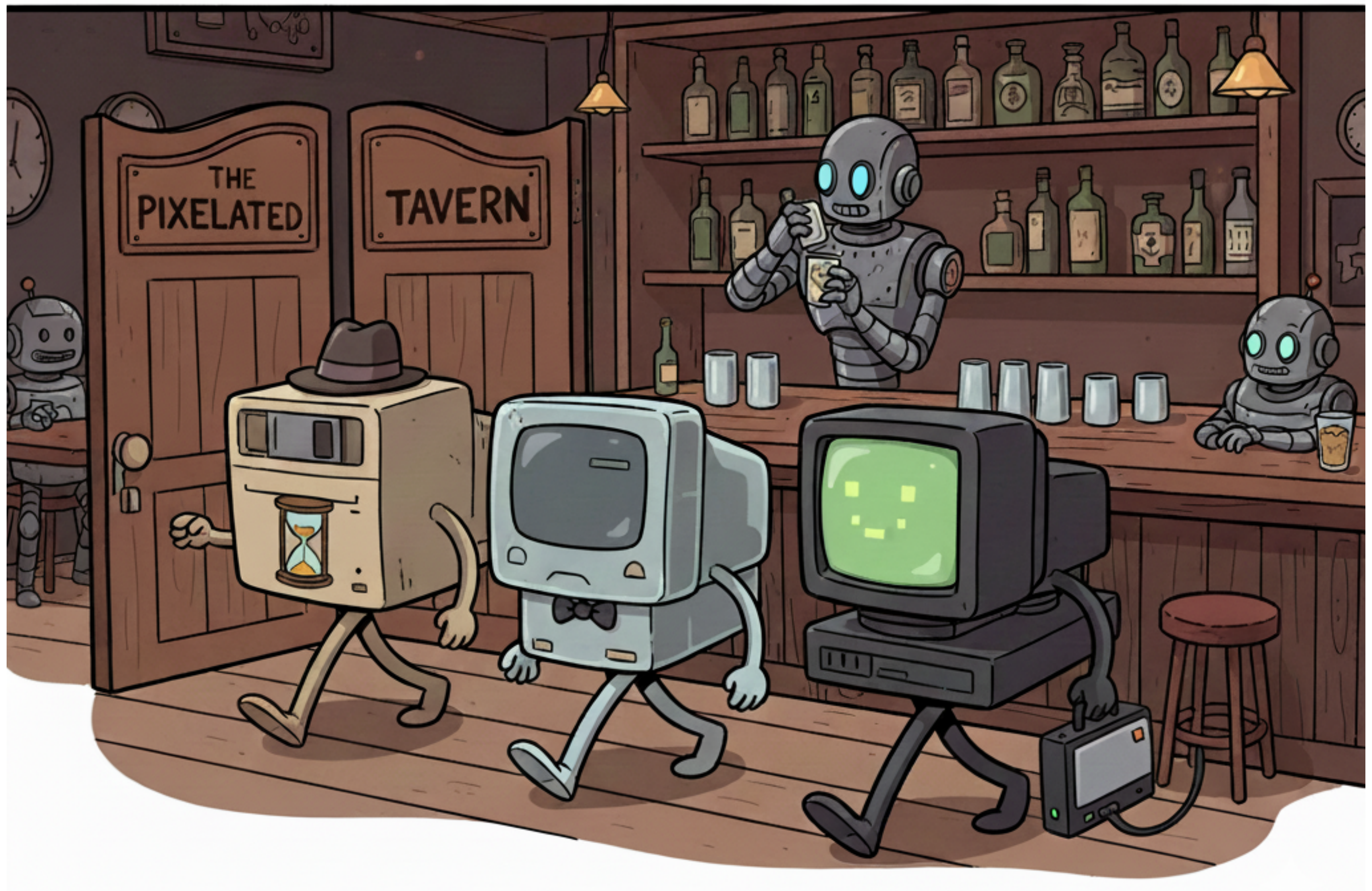

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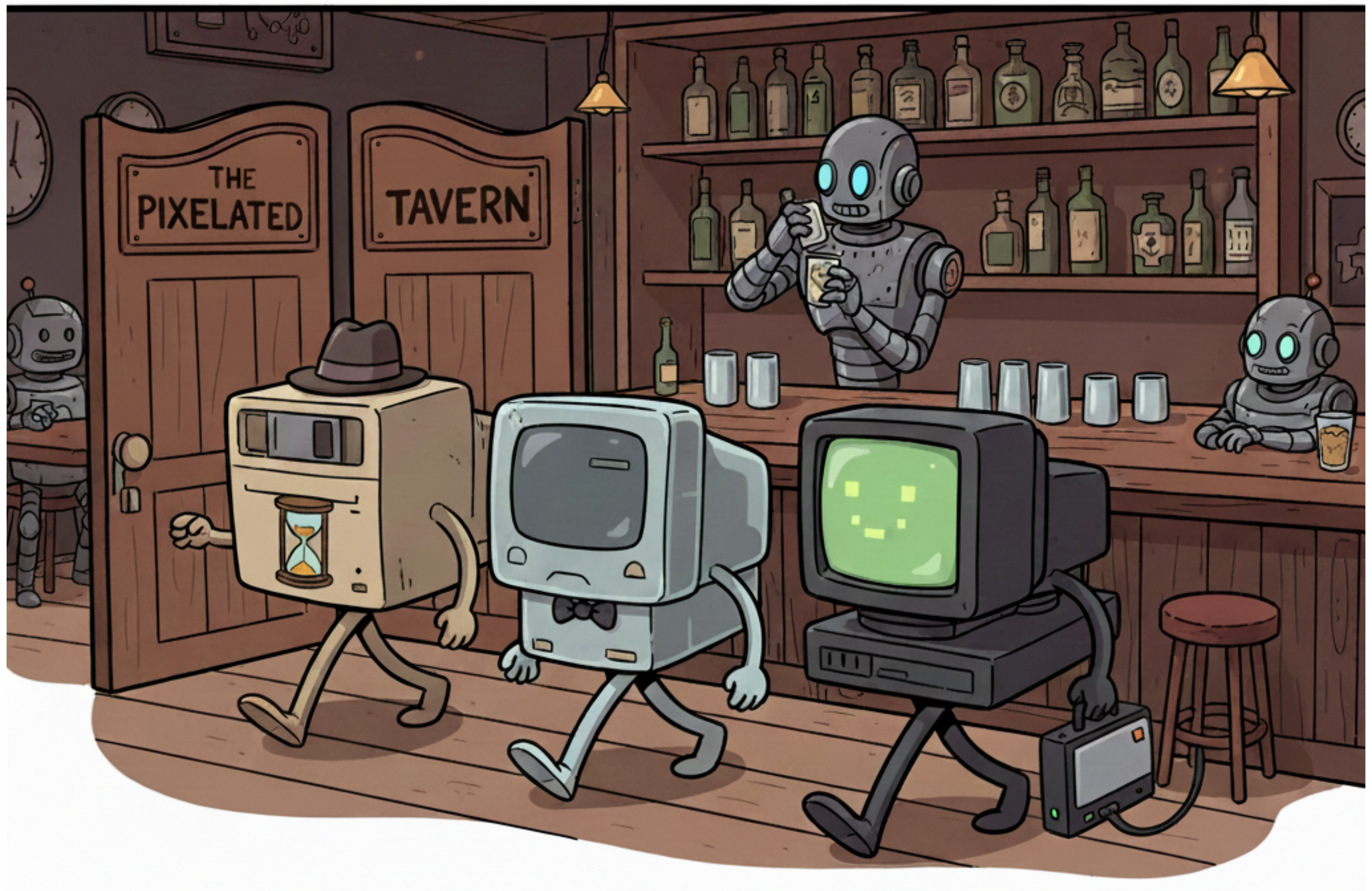

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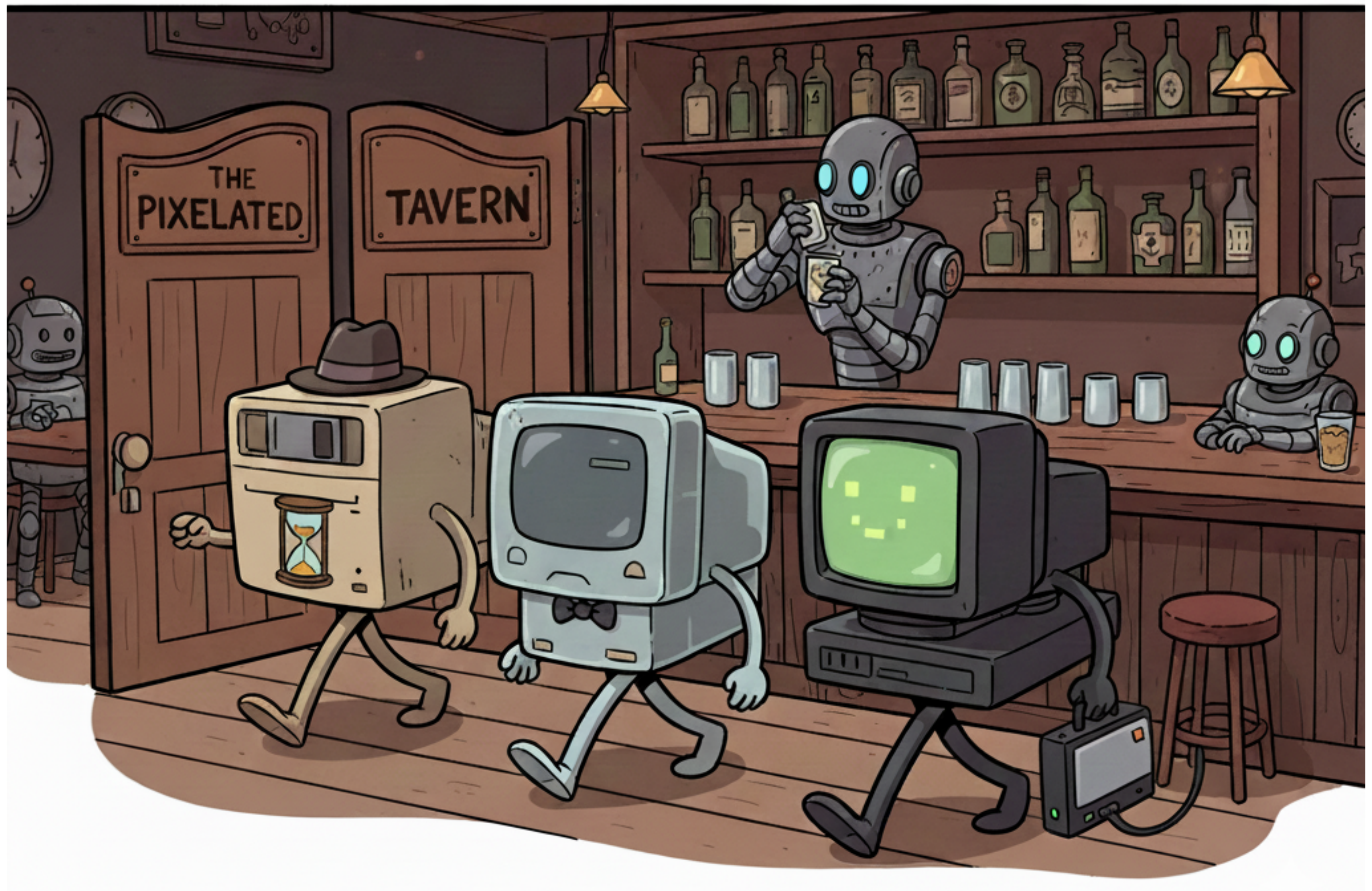

Q: How do the powers of these computers compare?

Is one of them "strictly weakest" i.e. the other two computers could do absolutely anything that it could?

50 Years of Time-Space Tradeoffs

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A: Computer A is strictly weakest

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TLDR: techniques from info-theoretic PIR connect to complexity theory in surprising ways!

