Factoring with a Quantum Computer: The State of the Art

Seyoon Ragavan
MIT

Part 1: A History of Quantum Factoring

Part 2: Factoring Some Integers in Sublinear Quantum Space

ARTIFICIAL INTELLIGENCE • STARTUPS • VENTURE

Quantum Computing Funding Hits Record High With Apparent AI Boost

Chris Metinko November 14, 2024

Sundar Pichai
@sundarpichai

Introducing Willow, our new state-of-the-art quantum computing chip with a breakthrough that can reduce errors exponentially as we scale up using more qubits, cracking a 30-year challenge in the field. In benchmark tests, Willow solved a standard computation in <5 mins that would take a leading supercomputer over 10^25 years, far beyond the age of the universe(!).

10:36 PM · Dec 9, 2024 · **18.8M** Views

Department of Energy Announces $65 Million for Quantum Computing Research

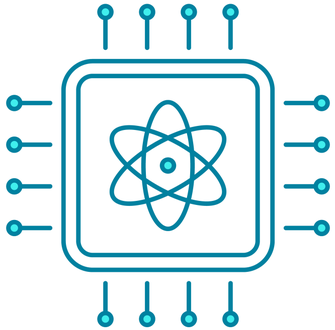
Projects span software, control systems, and algorithms for quantum computing

Office of Science

September 9, 2024

IBM's Quantum Heron Processor breaks speed records with 50x performance boost

Q: Why all this buzz around quantum computing?



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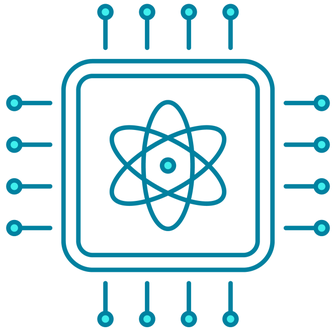
*A: Quantum computers **could** solve **some** problems that we believe to be completely intractable with a classical computer.*



Integer Factorisation

Quantum computing's poster child

Given an n-bit integer $N < 2^n$, find its prime factorisation in $\mathrm{poly}(n)$ time.

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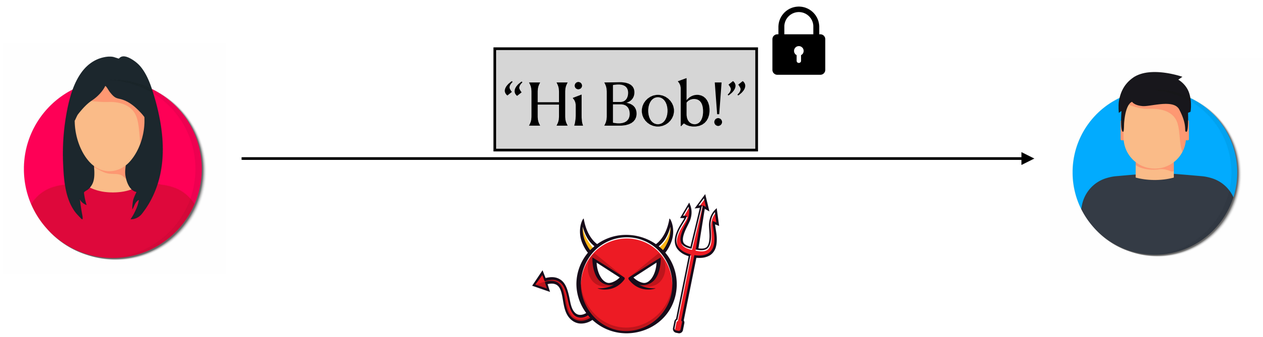
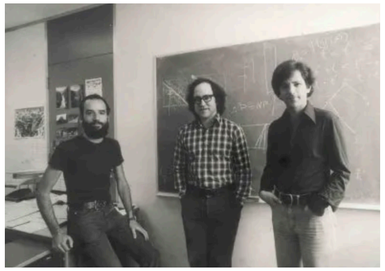
- **Quantum algorithms:** poly(n) **time!** (Shor 1994)

