Two-Server Private Information Retrieval in Sublinear Time and Quasilinear Space

Alexandra Henzinger and Seyoon Ragavan

Thanks Alexandra for some of these slides!

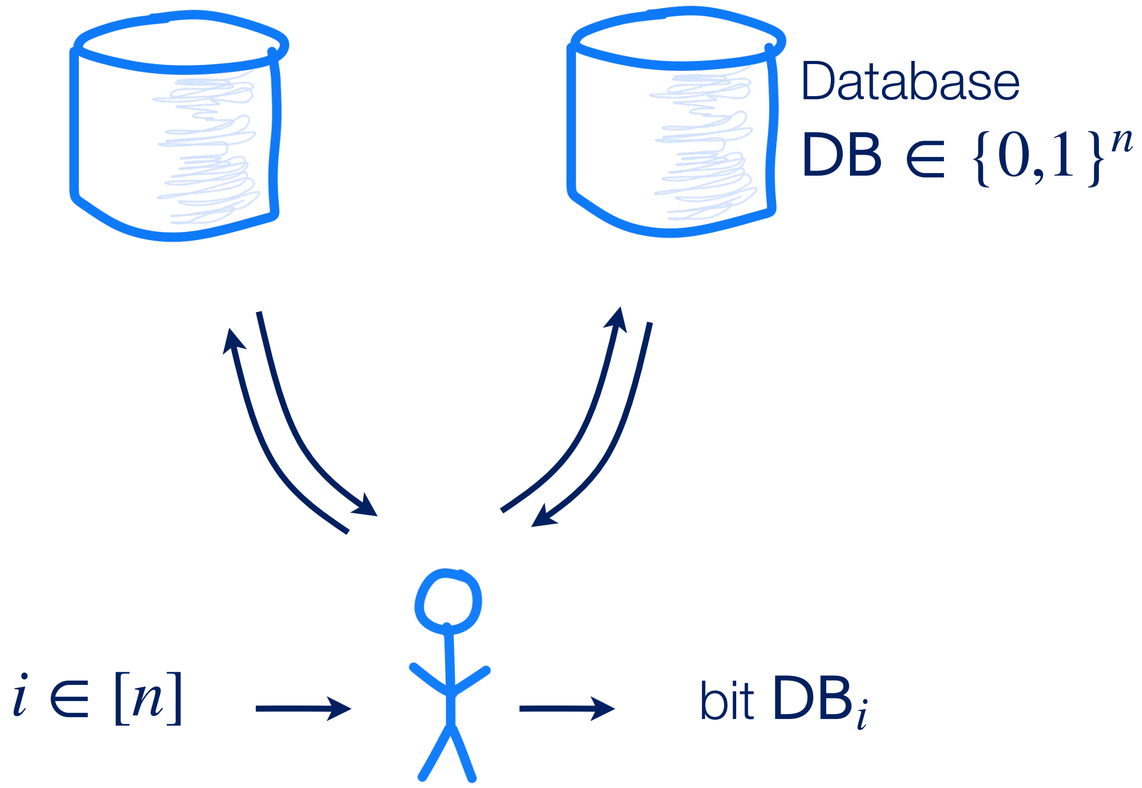
paper

code

ePrint: 2025/2008

Private Information Retrieval [CGKS95,KO97]

Goal: privately read an entry from a remote database



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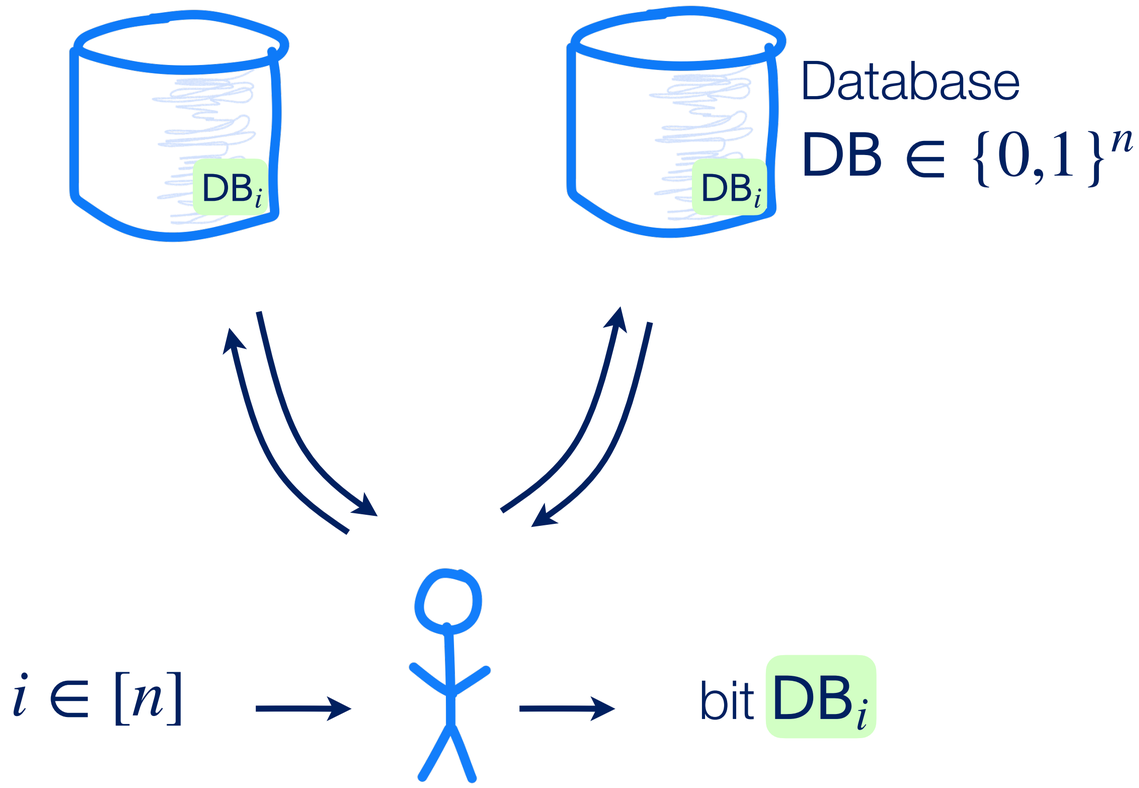

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for all DB $\in \{0,1\}^n$ and $i \in [n]$,
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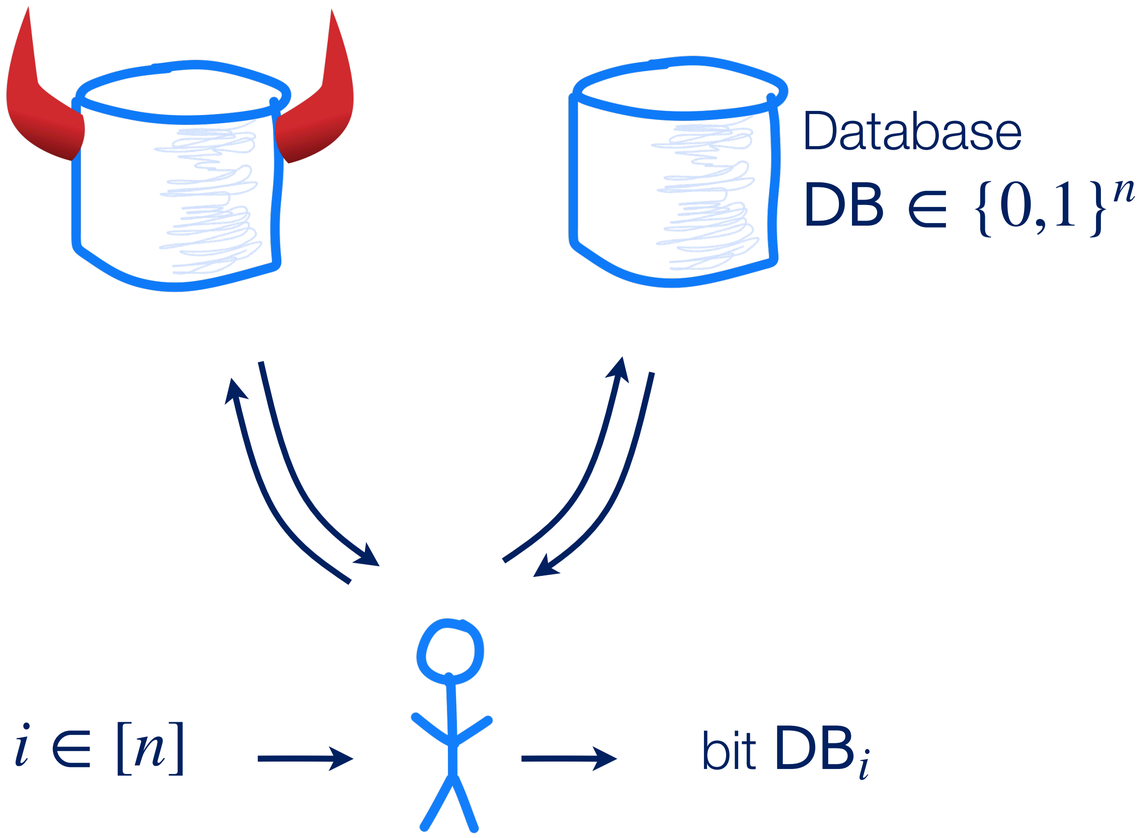

Correctness:
for all $\mathsf{DB} \in \{0,1\}^n$ and $i \in [n]$,
a user interacting with two honest servers learns DB_i.

Privacy:
an attacker compromising one server learns nothing about i,
even if malicious.

Formally: information-theoretic/computational indistinguishability of the server's view, on reads to any two indices i or i'.