Hopefully many more connections to come 😁

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- [HPR26] New results on catalytic space using techniques from information-theoretic PIR

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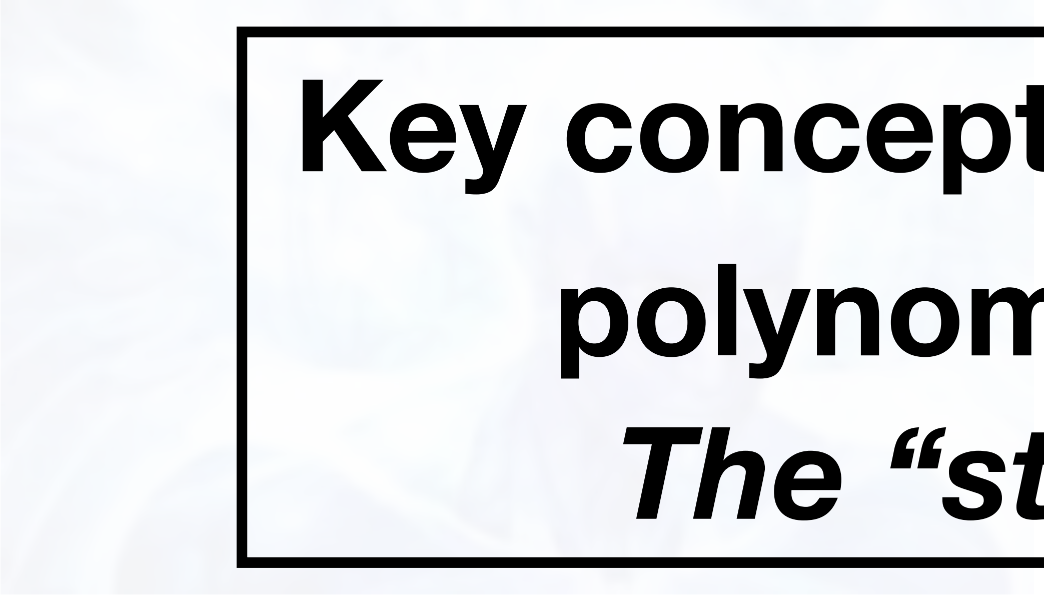


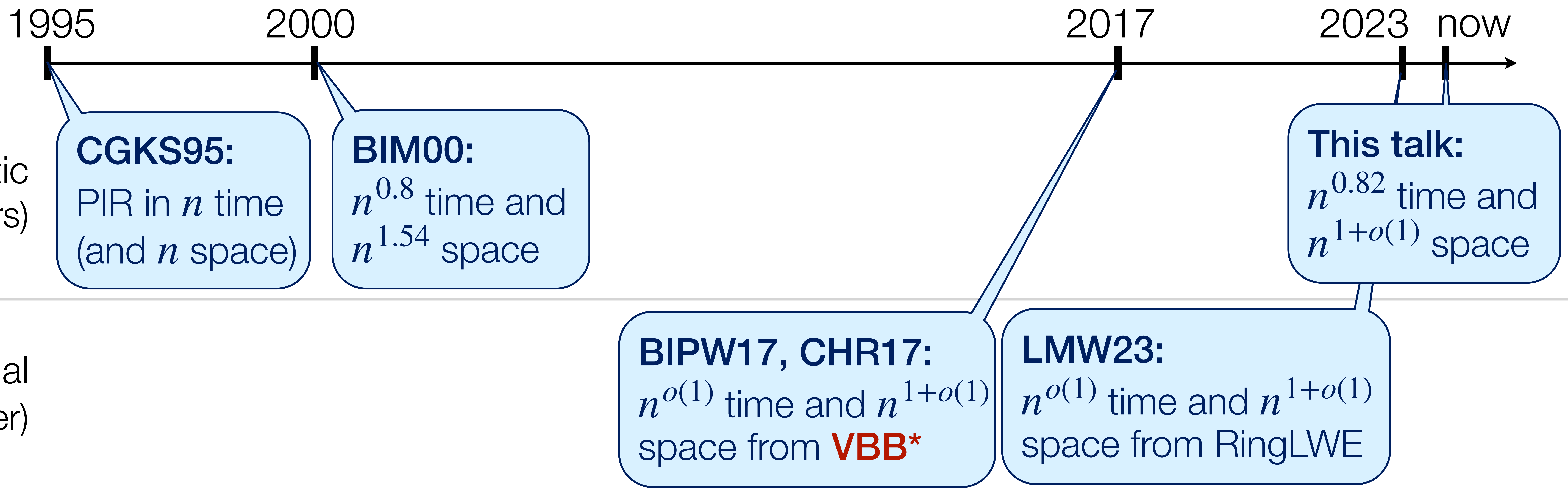
$O(1)$ servers, poly(n) storage, $n^{1-\Omega(1)}$ server time, communication $n^{o(1)}$?

Key conceptual challenge: better data structures for evaluating polynomials with many ($\gg \log n$ or even n^{ϵ}) variables.

The "store all evaluations" solution no longer cuts it!

$O(1)$ servers, poly(n) storage, $n^{o(1)}$ server time?

Thank you! Questions?



Info-theoretic (2 servers)

CGKS95: PIR in n time (and n space)

BIM00: $n^{0.8}$ time and $n^{1.54}$ space

This talk: $n^{0.82}$ time and $n^{1+o(1)}$ space

Computational (1 server)

BIPW17, CHR17: $n^{o(1)}$ time and $n^{1+o(1)}$ space from **VBB***

LMW23: $n^{o(1)}$ time and $n^{1+o(1)}$ space from RingLWE

Concrete storage (2 GB database)

8×10^5 TB **≈ $10 million in hard drives**

?

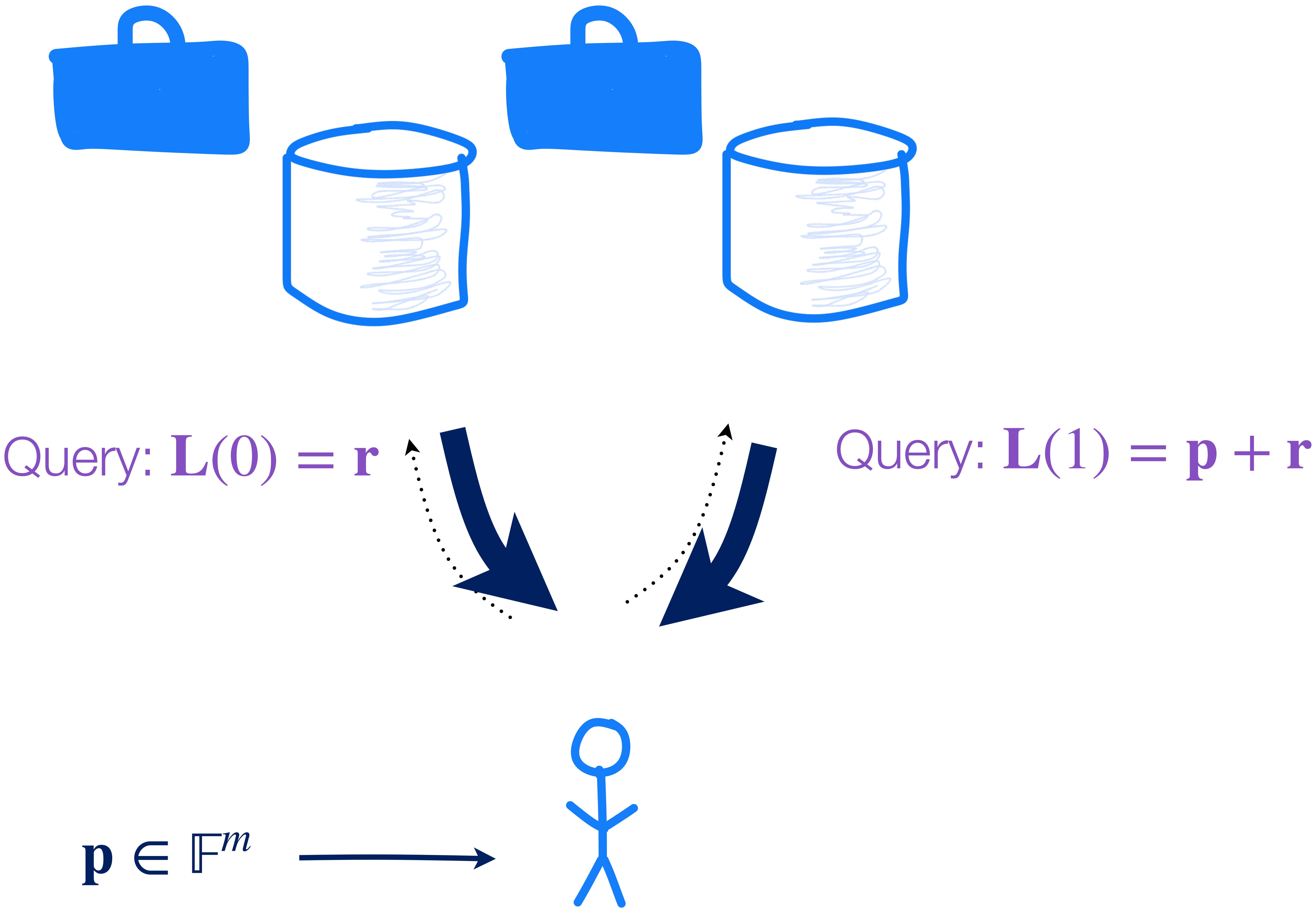
2×10^6 TB **≈ $20 million in hard drives**

1 TB **≈ $10 in hard drives**

*Ignoring polylog savings in time

Bonus Slides on ≥ 3 Servers

$s = 2$ Servers: A Refresher



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$\mathbf{p} \in \mathbb{F}^m$

Cheatsheet

Field: $\mathbb{F}_2$

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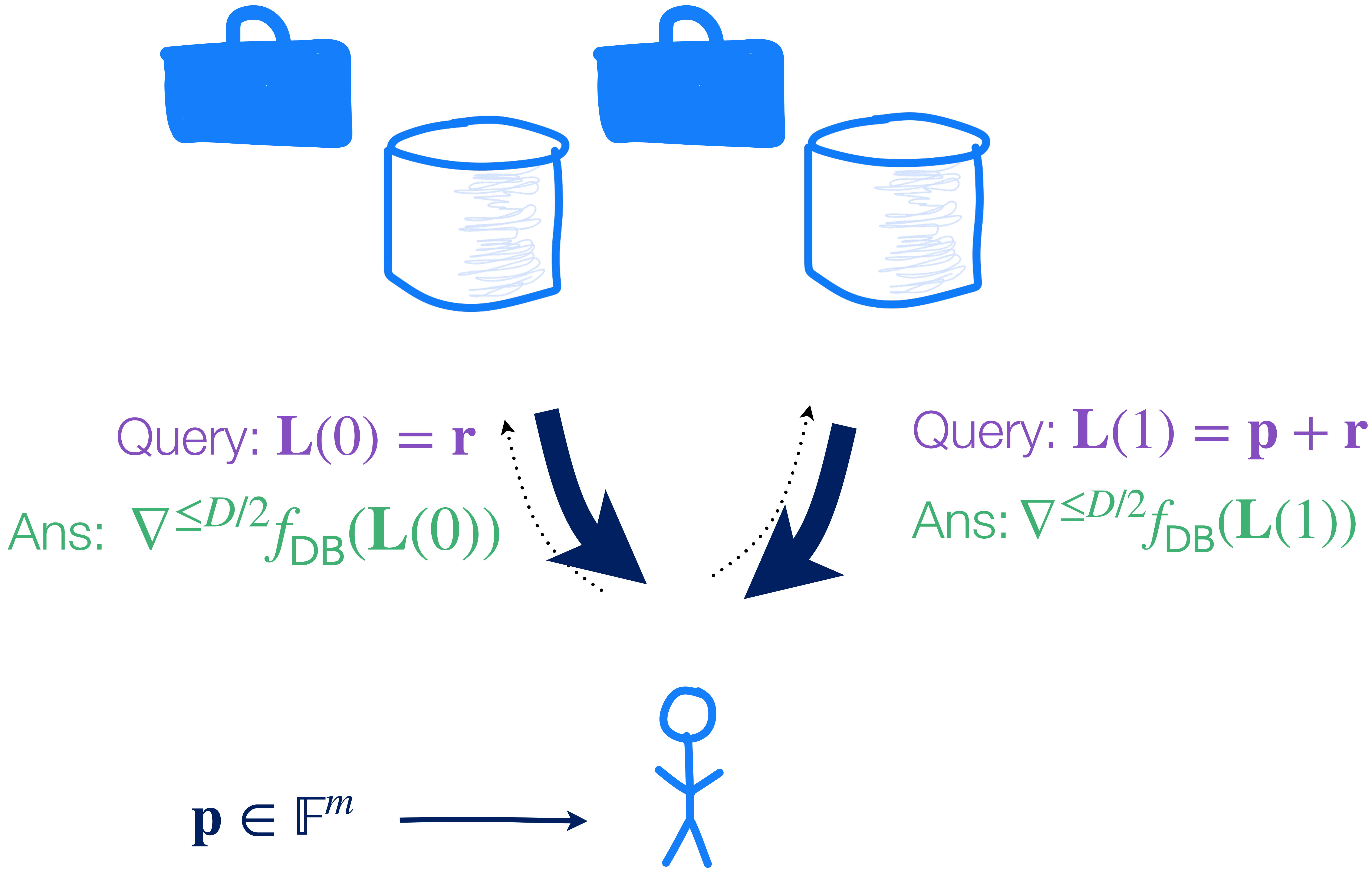