Implication 1: Breaking Cryptography

- RSA public-key cryptography:





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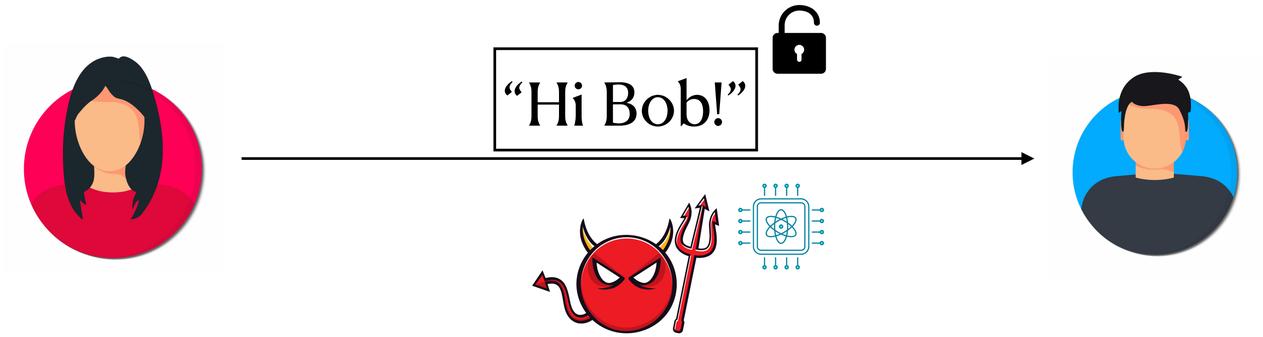
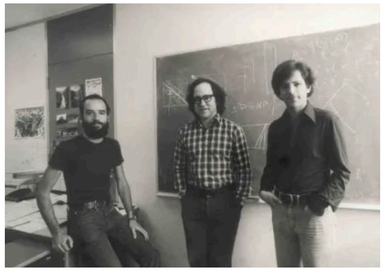
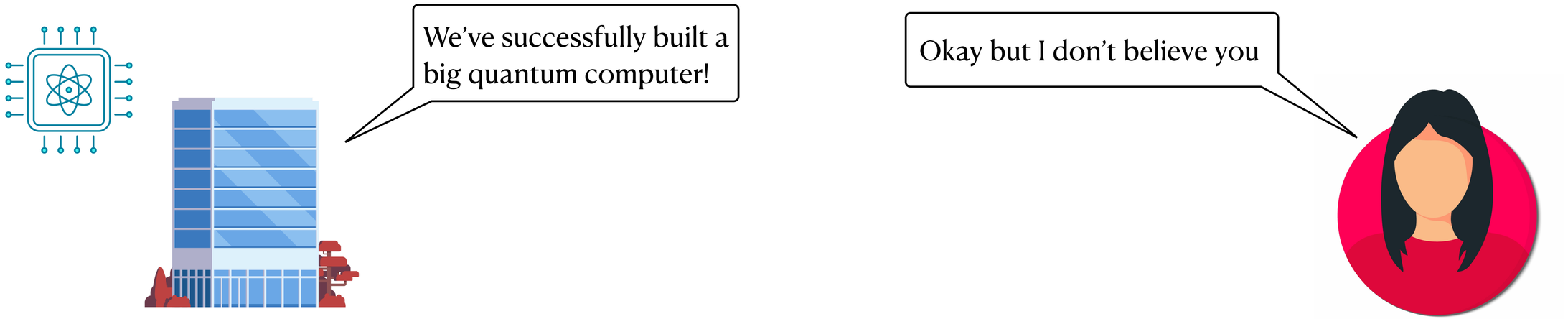




- **Completely broken if the eavesdropper has a large quantum computer (since then they could factor large integers)!**