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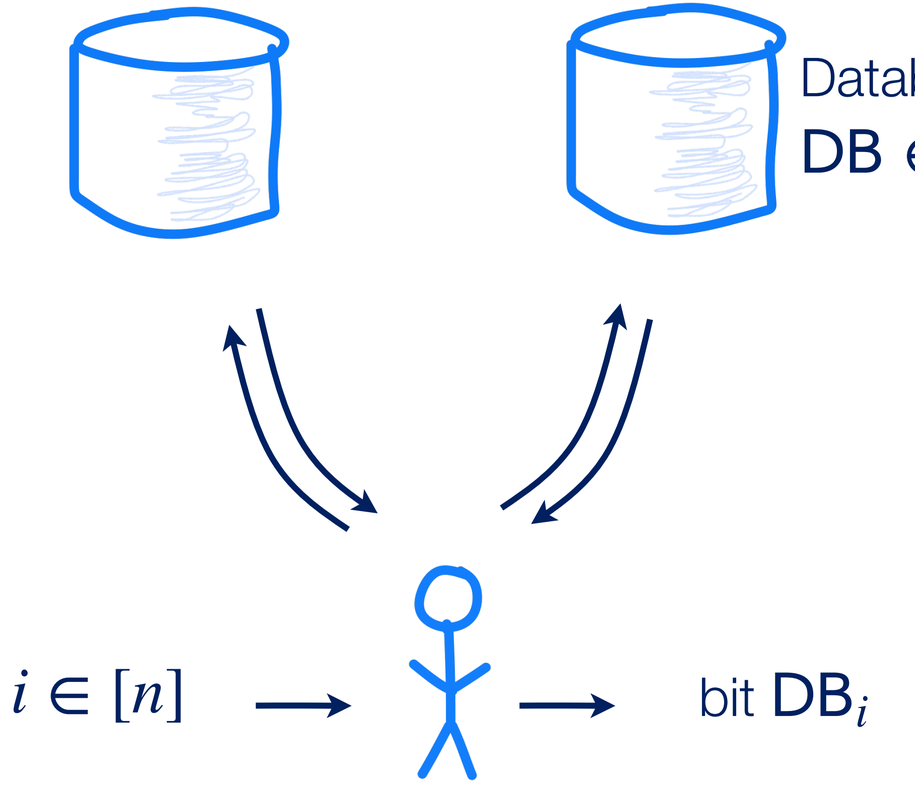

- Private web search [Coeus'21, Tiptoe'23, Wally'24]
- Private advertising [J'01,ObliviAd'12,AdVeil'20…]
- Private media delivery [Popcorn'16,PIRSONA'21,CrowdSurf'25,…]
- Metadata-hiding messaging [PIRTor'11,Pung'16,Talek'20,Addra'21,…]
- Private auditing and blocklists [Checklist'20,SimplePIR'23,HPW'25,…]
- Private file storage [Dory'20,Peer2PIR'25,…]
- **On your iPhone:** Private caller-ID

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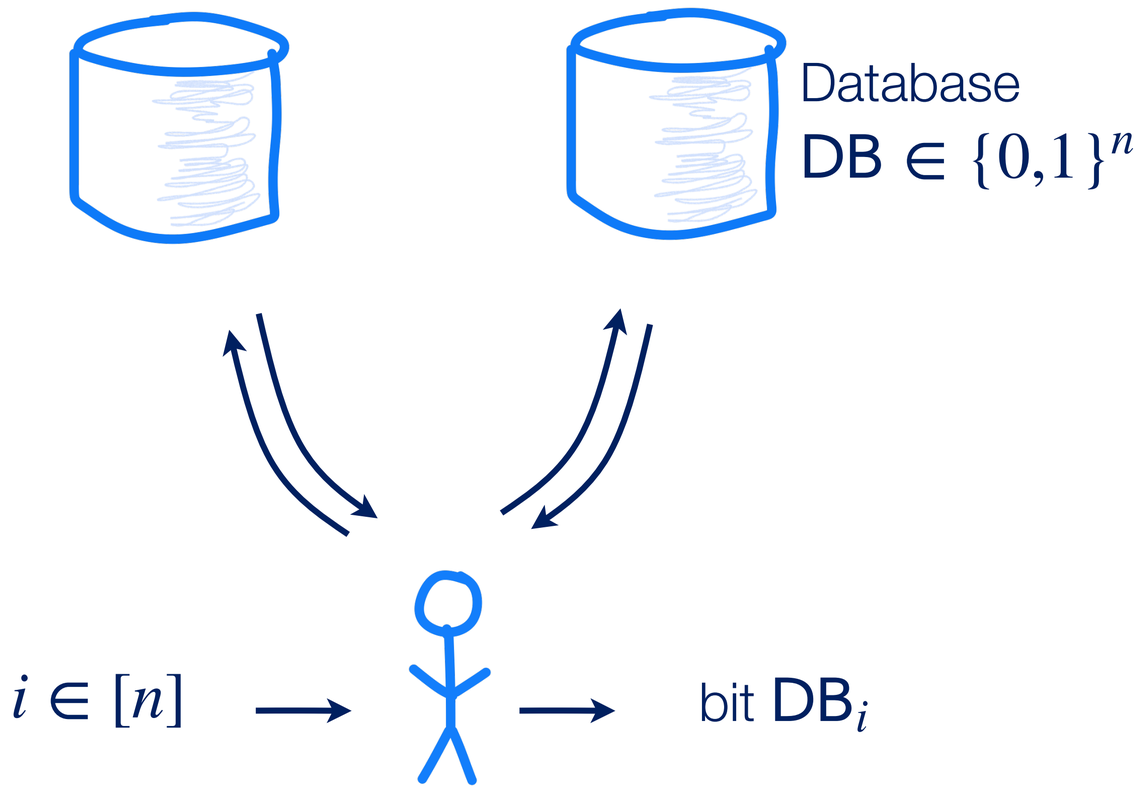

We have PIR constructions with very little communication:

- No privacy: $\log n + 1$
- Info-theoretic privacy: $n^{o(1)}$
- Comp. privacy: $O(\lambda \cdot \log n)$

[KO97,CMS99,DG16,BGI16]

…but these require lots of server work:

- No privacy: $O(1)$ time
- With privacy: $\Omega(n)$ time

[BIM00, PY22]

Solution: PIR with Preprocessing [BIM00]

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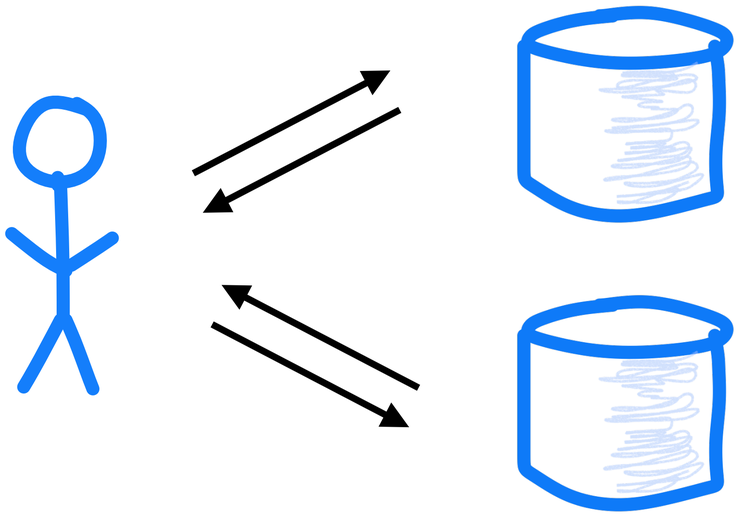

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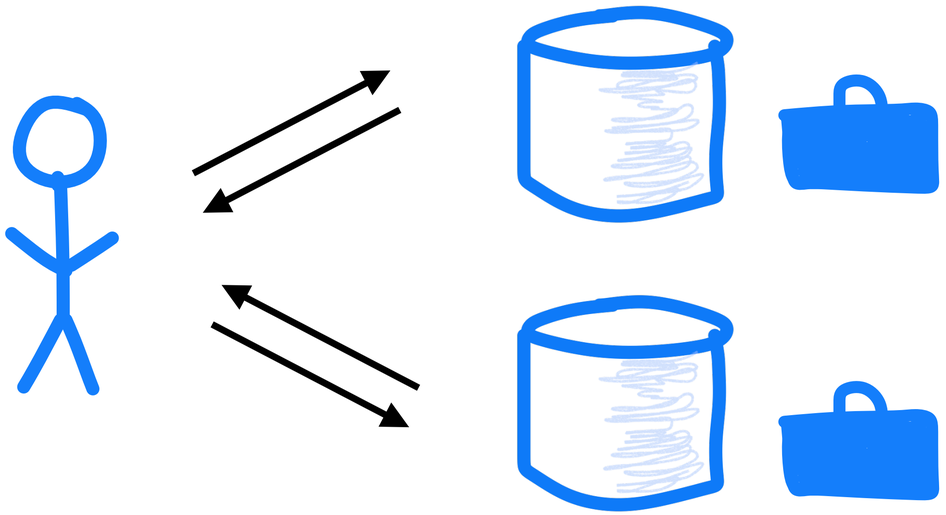

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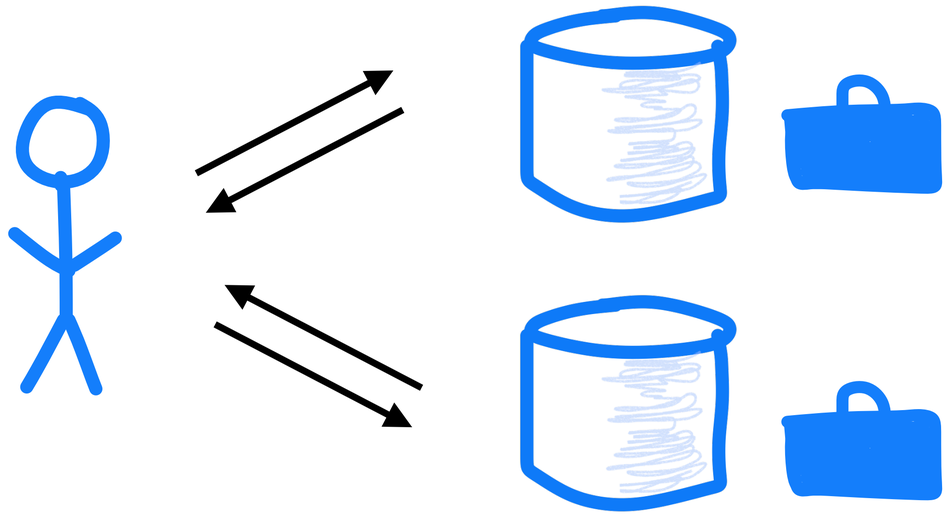