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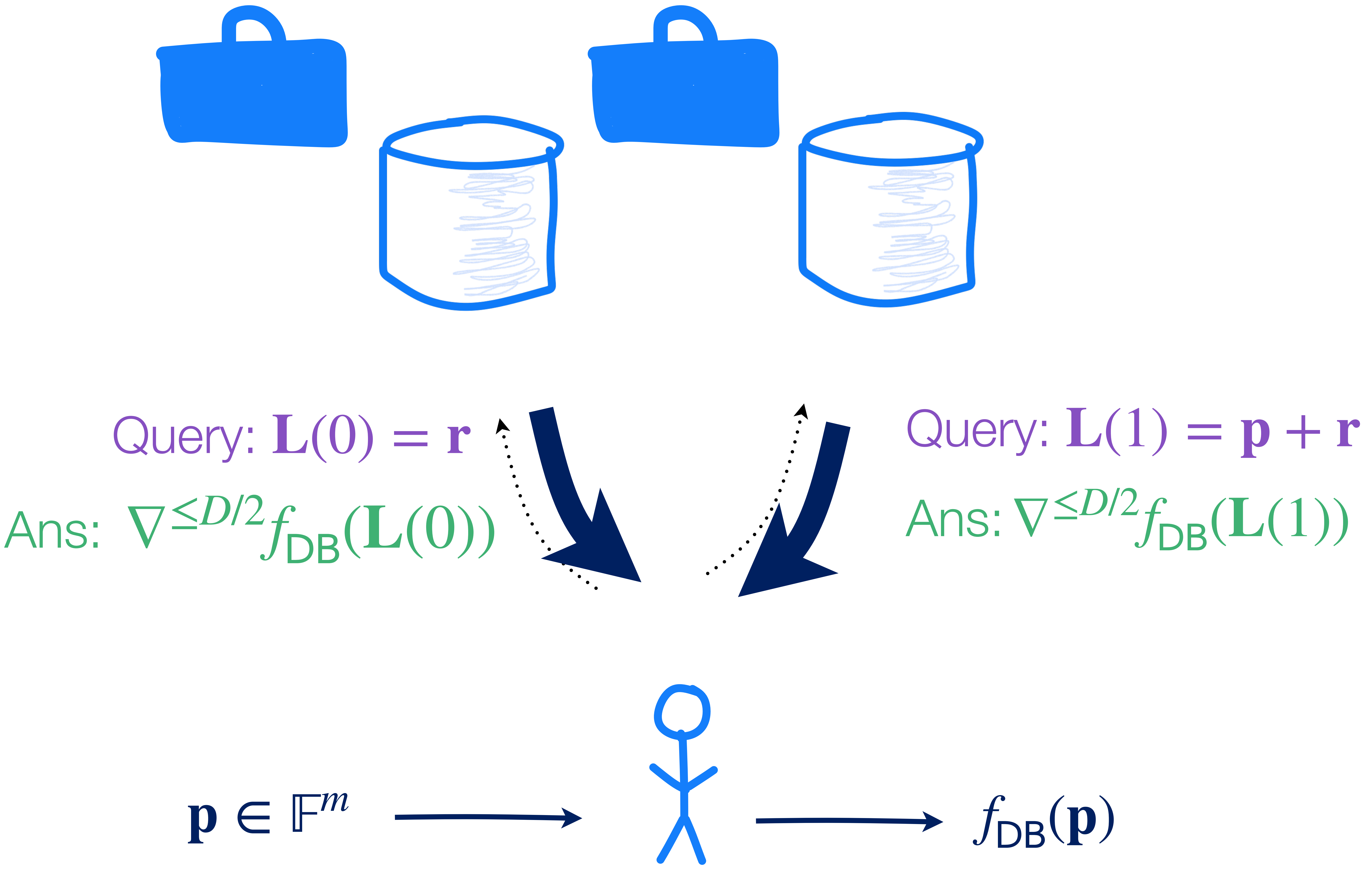

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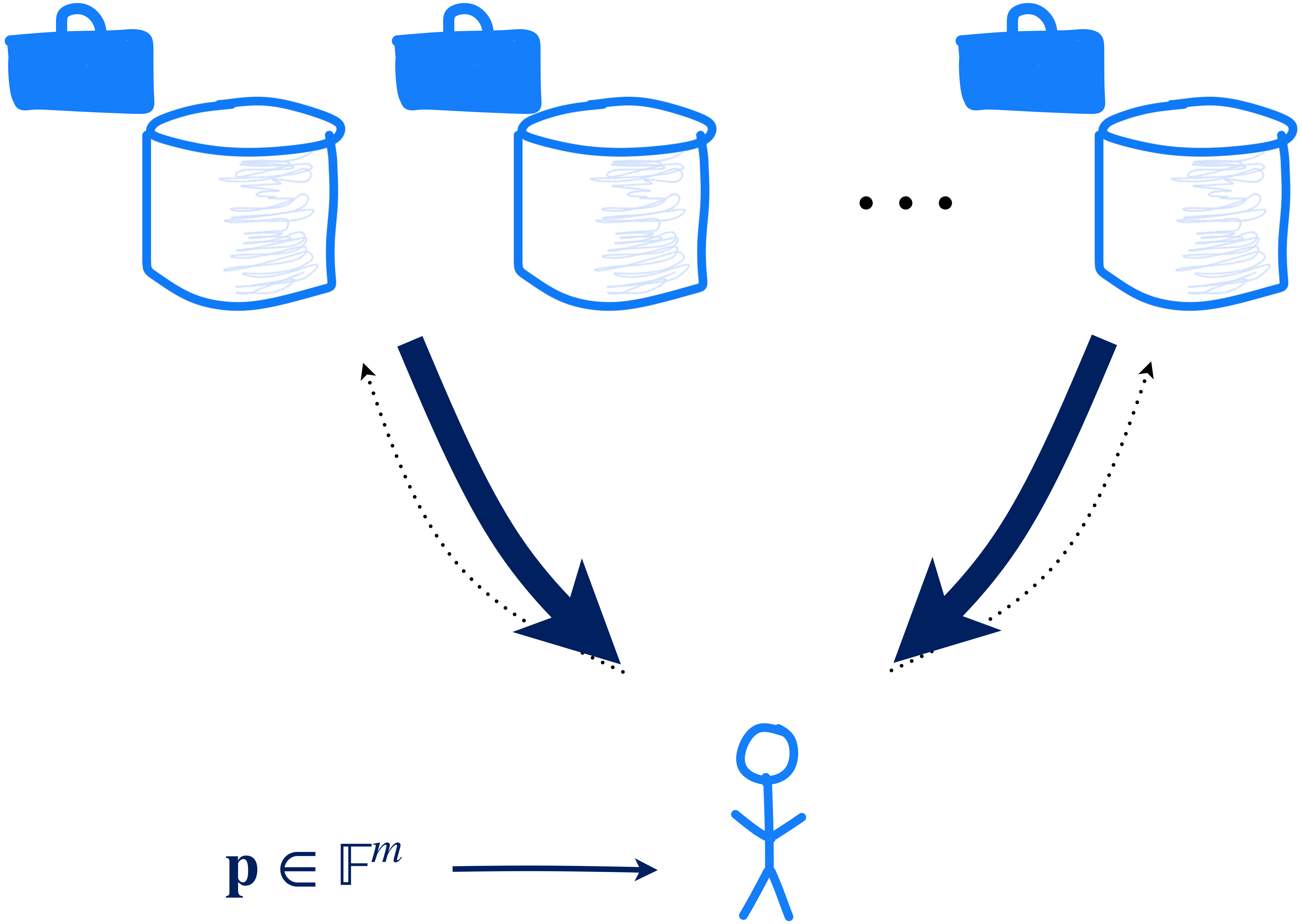

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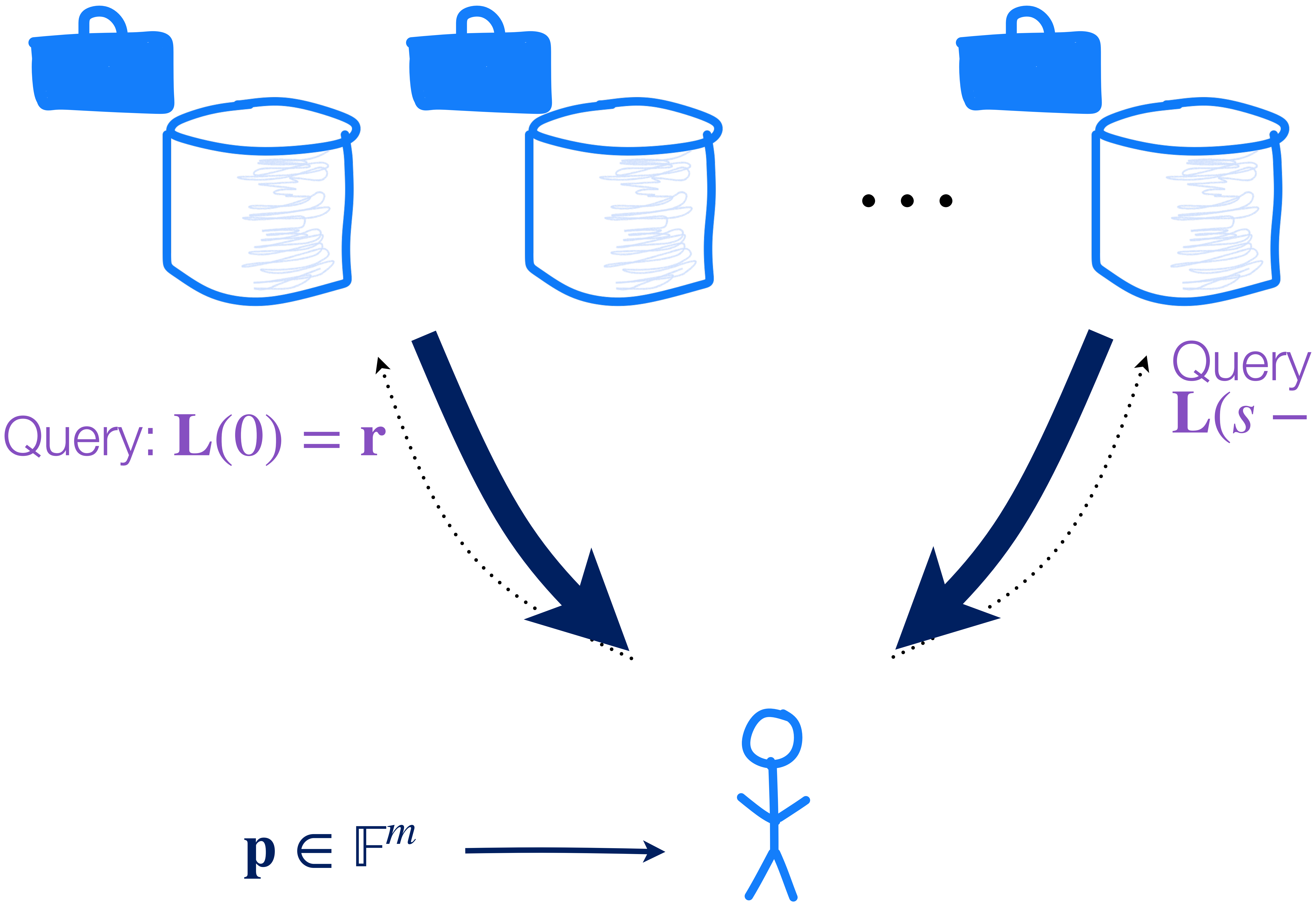

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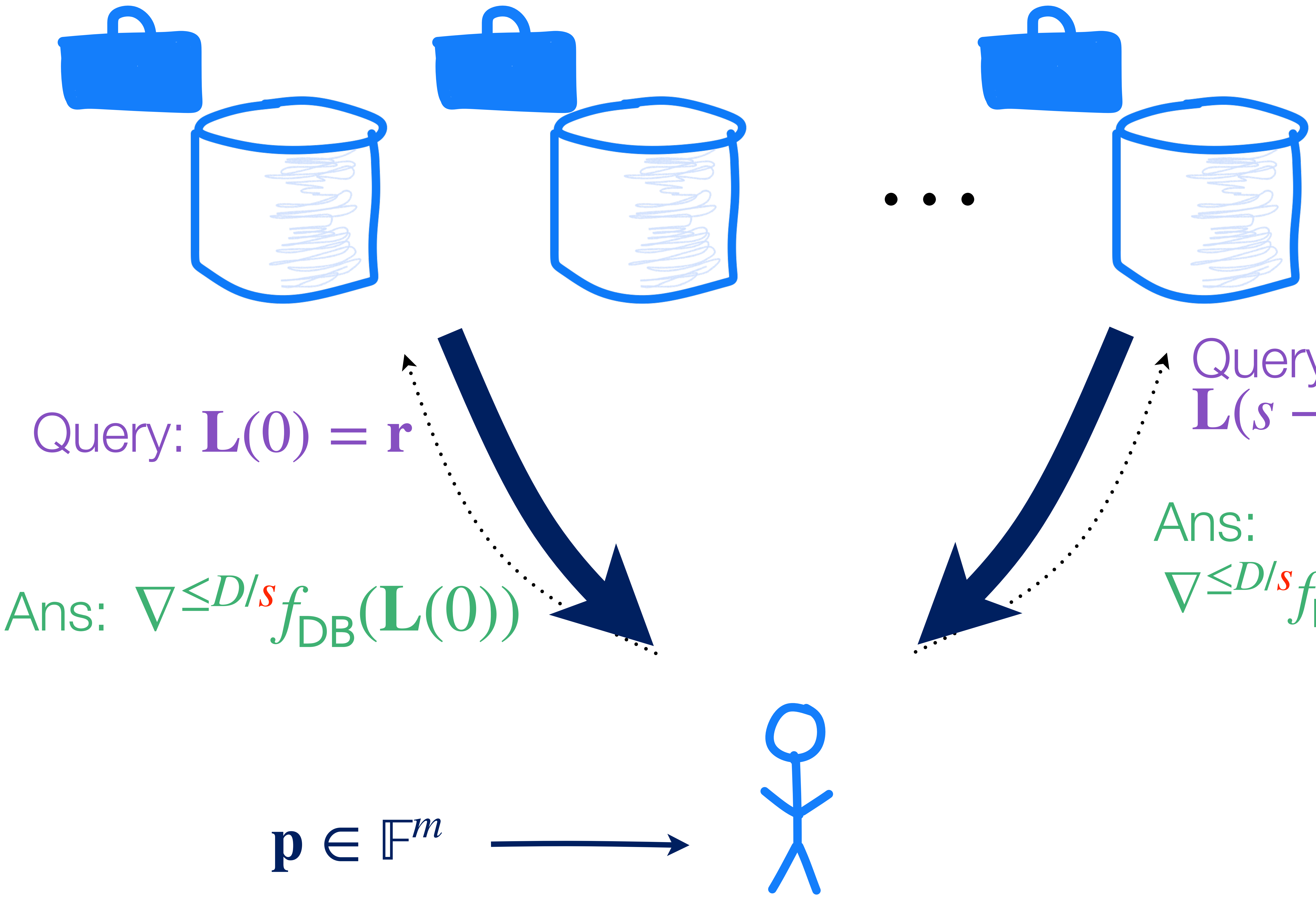