Implication 2: Proofs of Quantumness



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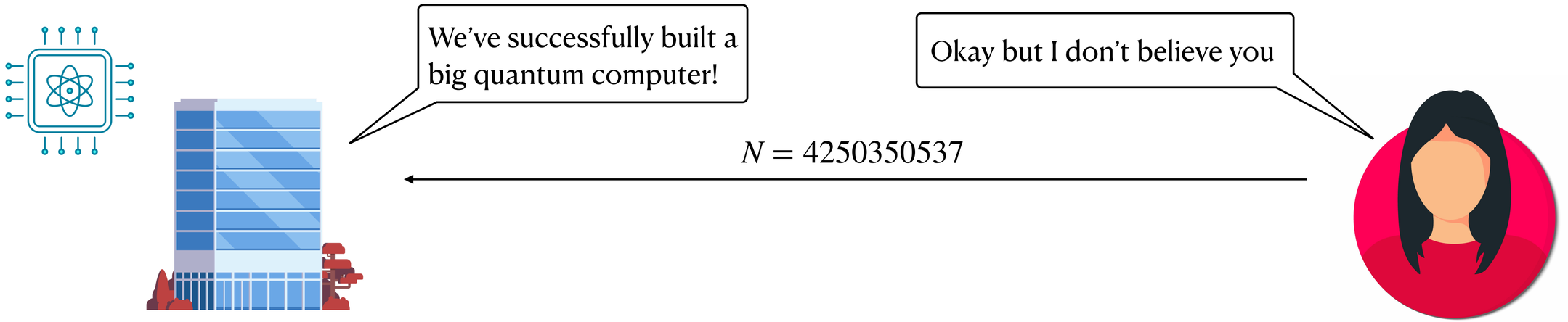

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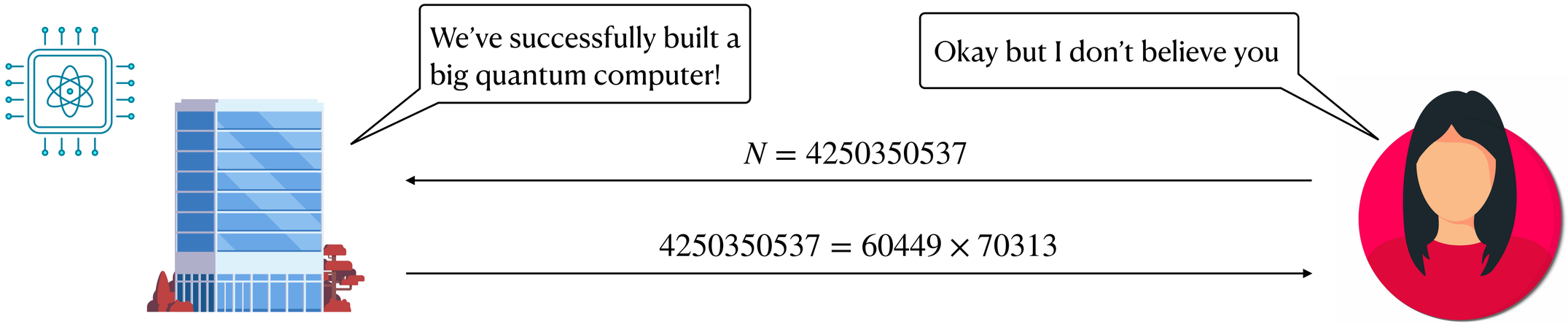