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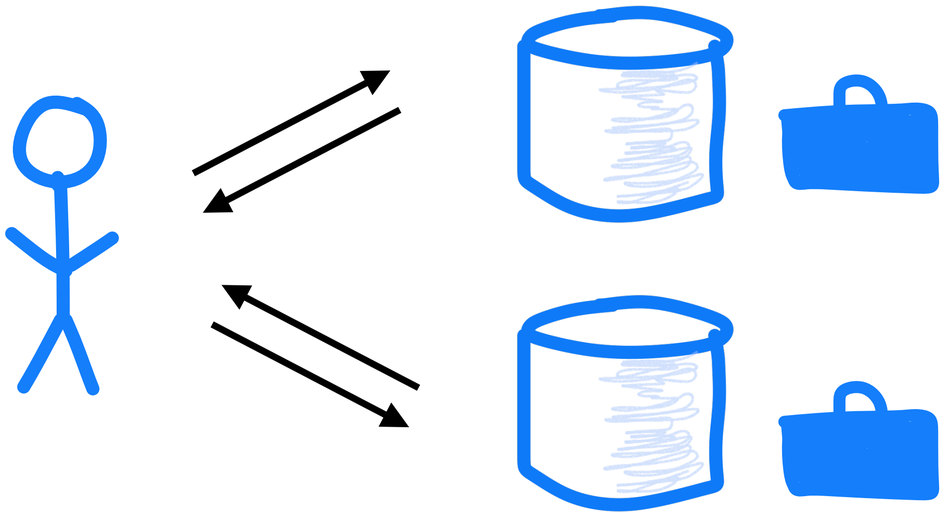

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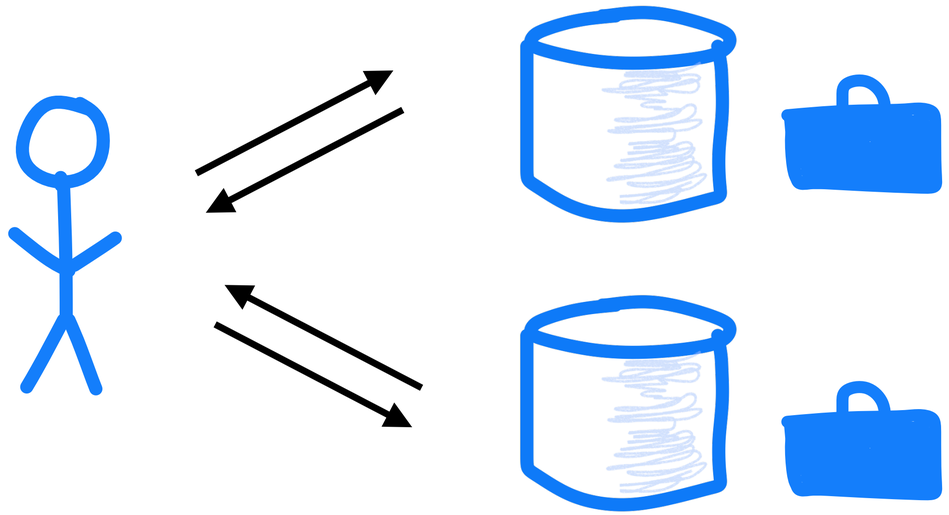

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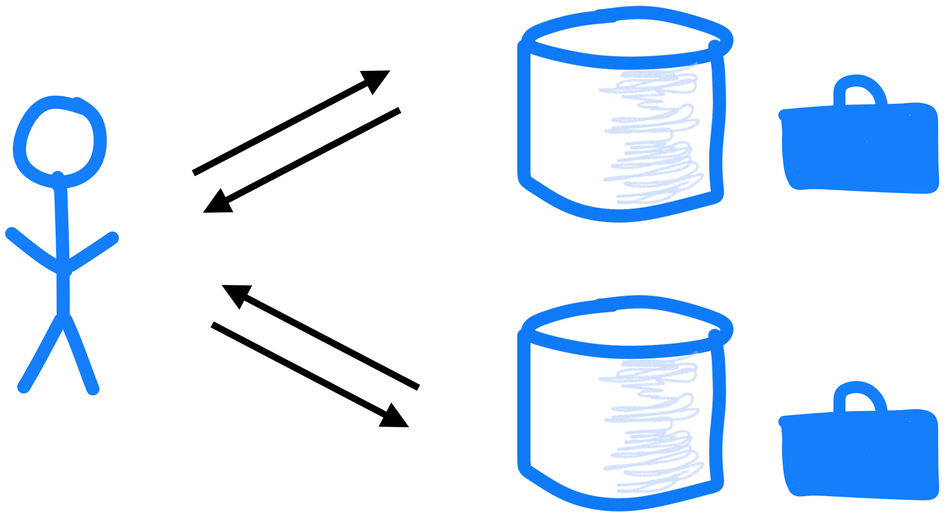

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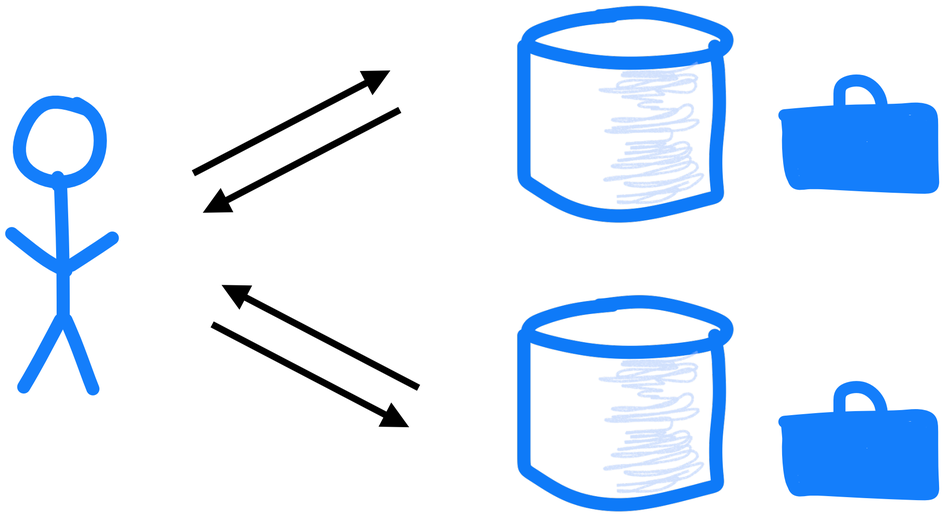

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Disclaimer: most constructions (including ours) sacrifice communication efficiency for sublinear server time

25 years of work on PIR with preprocessing

Construction	Setting	Time per query	Storage
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Our work	2 servers Info-theoretic	$n^{0.82}$	$n^{1+o(1)}$	1 TB (one $10 hard drive)

Theorem. For any database of $n > 10^6$ bits, there exists information-theoretic, two-server PIR with preprocessing with:

- $1.5 \cdot \sqrt{\log_2 n} \cdot n$ bits of storage,
- $12 \cdot n^{0.82}$ server RAM lookups per query,
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First info-theoretic PIR with

1. constant number of servers,
2. quasilinear storage, and
3. polynomially-sublinear time.

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Previous Work [BIM00, GLM+25]: PIR with Preprocessing from Polynomials and Derivatives

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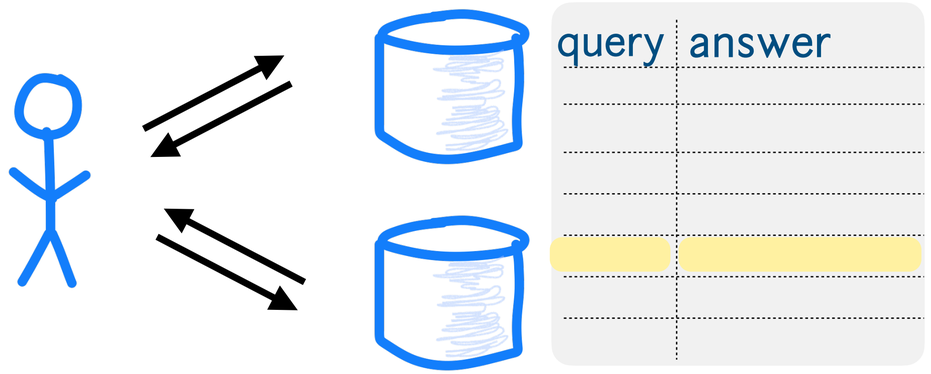

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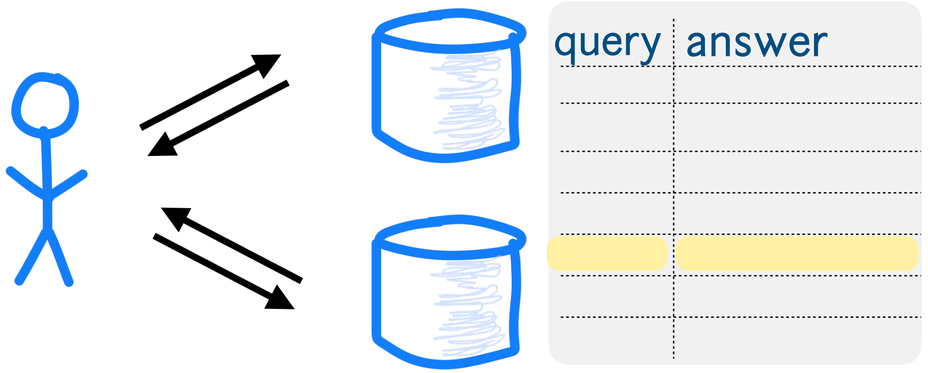