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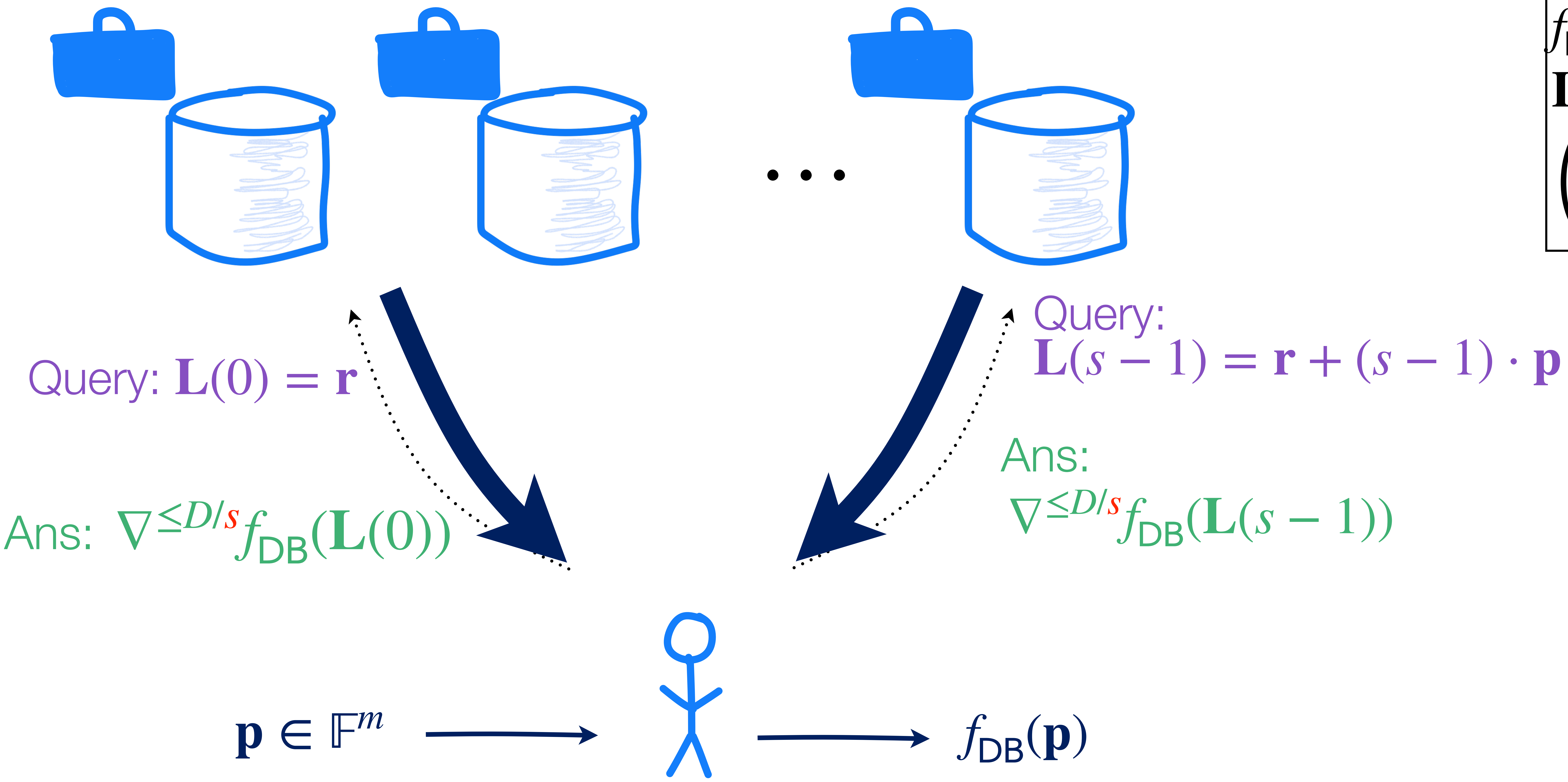

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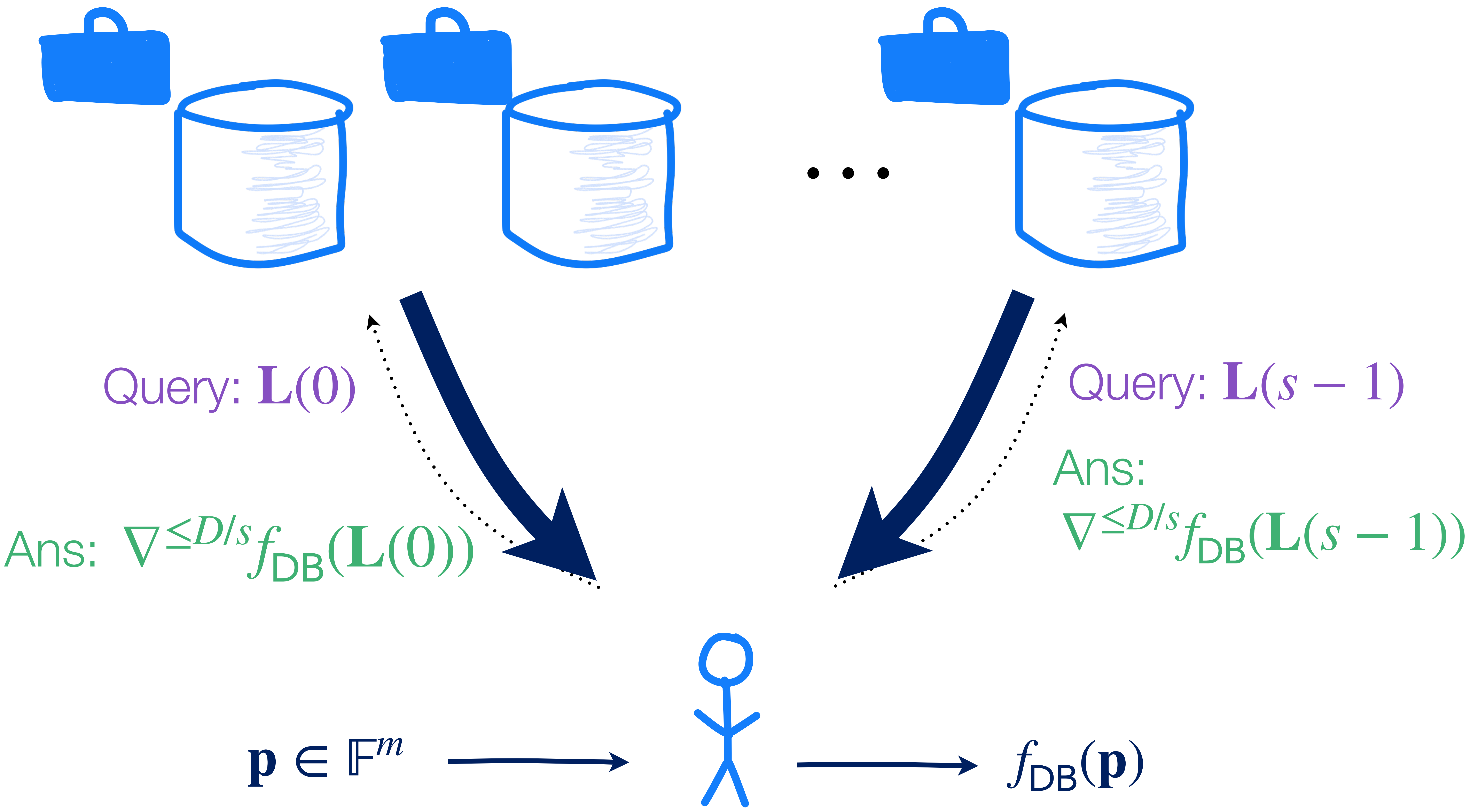

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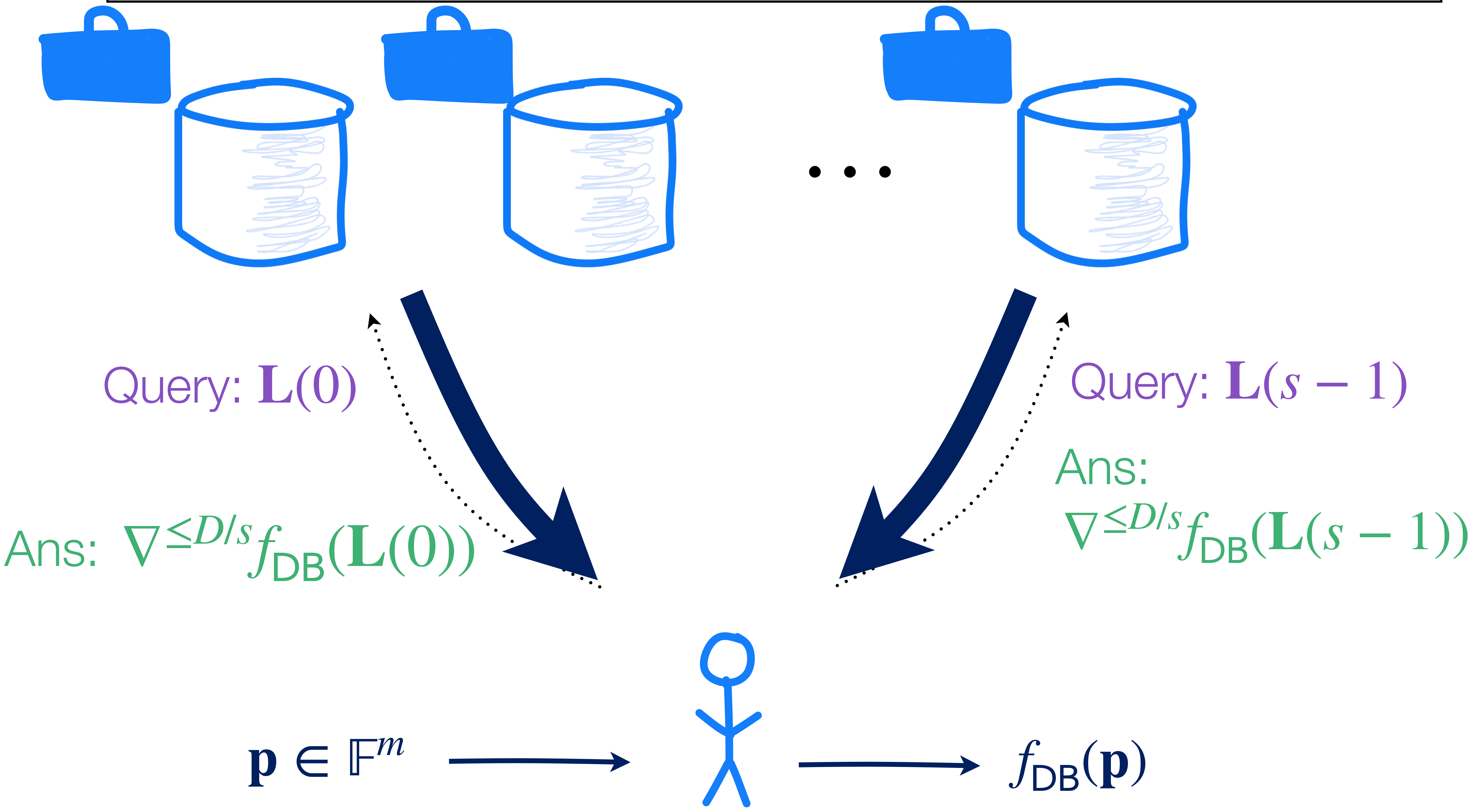