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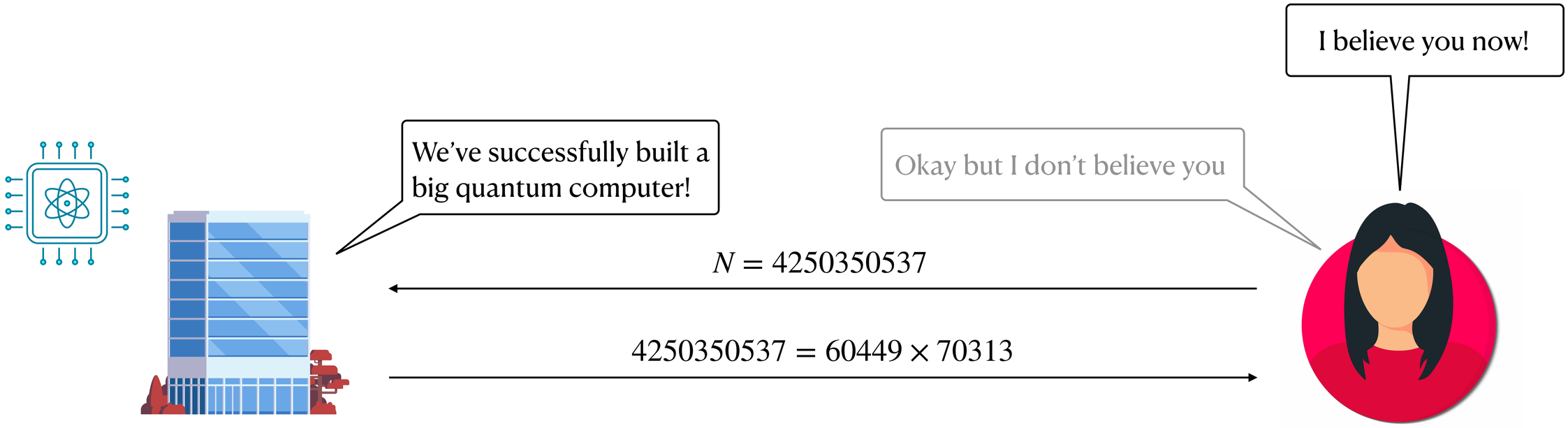

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- Are there special types of integers that are especially easy to factor quantumly?
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Coming up: some answers to these questions!

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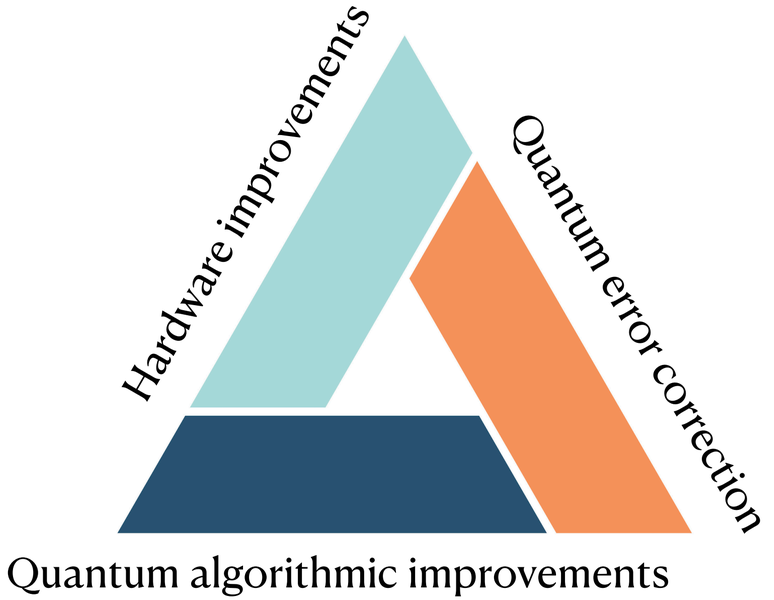