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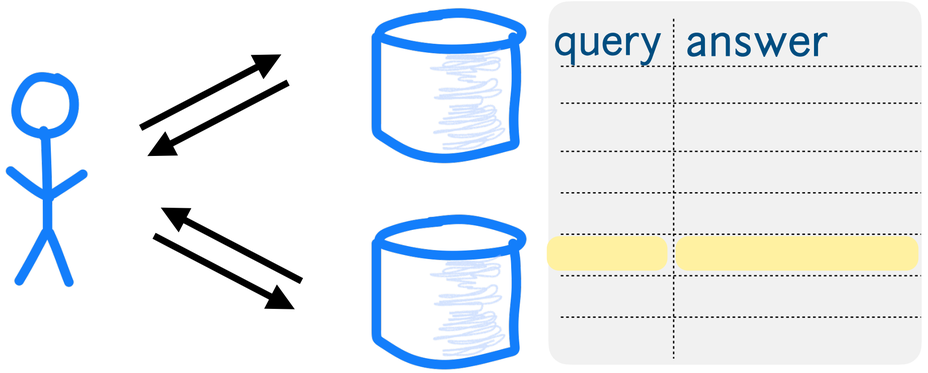

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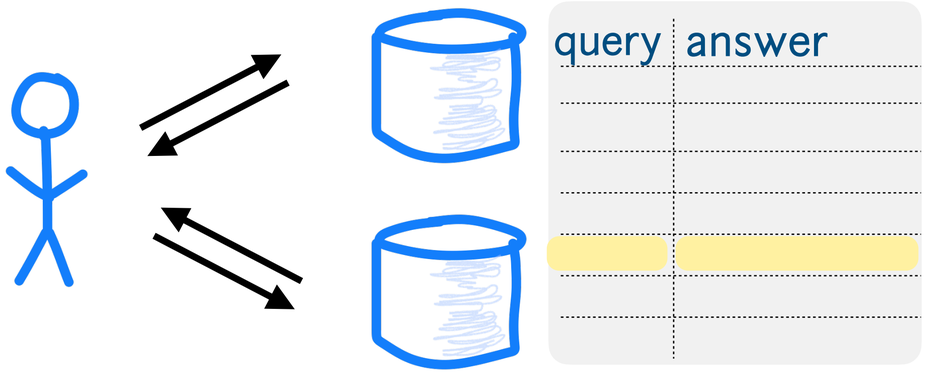

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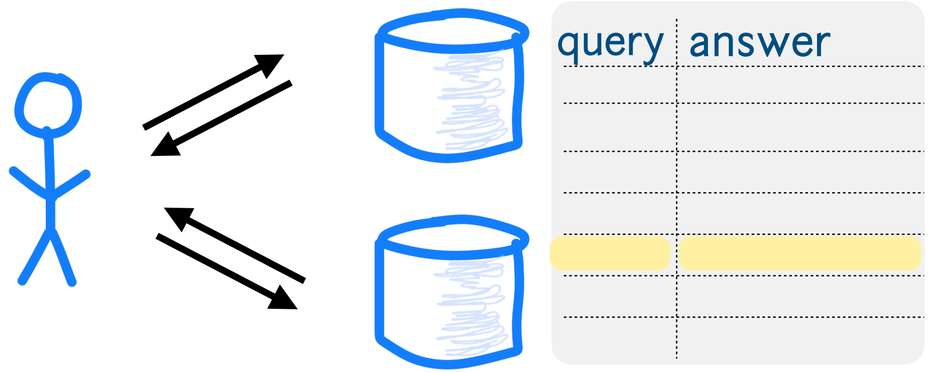
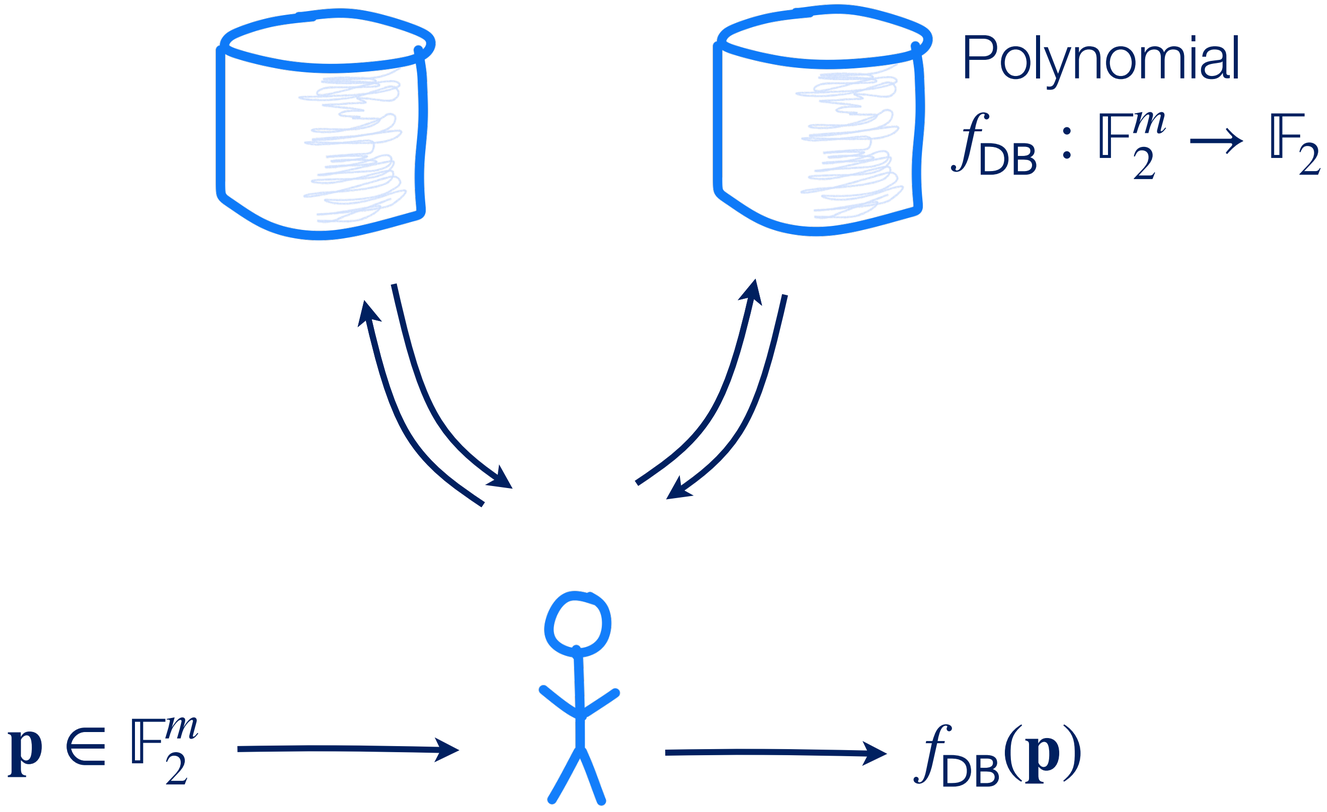


Moral: we should look for PIR protocols with query length $\ell_q \leq O(\log n)$

Spoiler: our construction will deviate from this brute force paradigm!
(but we will still need query length $\ell_q \leq O(\log n)$)

Abstraction: PIR from private polynomial evaluation

A common step in [BIM00,BIKR02,BIK05,WY05,BV11…]



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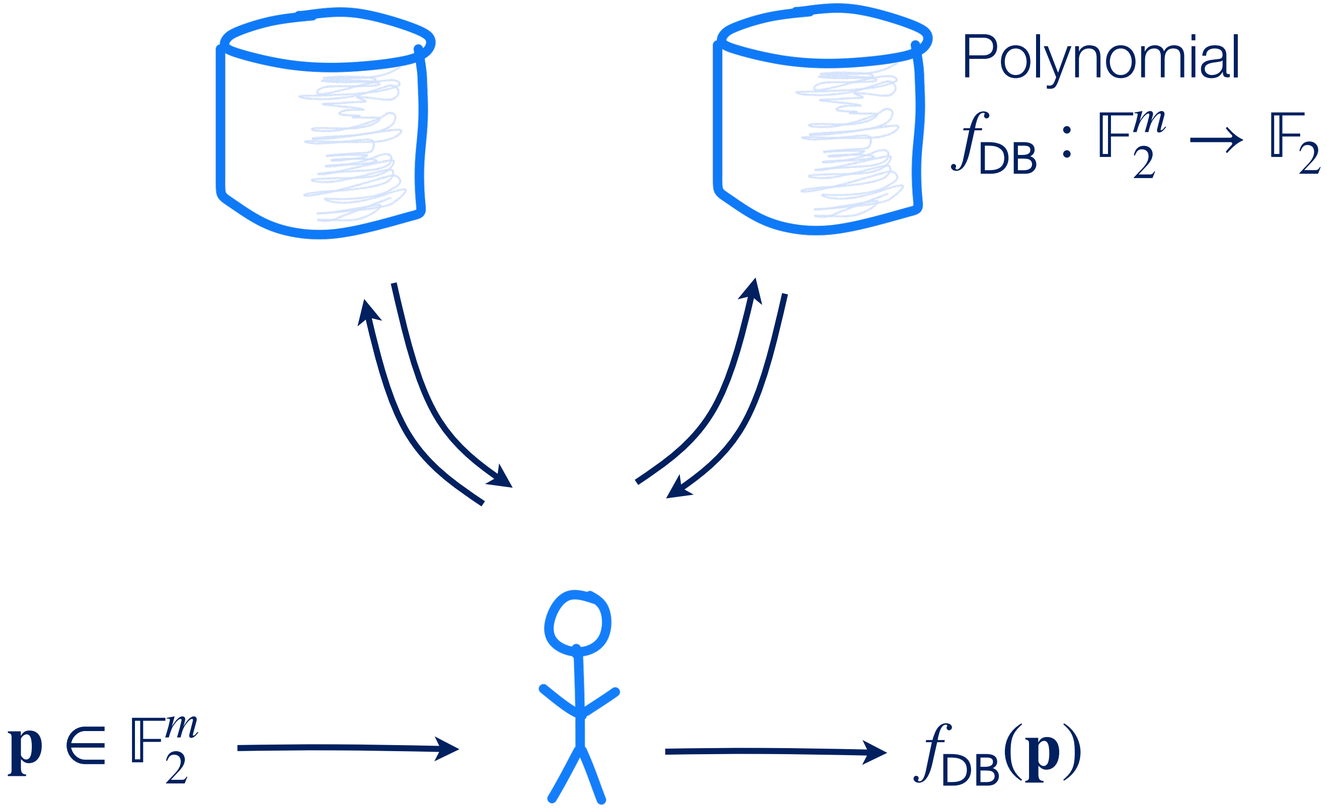