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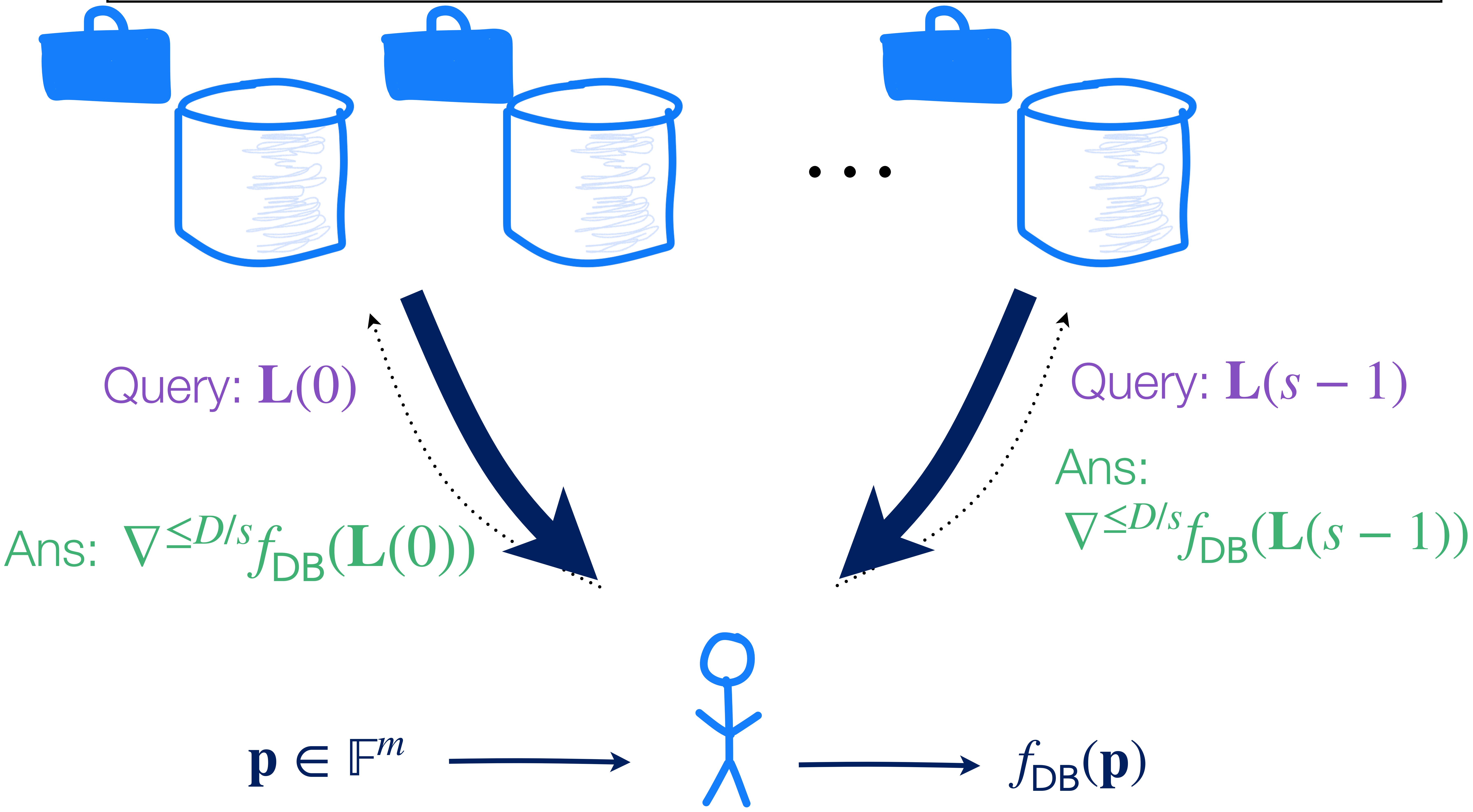

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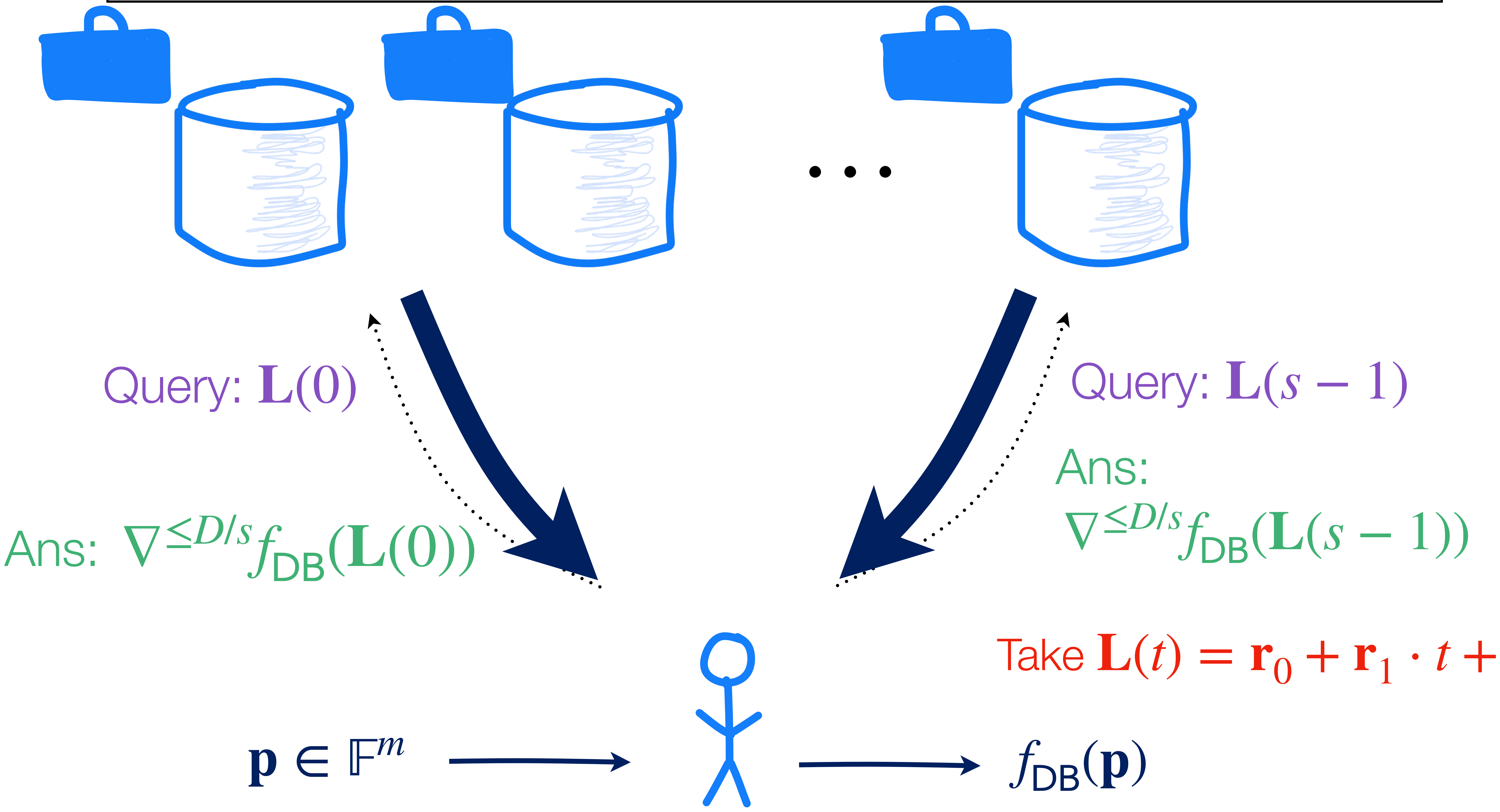

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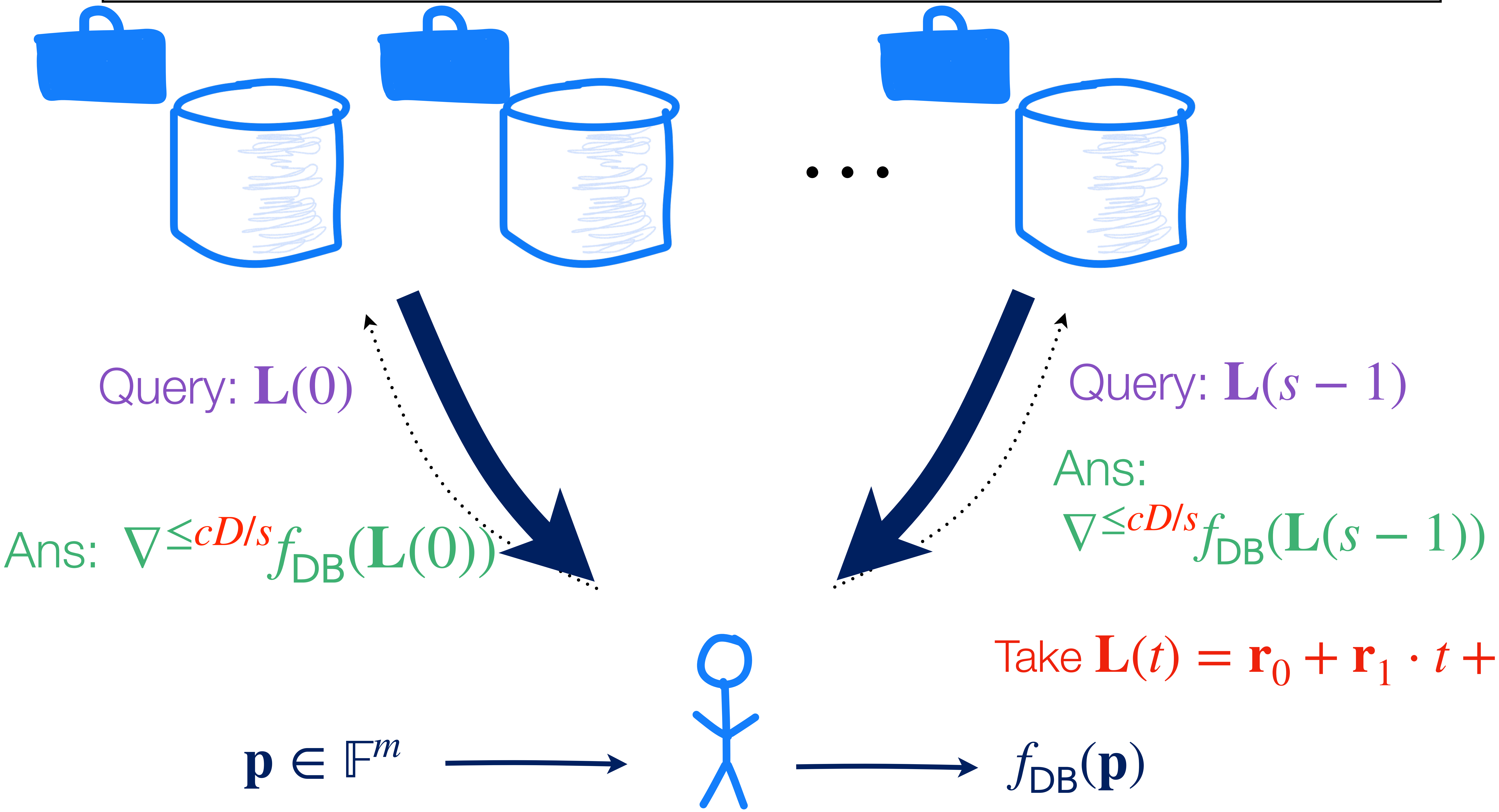

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- **TLDR: the sweet spot for d increases as the number of servers increases**