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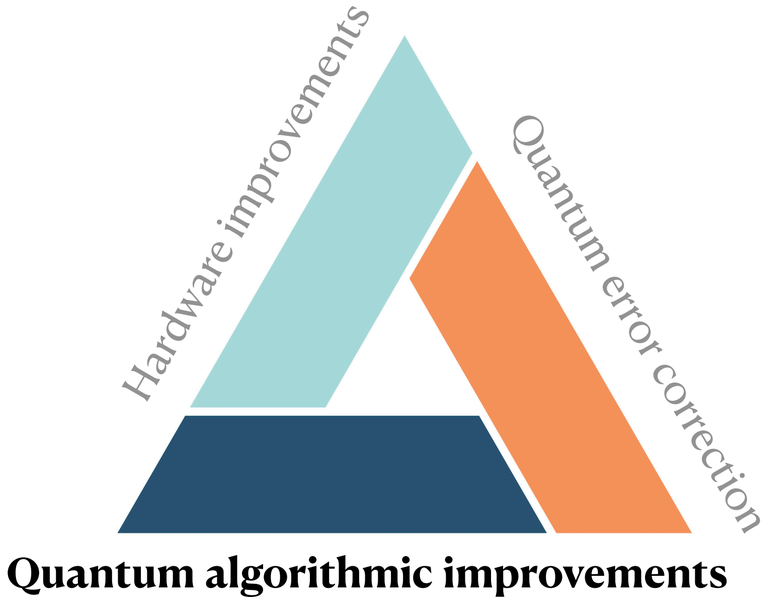
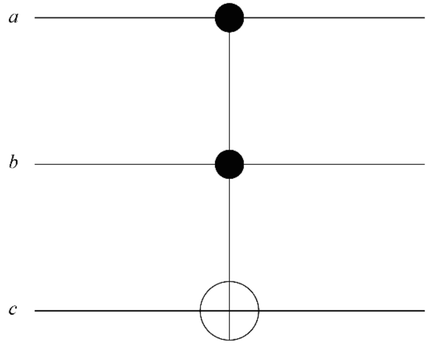


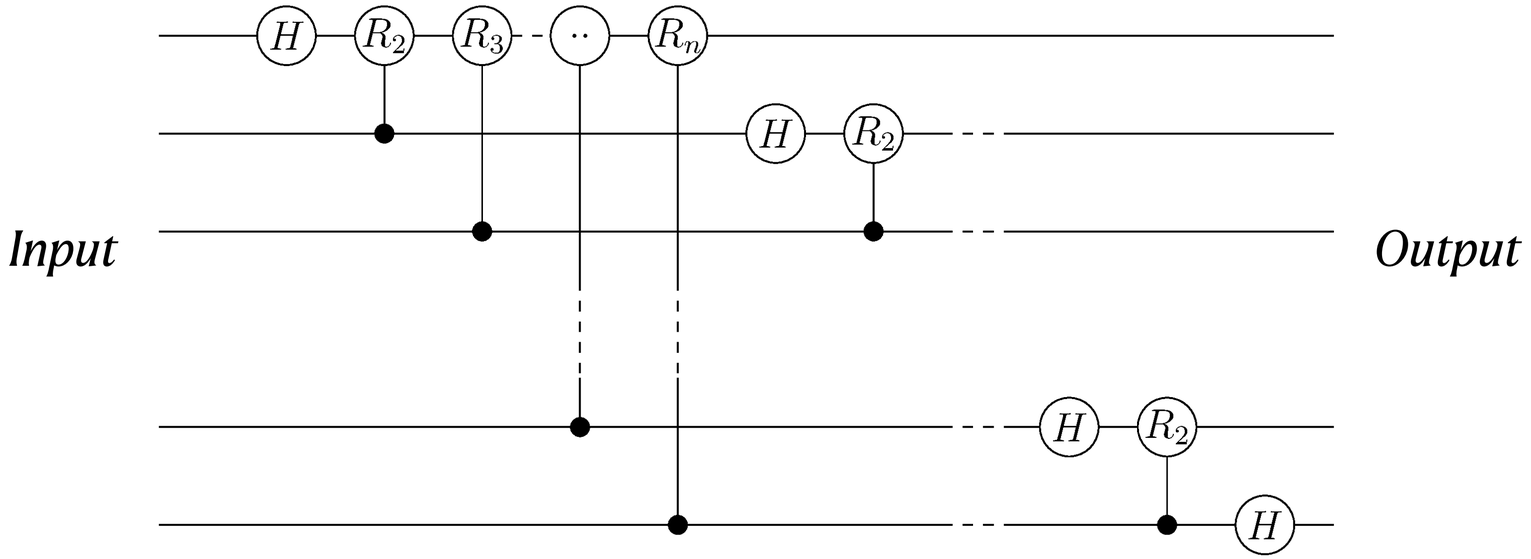
Efficiency Metrics for Quantum Circuits

- Like a classical circuit, but uses reversible logic gates on a fixed number of wires