We need:

1. f_{DB} is homogenous and degree-D
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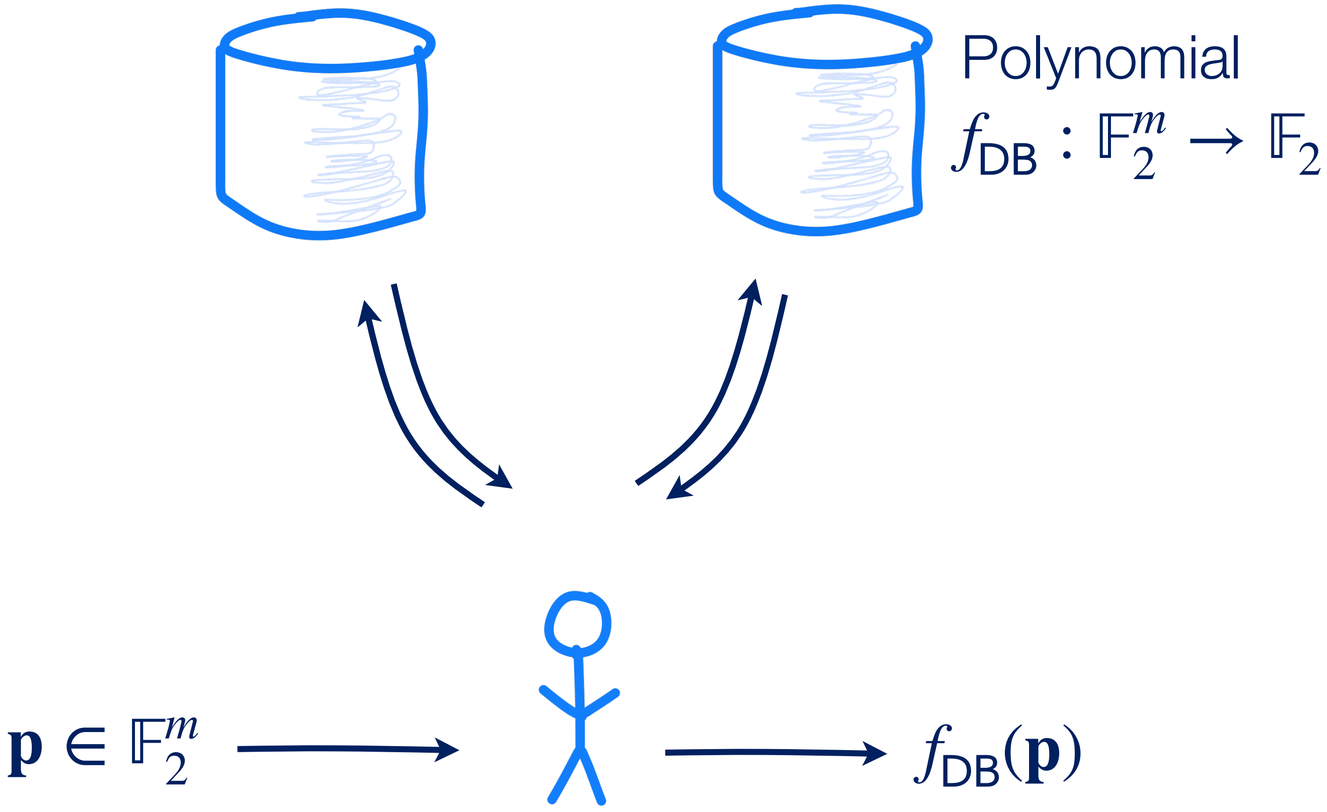

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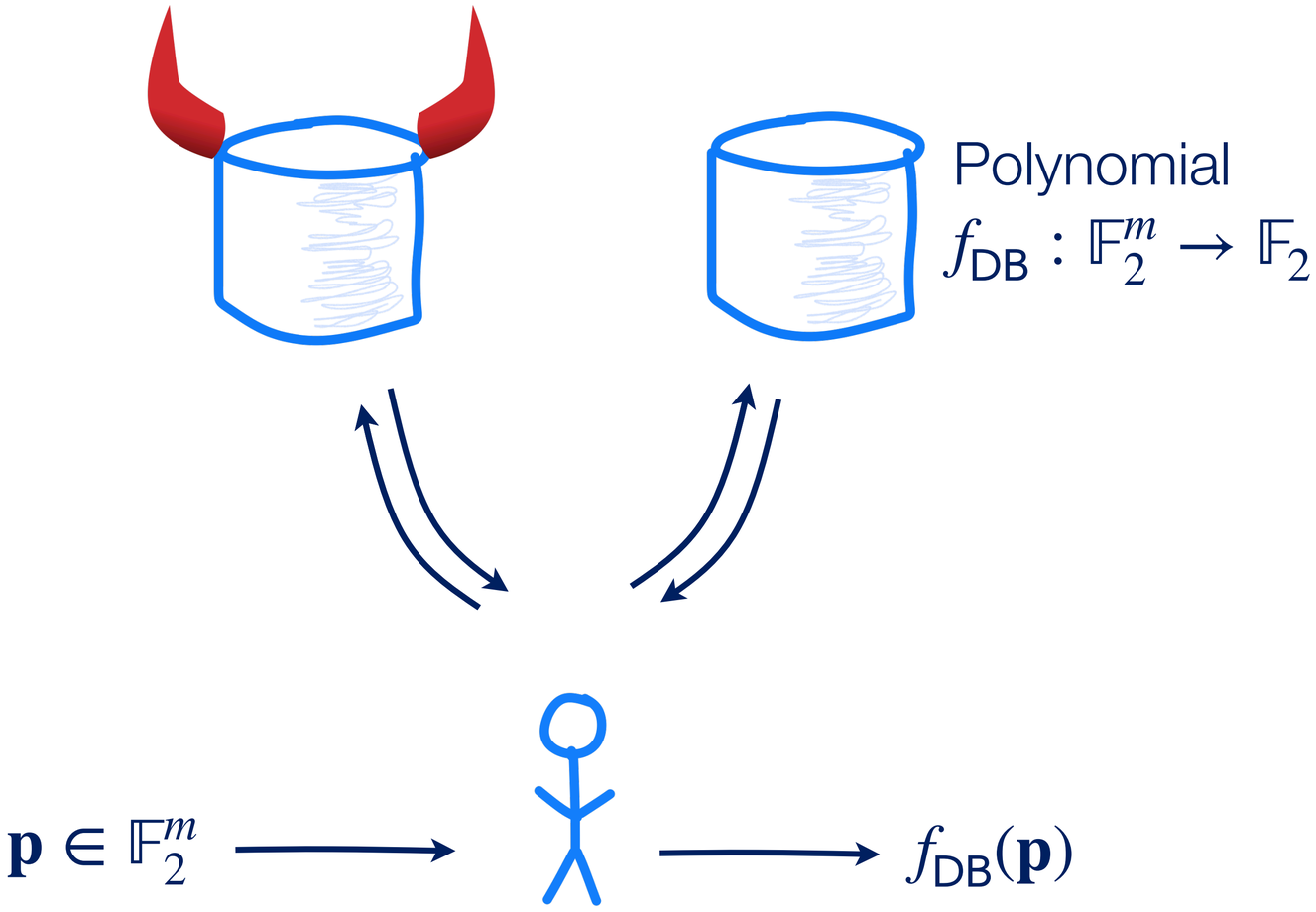

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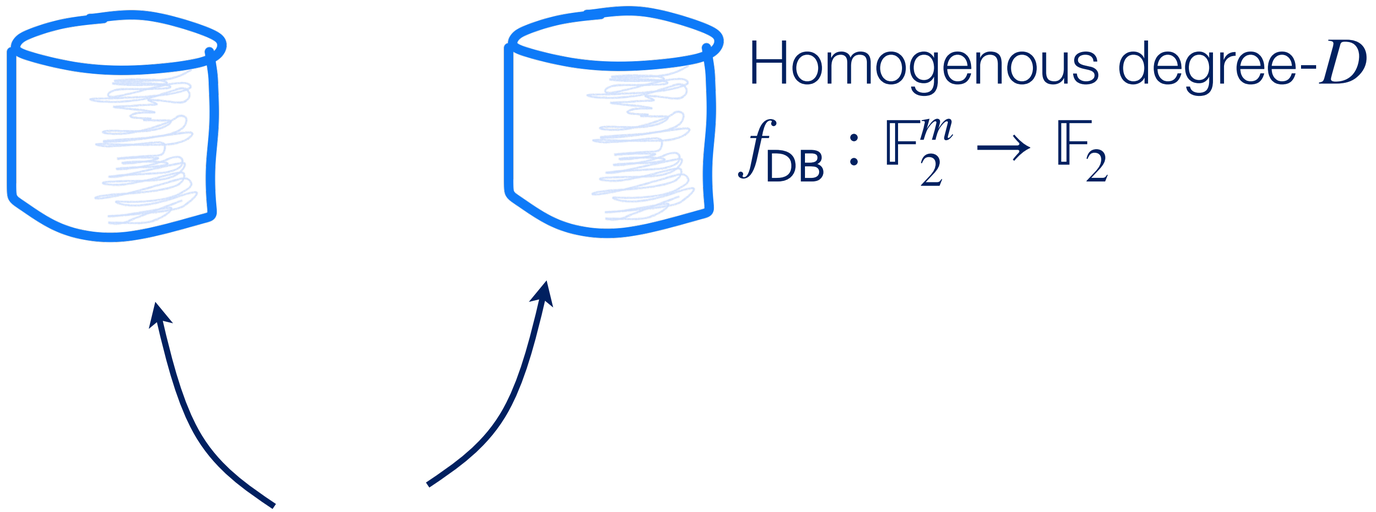
PIR from derivatives [WY05]