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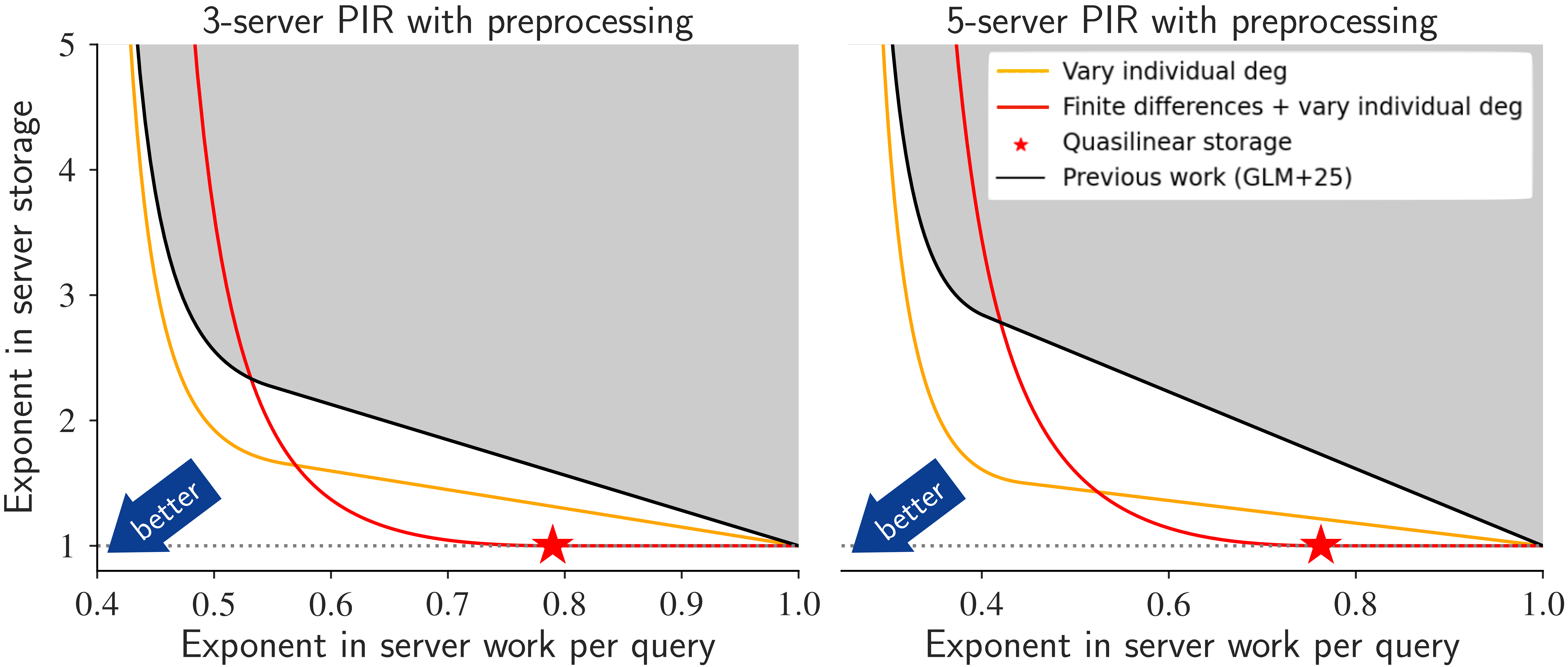
3-server PIR with preprocessing
5-server PIR with preprocessing
Exponent in server storage
5
4
3
2
1
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Exponent in server work per query
0.4
0.6
0.8
1.0
Exponent in server work per query
better
better
Vary individual deg
Finite differences + vary individual deg
Quasilinear storage
Previous work (GLM+25)

Bonus Slides on Locally Decodable Codes

Smooth Locally Decodable Codes [KT00]

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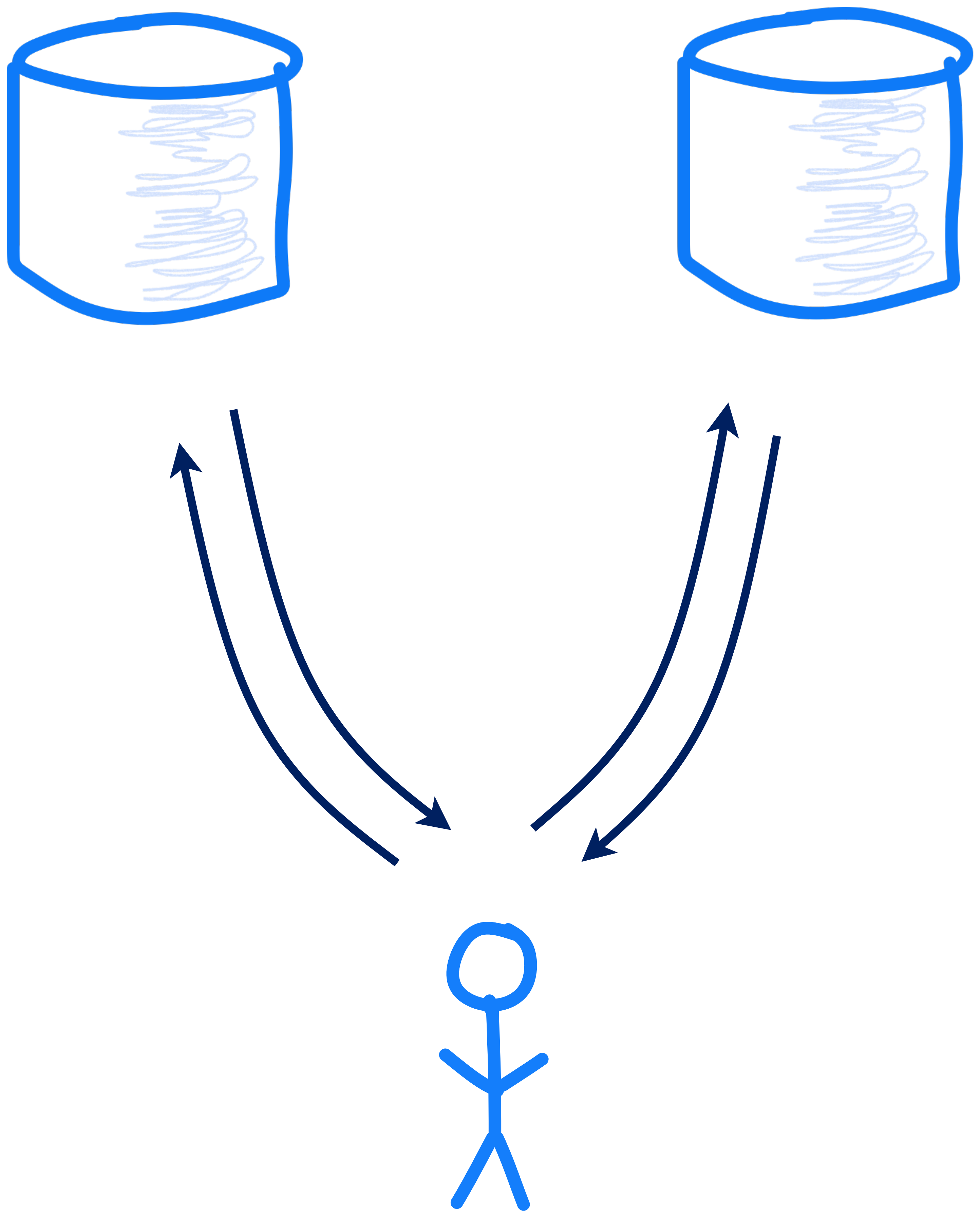
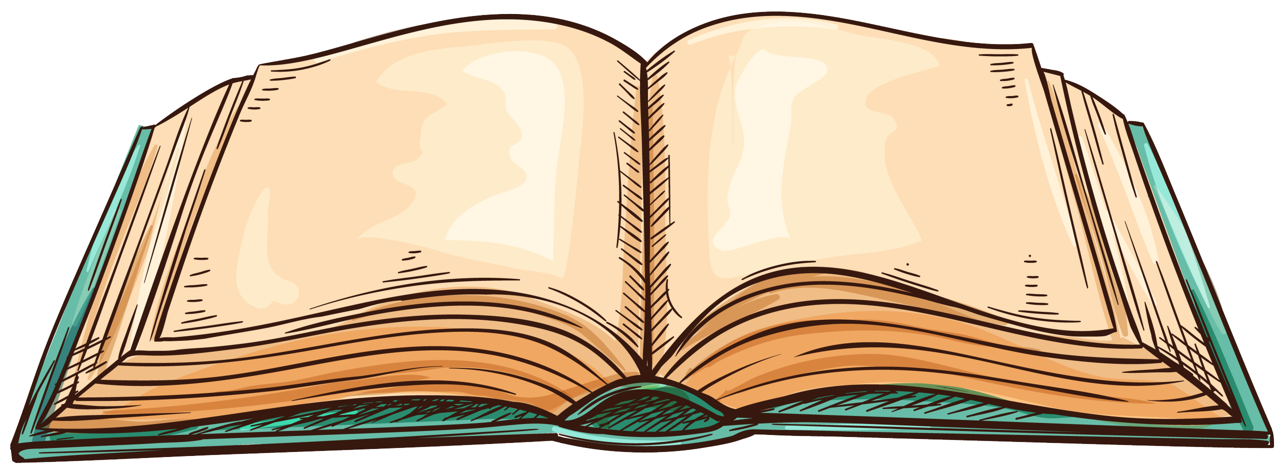
2-query LDCs → 2-server PIR





LDCs	PIR
Message	DB
Encoding	Preprocessing
Decoding	Reconstruction

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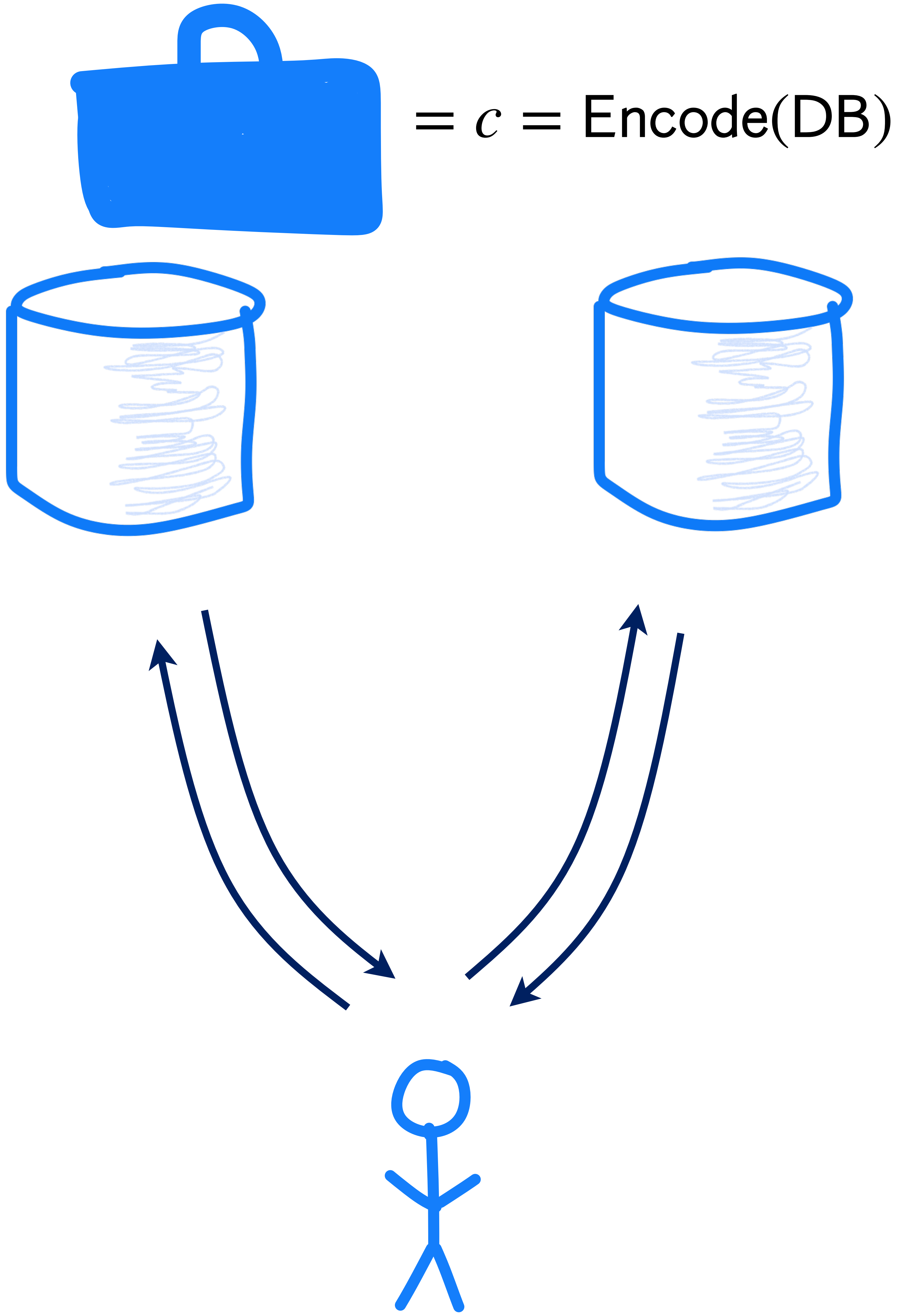
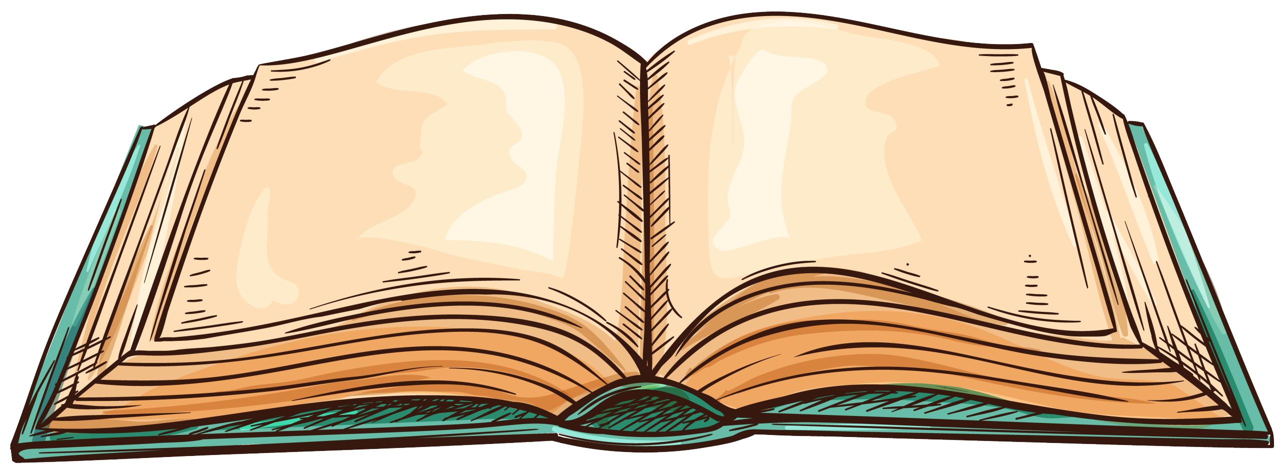