Gates: the number of elementary operations (circles)

Space: the number of wires (lines)





Quantum Factoring Algorithms: A History

Authors	Types of inputs	Gates	Space
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$

n is the number of bits in the input N

$\tilde{O}(\cdot)$ hides constant and $\text{poly}(\log n)$ factors

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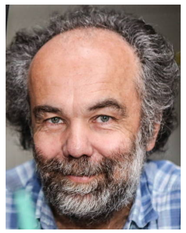

Any input: would break RSA cryptography, and suffice as a proof of quantumness

$N = P^2Q$: only suffices as a proof of quantumness

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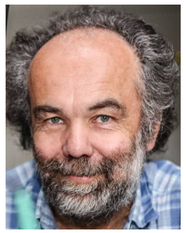

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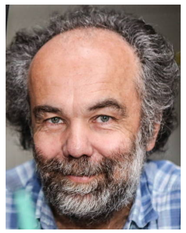

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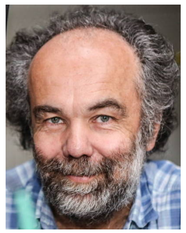



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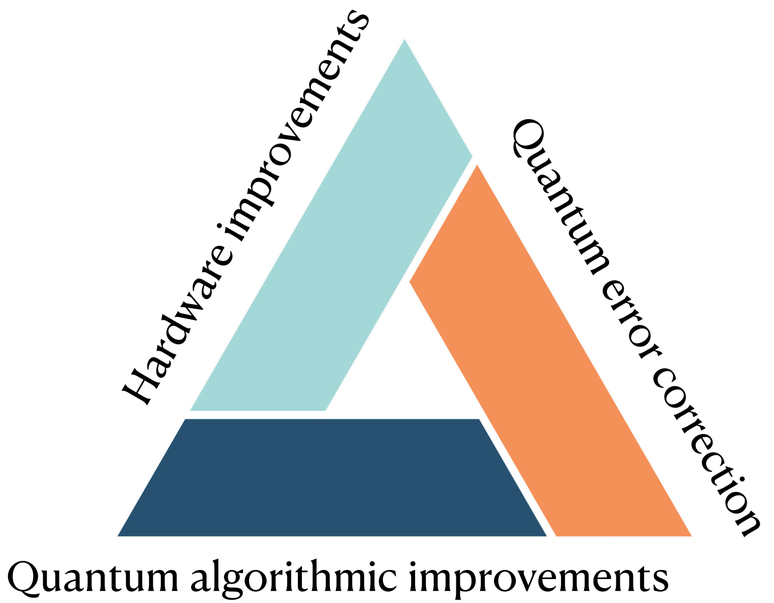
- Interdisciplinary collaboration with hardware, error correction researchers needed for further improvements



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$2040 says we'll be able to quantumly factor 2040-bit integers by 2040

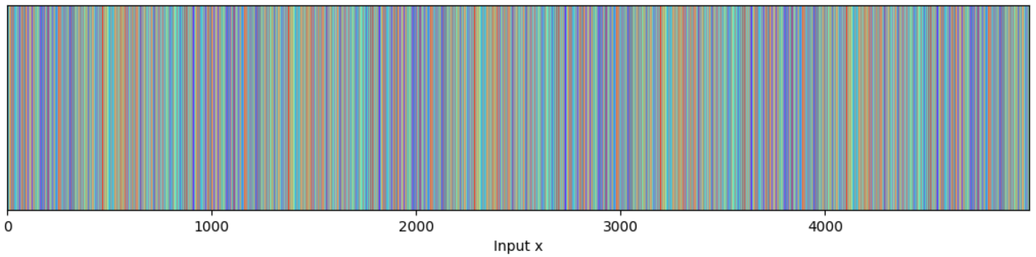
Part 2: Proofs of Quantumness from Factoring P^2Q

Gregory D. Kahanamoku-Meyer*, Seyoon Ragavan*, Vinod Vaikuntanathan*, Katherine Van Kirk†