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$f_{\mathsf{DB}} : \mathbb{F}_2^m \to \mathbb{F}_2$

Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line
$\mathbf{L}(t) = \mathbf{r} + t \cdot \mathbf{p}$

Fact: *leading coefficient of* $f_{\mathsf{DB}}(\mathbf{L}(t))$ *is* $f_{\mathsf{DB}}(\mathbf{p})$

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- ➡ $D = 3$
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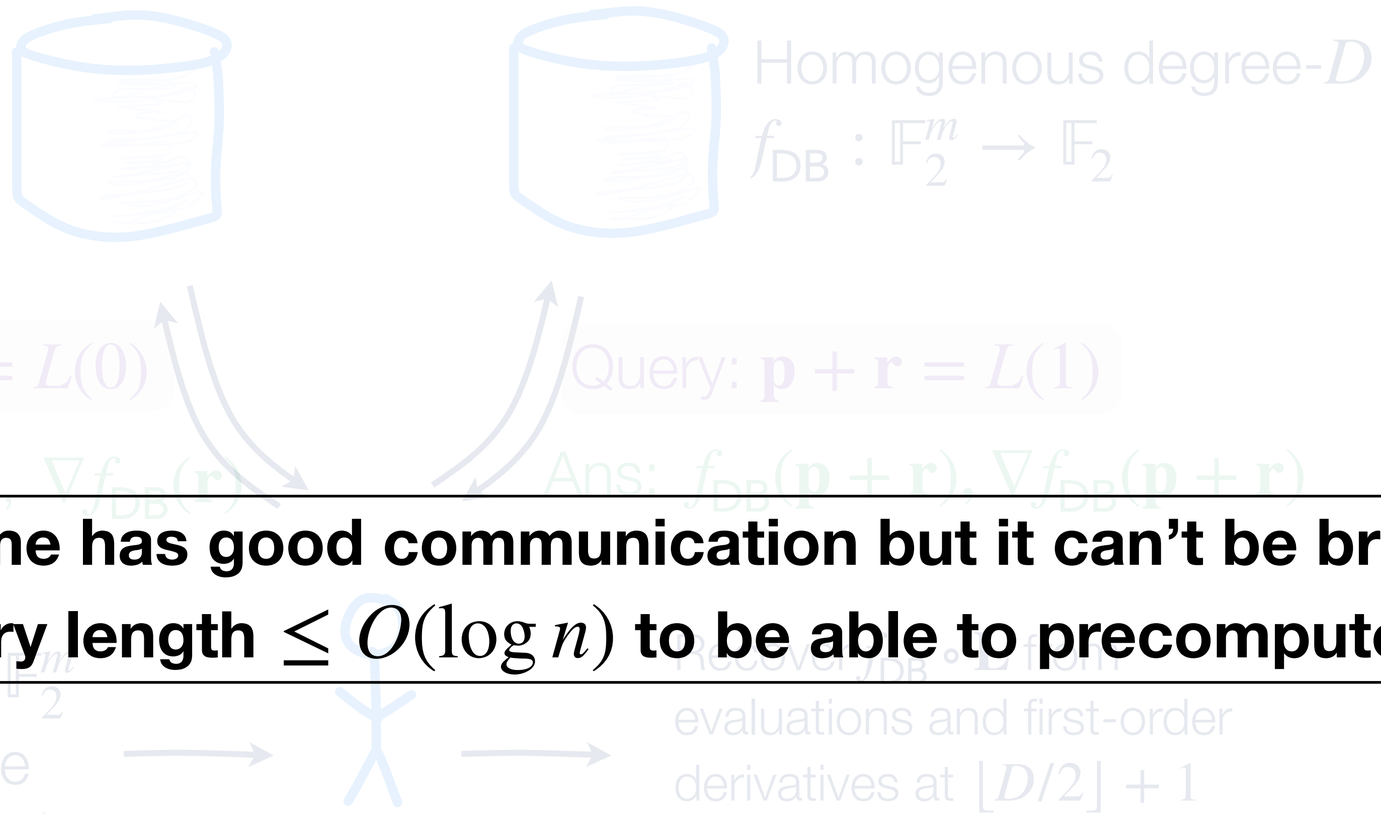

This scheme has good communication but it can't be brute-force preprocessed!

Need query length $\leq O(\log n)$ to be able to precompute answers to all queries

Fix: even more derivatives! [BIM00,GLM+25]

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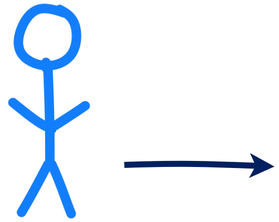

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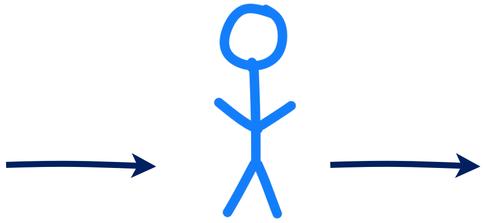


Recover $f_{\mathsf{DB}} \circ \mathbf{L}$ from evaluations and $\leq \lfloor D/2 \rfloor$-order derivatives at 2 points via Hermite interpolation

With 2 servers, gives "imbalanced" PIR with

- ➡ $m = (1 + o(1)) \cdot \log n$
- ➡ $D = m/2$
- ➡ query $m \approx \log n$ and answer $\binom{m}{\lfloor D/2 \rfloor} \approx n^{0.82}$ (partial derivatives to order $\leq \lfloor D/2 \rfloor$)

Brute Force → PIR with Preprocessing

Precomputing answers for every query

Homogenous deg-D

$f_{\mathsf{DB}} : \mathbb{F}_2^m \to \mathbb{F}_2$

Query: $\mathbf{r}$

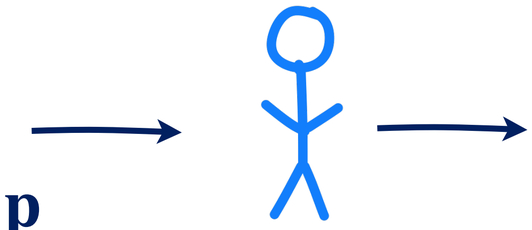
Ans: $f_{\mathsf{DB}}(\mathbf{r}),\ \nabla f_{\mathsf{DB}}(\mathbf{r}),\ \ldots,\ \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{r})$

Query: $\mathbf{p} + \mathbf{r}$

Ans: $f_{\mathsf{DB}}(\mathbf{p}+\mathbf{r}),\ \nabla f_{\mathsf{DB}}(\mathbf{p}+\mathbf{r}),\ \ldots,\ \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{p}+\mathbf{r})$

Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line $\mathbf{L}(t) = \mathbf{r} + t \cdot \mathbf{p}$



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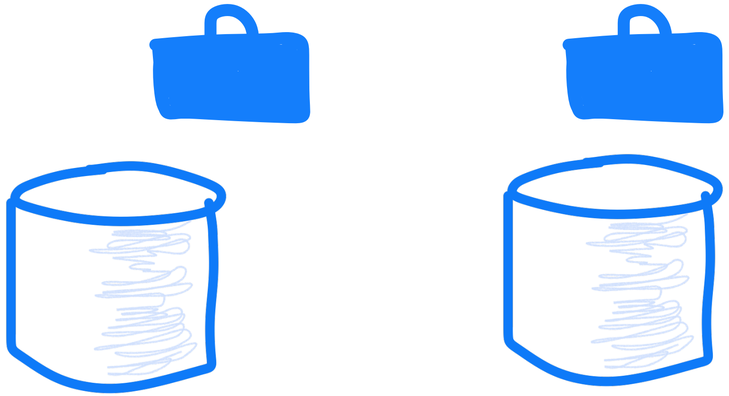

$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\mathsf{DB}}(\mathbf{2}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathsf{DB}}(\mathbf{2})$
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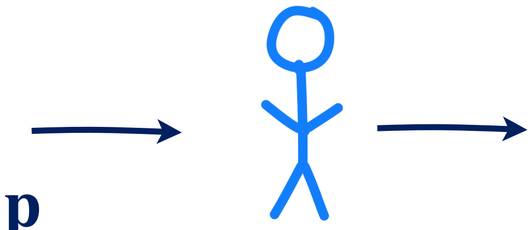