$= c =$ Encode(DB)



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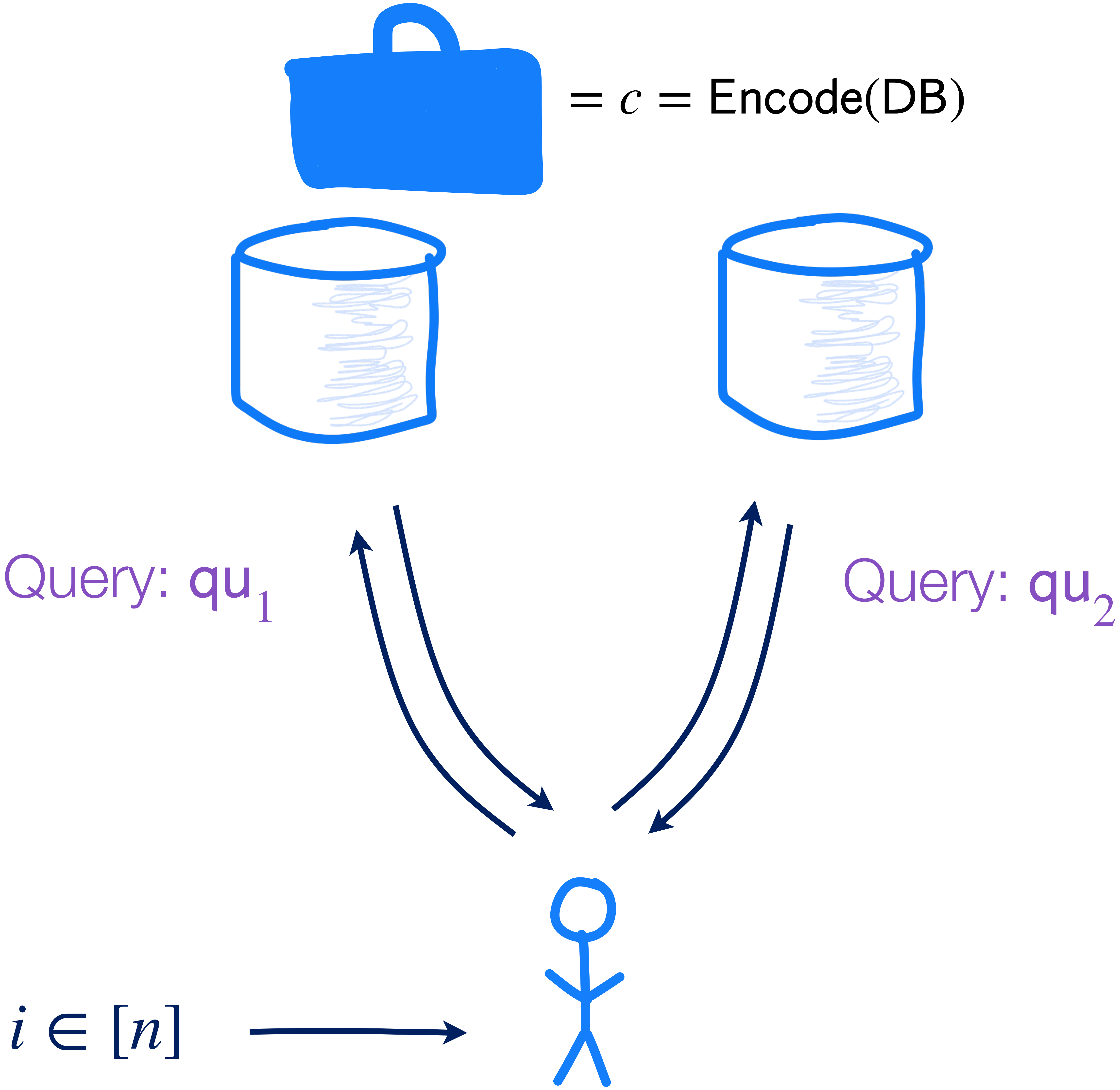

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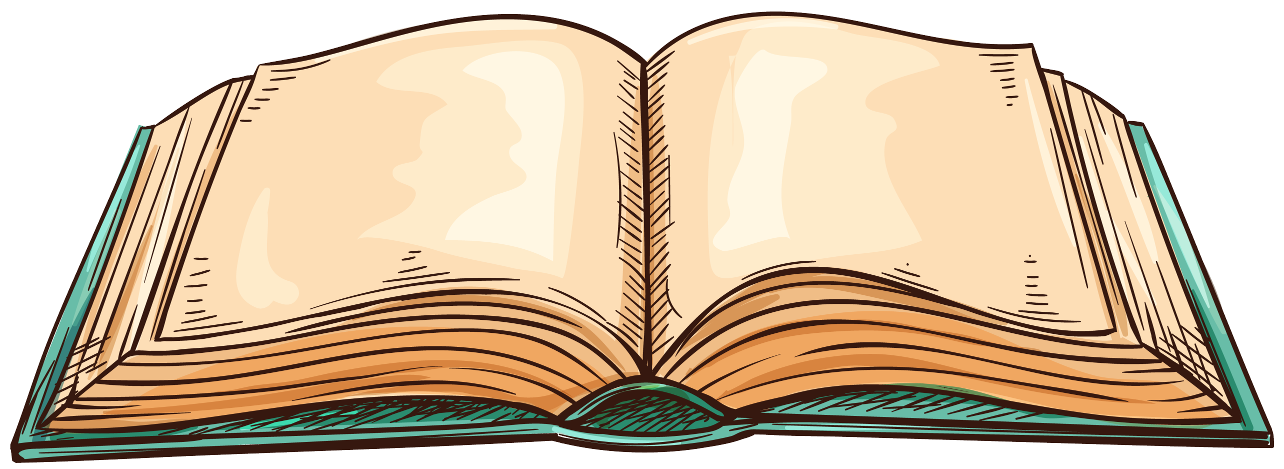

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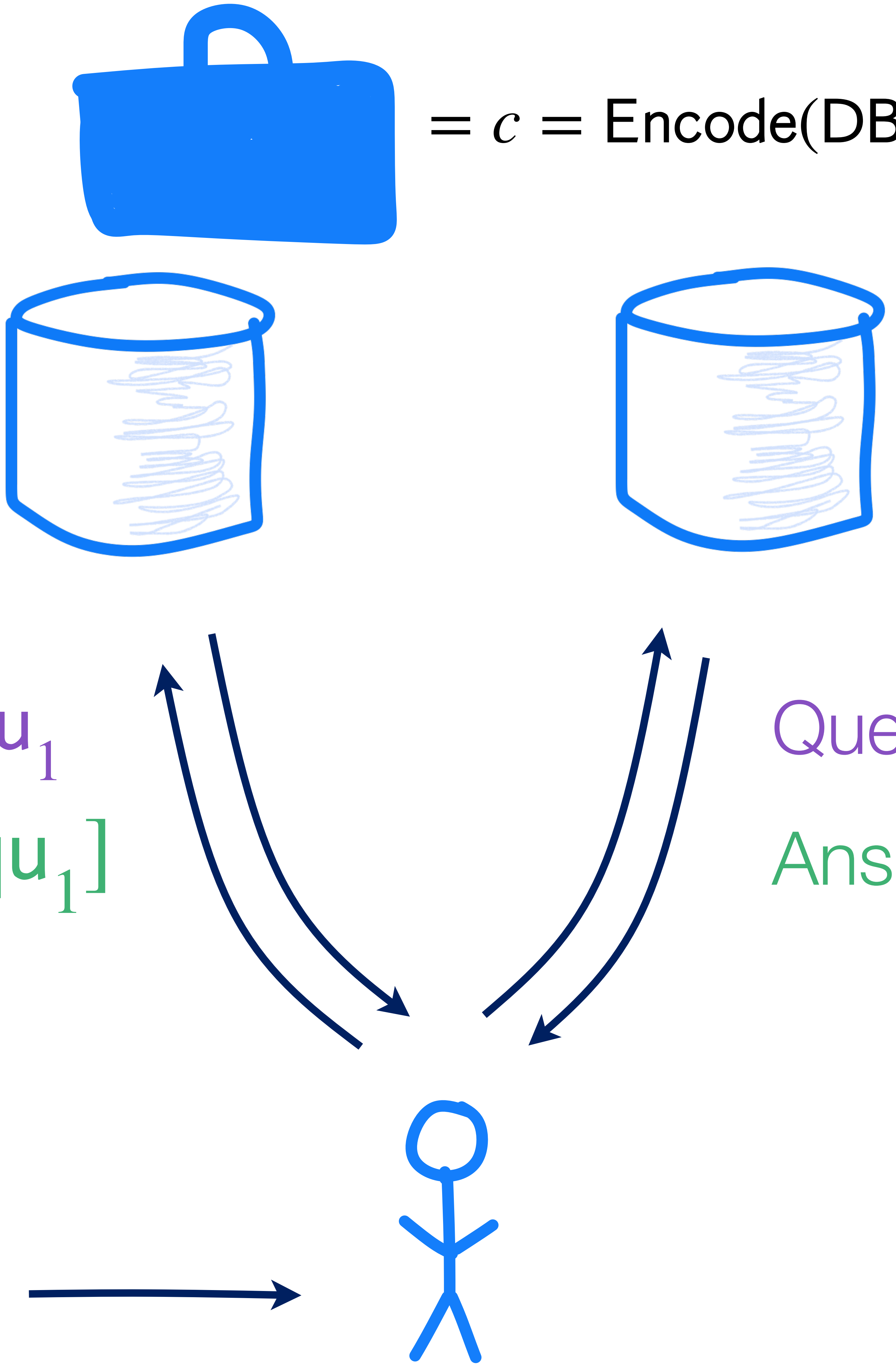

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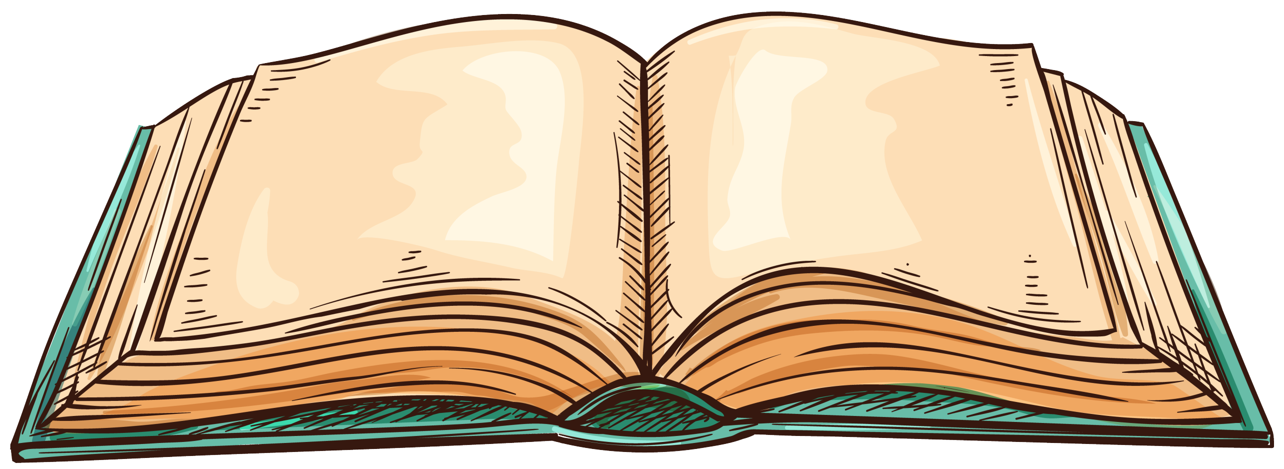