*MIT, †Harvard


Preliminary: Quantum Period Finding

- Strictly periodic function $f : \mathbb{Z} \to \mathbb{Z}$ with unknown period T
  - $x \equiv y \pmod{T} \Leftrightarrow f(x) = f(y)$



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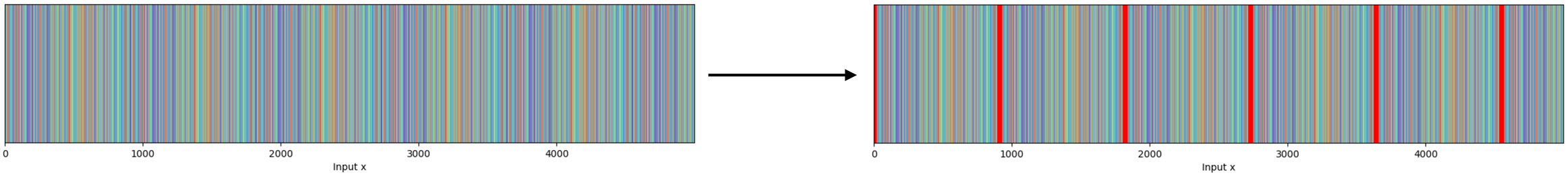

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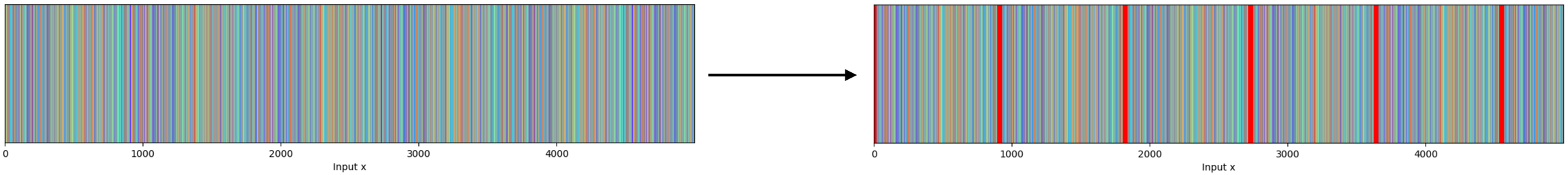
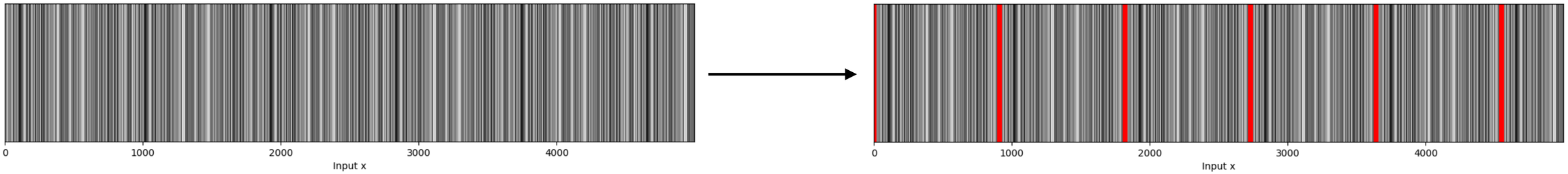
- Also the centrepiece of Shor's quantum factoring algorithm



Preliminary: General Quantum Period Finding

May and Schlieper (2022)

- ~~Strictly~~ periodic function $f : \mathbb{Z} \to \mathbb{Z}$ with unknown period T
  - $x \equiv y \pmod{T} \Rightarrow f(x) = f(y)$
- Informal theorem* statement: can quantumly recover T using essentially only the gates/space needed to compute $f(x)$ for $|x| \leq \text{poly}(T)$
- In fact, the algorithm is exactly the same!