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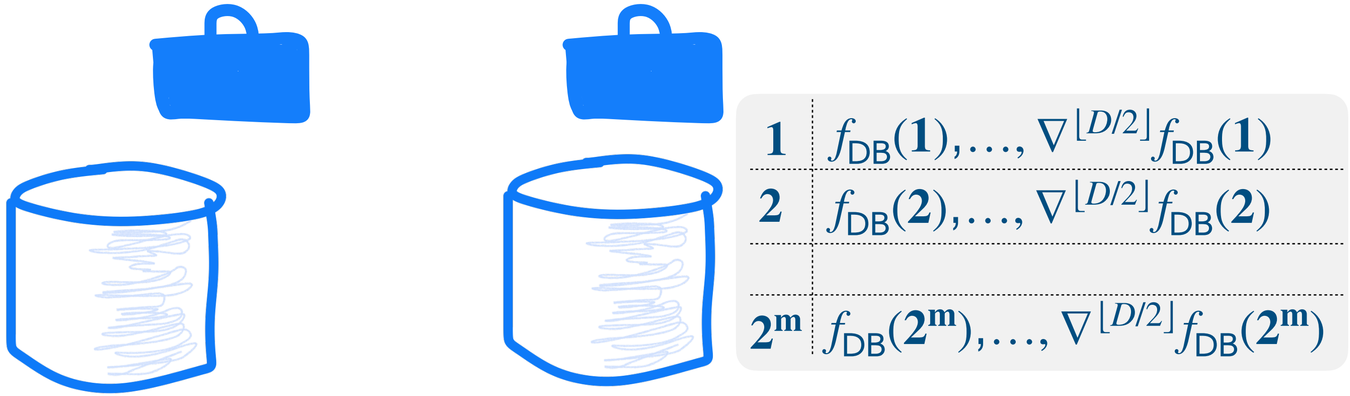

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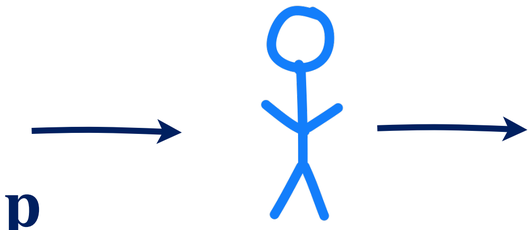

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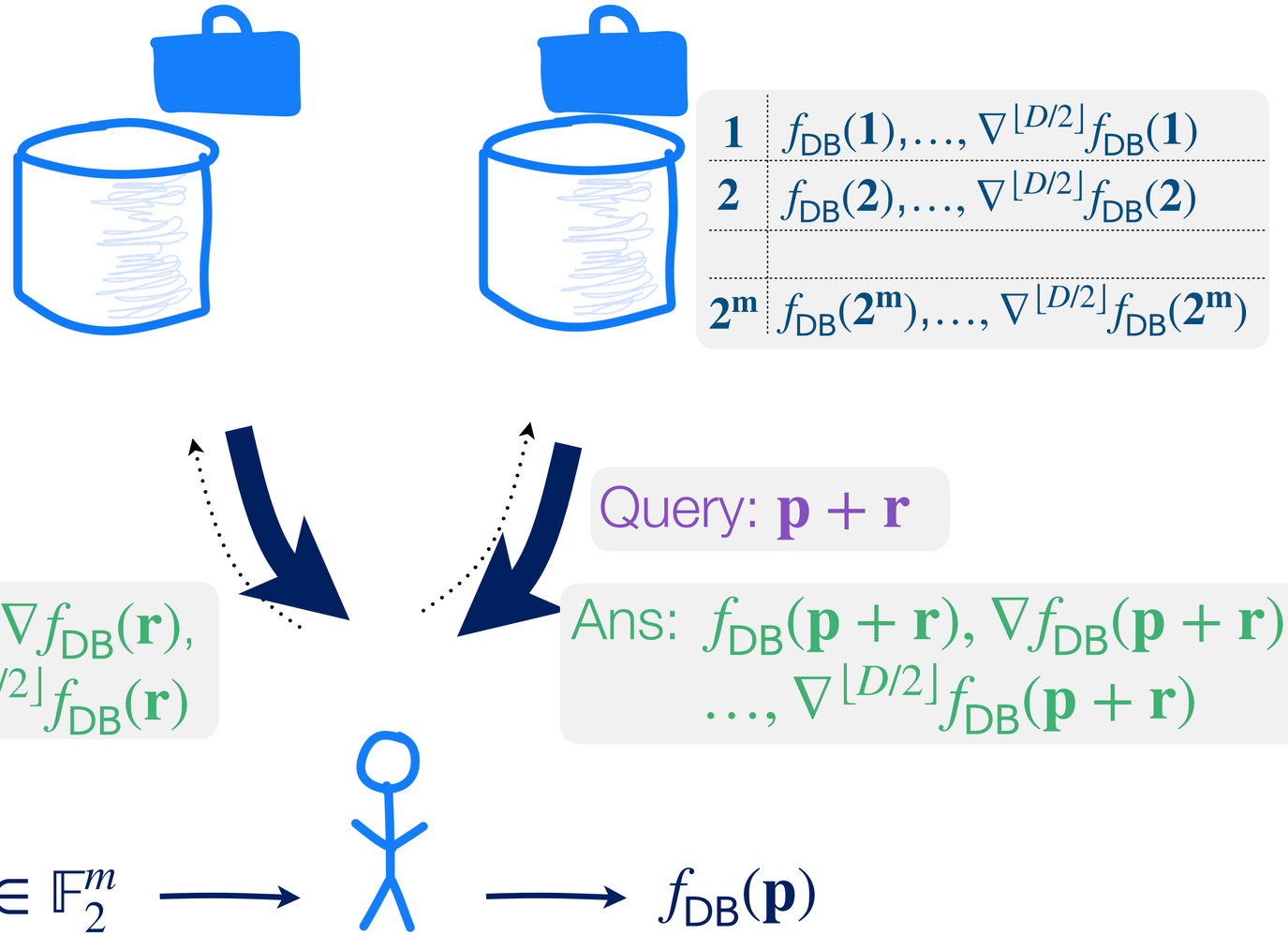
With 2 servers, gives preprocessing PIR with

- ➡ Query $O(\log n)$ and answer $n^{0.82}$
- ➡ $2^m \cdot n^{0.82} = n^{1.82+o(1)}$ server storage
- ➡ $n^{0.82}$ server time

Our Work: Preprocessing Better than Brute Force

Finding Redundancy in the Brute-Force Data Structure



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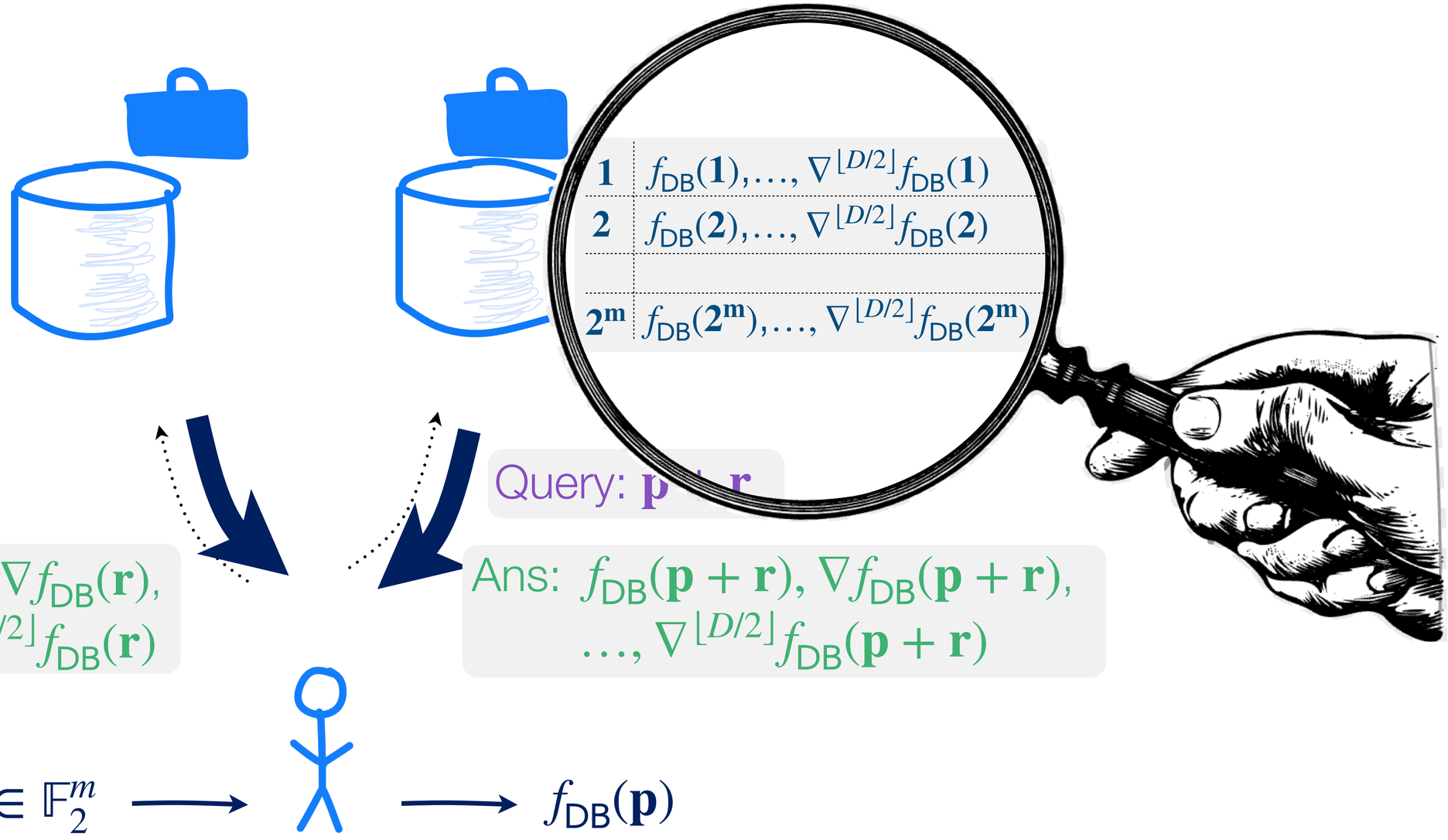