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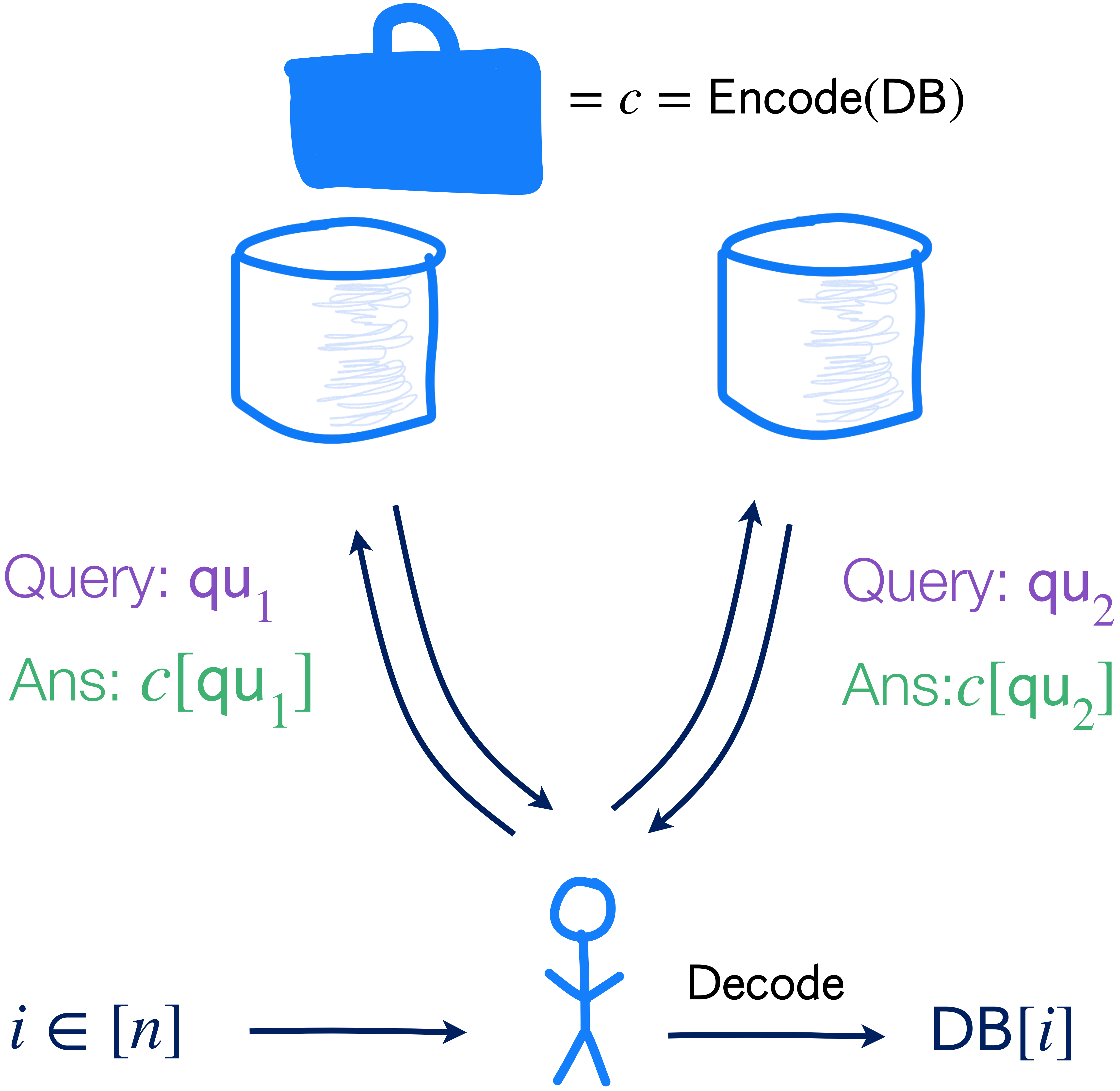

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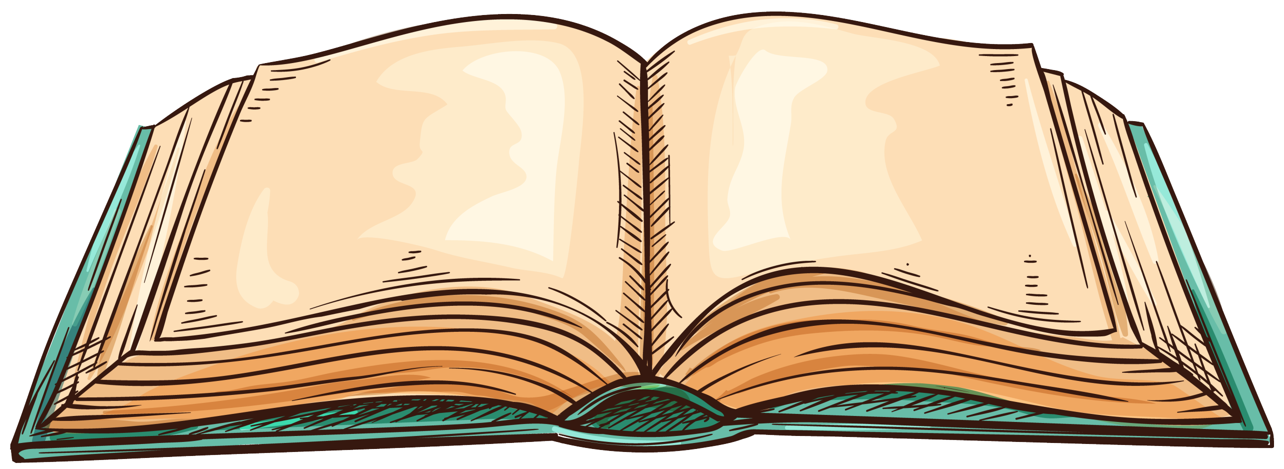
Decode

DB$[i]$



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Casting BIM00 PIR as an LDC

$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\mathsf{DB}}(\mathbf{2}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{2})$
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- Encode($\mathsf{DB} \in \{0,1\}^n$):
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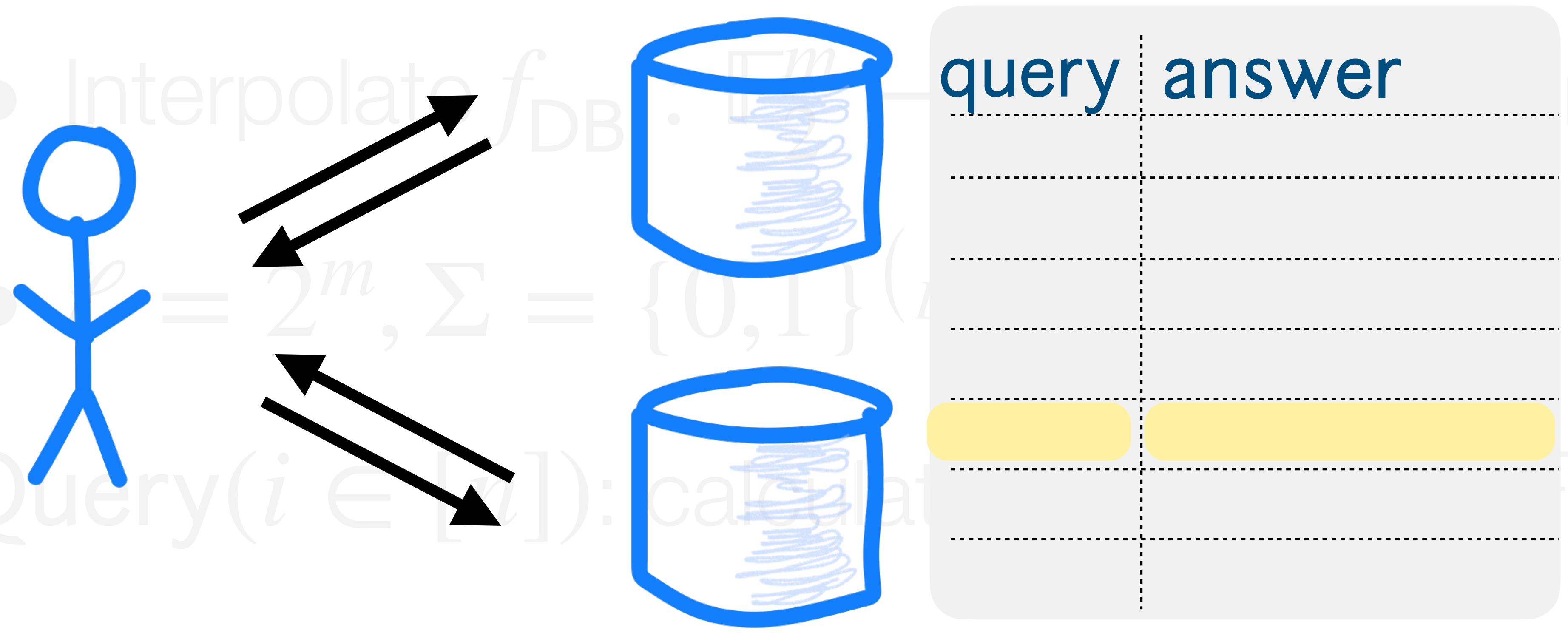
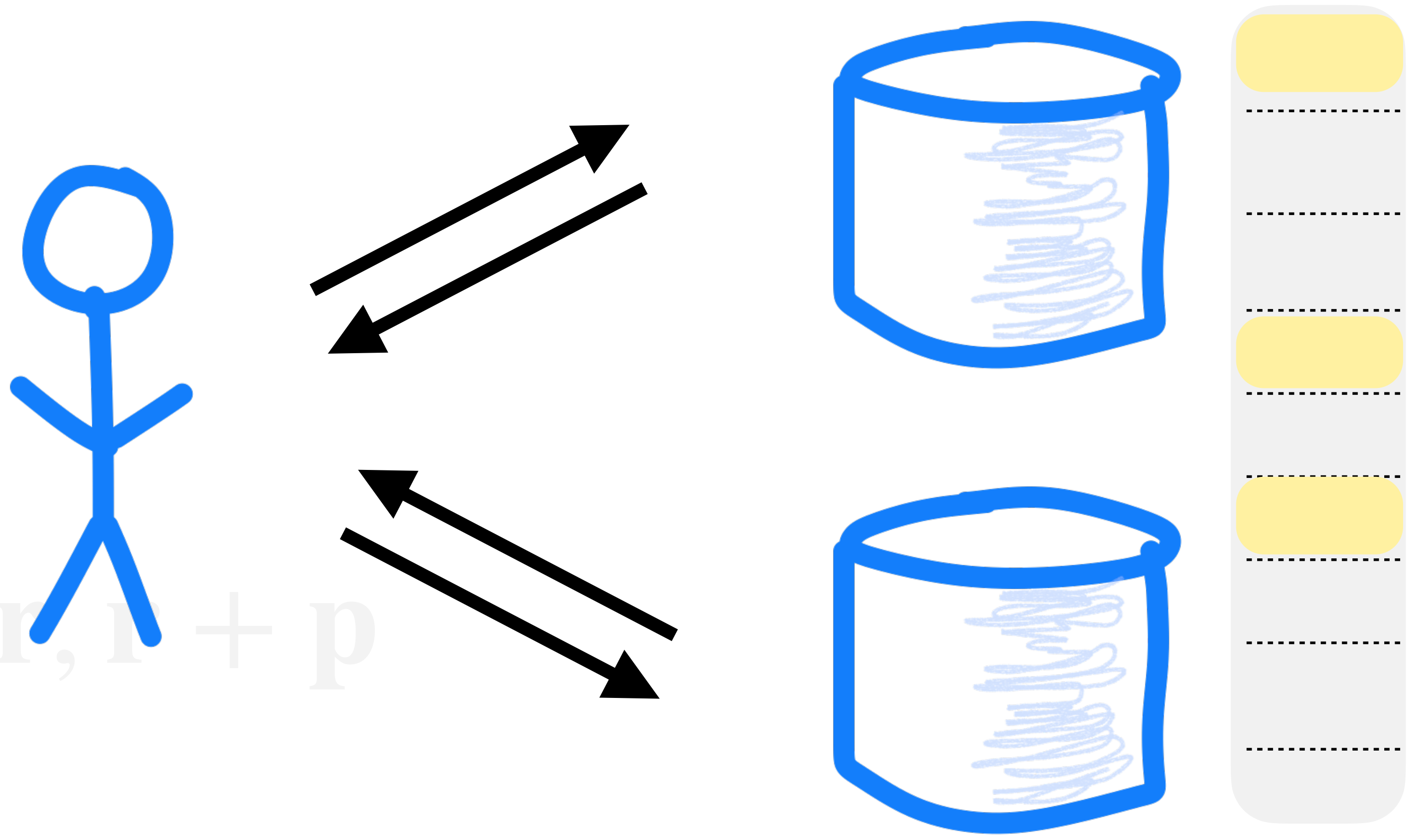
LDCs and BIM00 PIR



Our PIR



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Defining Batch-Smooth LDCs

Cheatsheet

n: message length

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Σ: alphabet

q: locality

b: number of batches

Defining Batch-Smooth LDCs

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2	$\mathsf{qu}_{q/b+1}, \ldots, \mathsf{qu}_{2q/b}$
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Then each row's distribution should be independent of the index i being queried

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BIM00 PIR as a Batch-Smooth LDC

Cheatsheet

n: message length

$\binom{m}{D} \geq n$

BIM00 PIR as a Batch-Smooth LDC

- Number of batches: $b = 2$

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BIM00 PIR as a Batch-Smooth LDC

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Our PIR as a Batch-Smooth LDC

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Consequence: the first batch-smooth LDC with constant alphabet size, constant number of batches b, codeword length $n^{1+o(1)}$, and polynomially sublinear number of queries $q = n^{1-\Omega(1)}$