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- Legendre symbol essentially indicates whether this is the case:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a nonzero quadratic residue modulo } p; \text{ and} \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p; \text{ and} \\ 0, \text{ if } a \text{ divisible by } p\,. \end{cases}$$

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  - Could have $\left(\frac{a}{N}\right) = 1$ without a being a quadratic residue modulo N

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- Useful property: $a \equiv b \pmod{N} \Rightarrow \left(\frac{a}{N}\right) = \left(\frac{b}{N}\right)$
- Theorem (Schönhage 1971): can compute $\left(\frac{a}{N}\right)$ in time $\tilde{O}(\log N)$

Quantumly Factoring $N = P^2 Q$

Li, Peng, Du, Suter (2012)

- Simple observation:*

$$\left(\frac{a}{N}\right) = \left(\frac{a}{P}\right)^2 \left(\frac{a}{Q}\right) = \left(\frac{a}{Q}\right), \text{ which is periodic with period } Q!$$

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- Space (if naively implemented): also $\tilde{O}(\log N)$

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Kahanamoku-Meyer, R, Vaikuntanathan, Van Kirk (2024)

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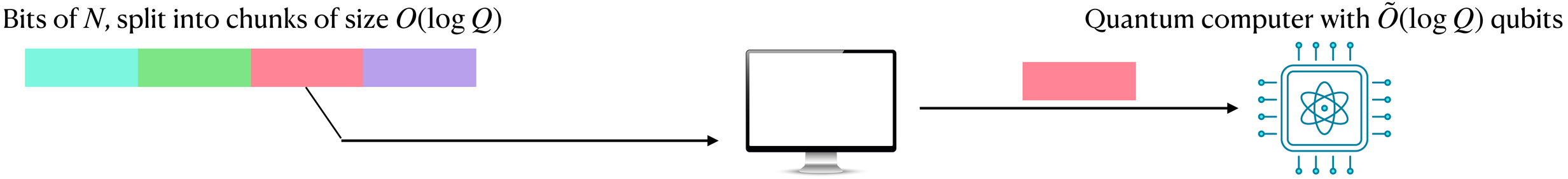
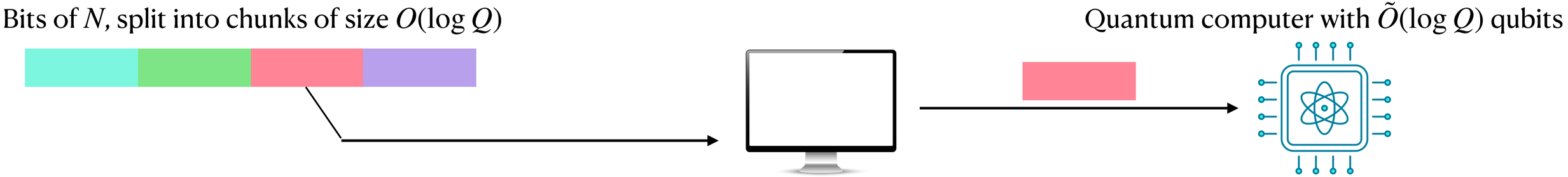


Pushing Space Down to $\tilde{O}(\log Q)$!

Kahanamoku-Meyer, R, Vaikuntanathan, Van Kirk (2024)

- A new way to compute $\left(\frac{a}{N}\right)$, exploiting two observations:
  - $\log a = O(\log Q)$ could potentially be much less than $O(\log N)$
  - N is classically known → could quantumly "stream" through bits of N to save space
- Resulting circuit: gates still $\tilde{O}(\log N)$ and space is now $\tilde{O}(\log Q)$!



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- Sweet spot: $\log Q = \tilde{O}((\log N)^{2/3}) \rightarrow$ space $\tilde{O}((\log N)^{2/3})$, gates $\tilde{O}(\log N)$

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More Open Questions

- Other quantum factoring algorithms exploiting special structure in N? (Many such algorithms in the classical world)
- Can these algorithms be boosted (e.g. using number theory magic) to factor generic integers N, or at least RSA integers $N = PQ$, more efficiently?
  - Current generalisation: can **completely** factor $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ for distinct $\alpha_1, \ldots, \alpha_r$