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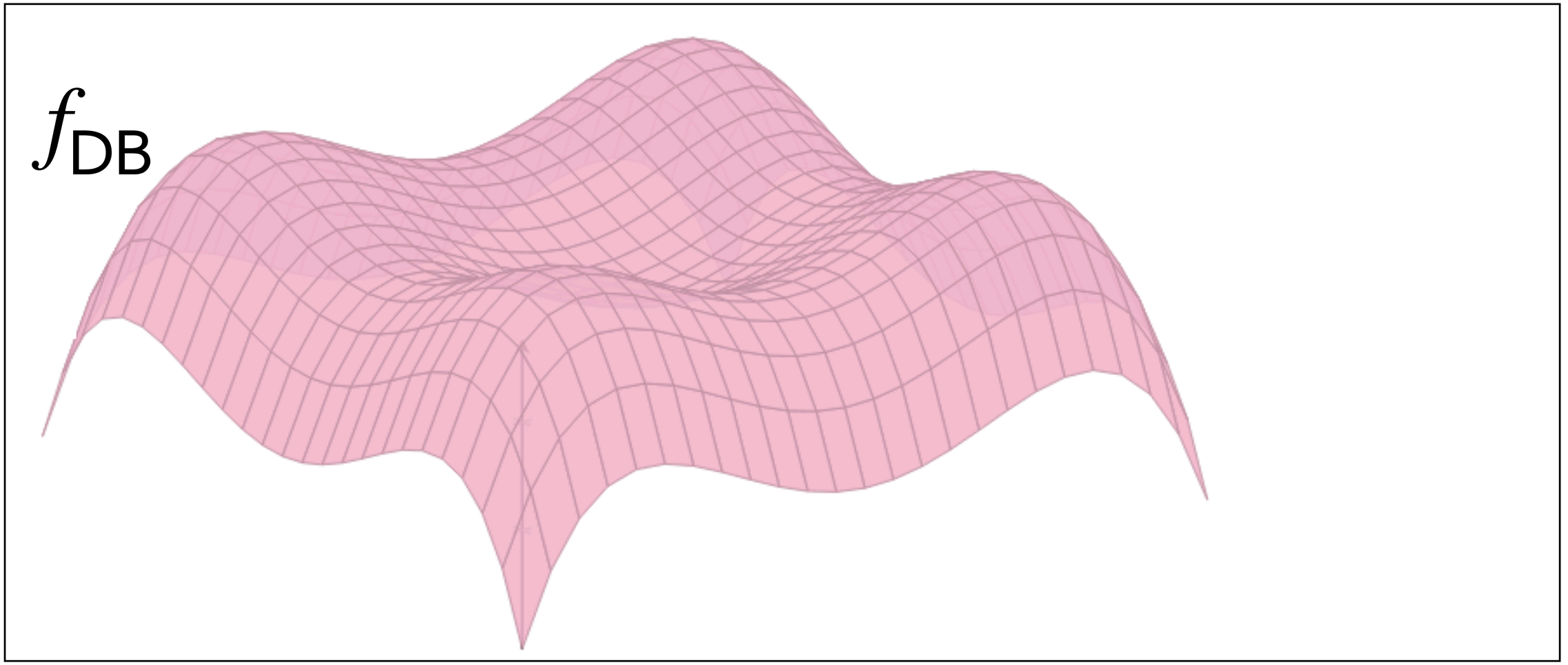

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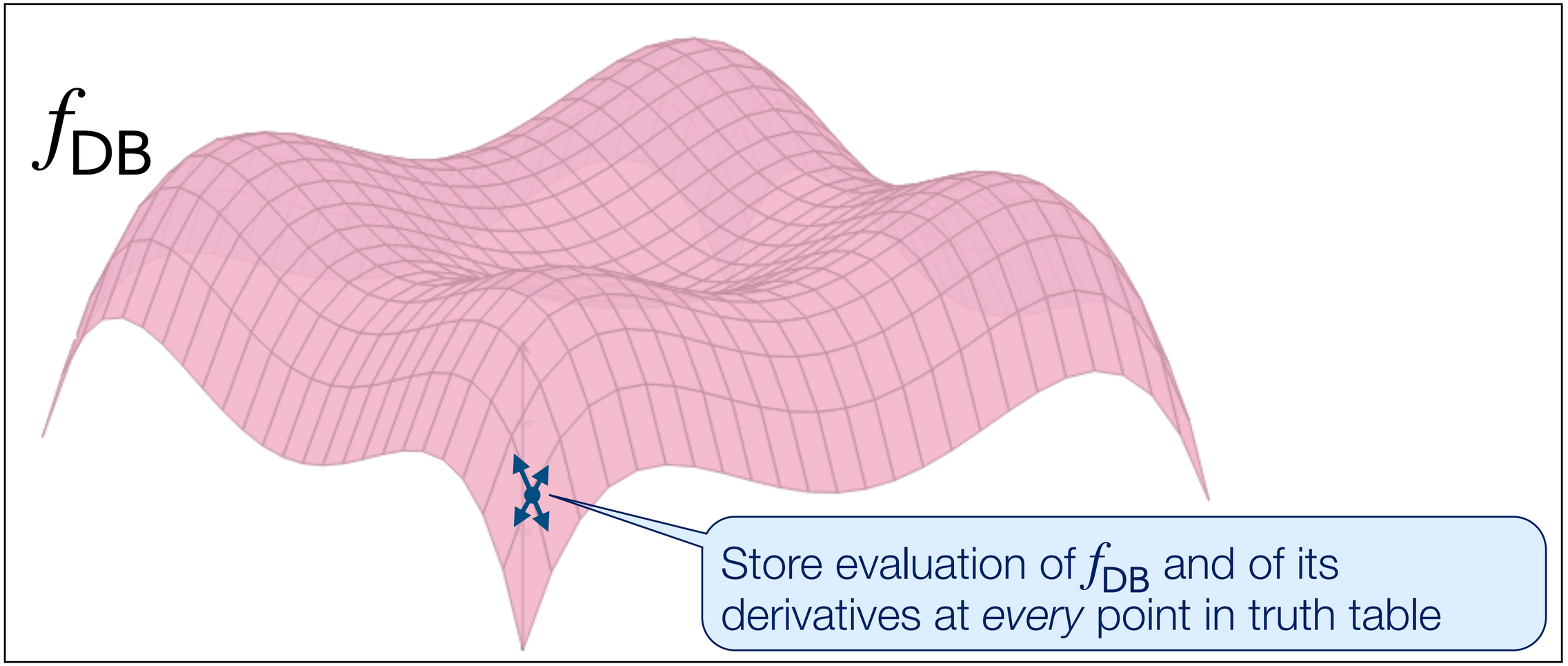

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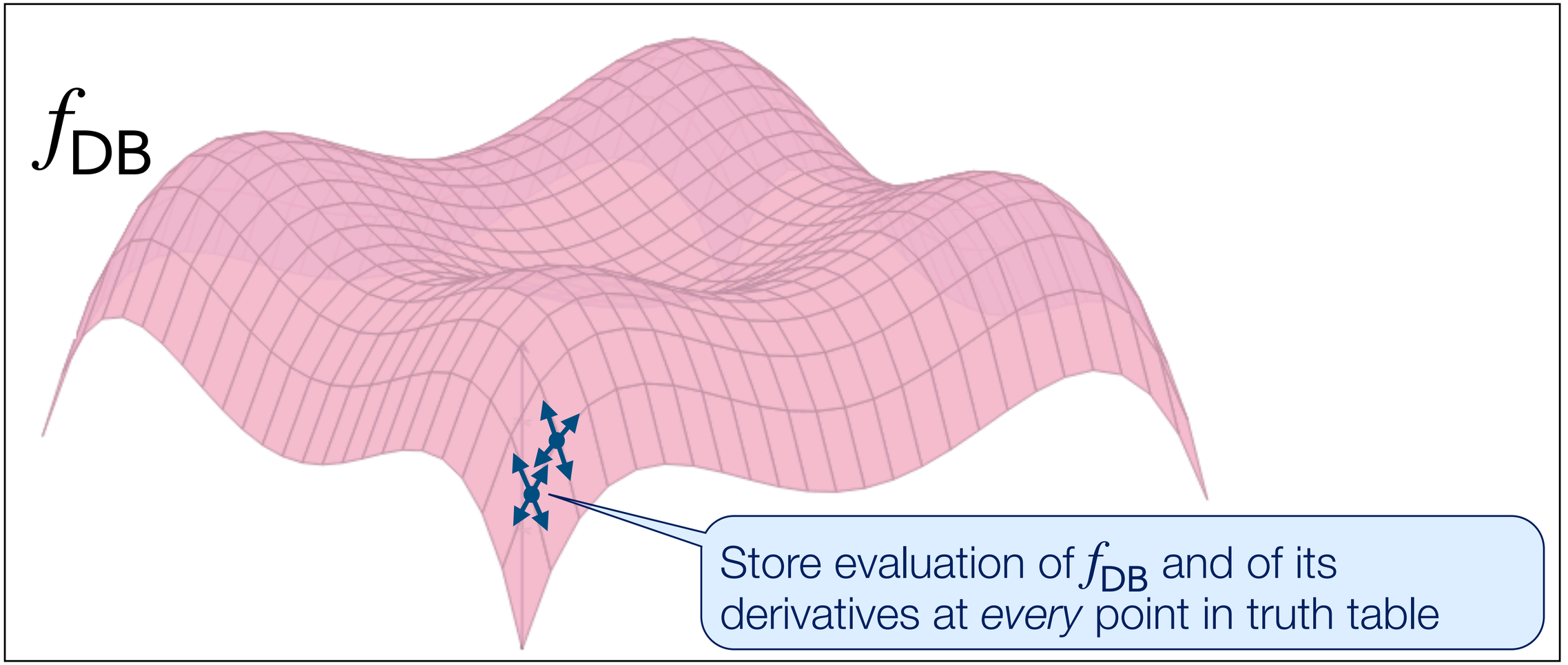

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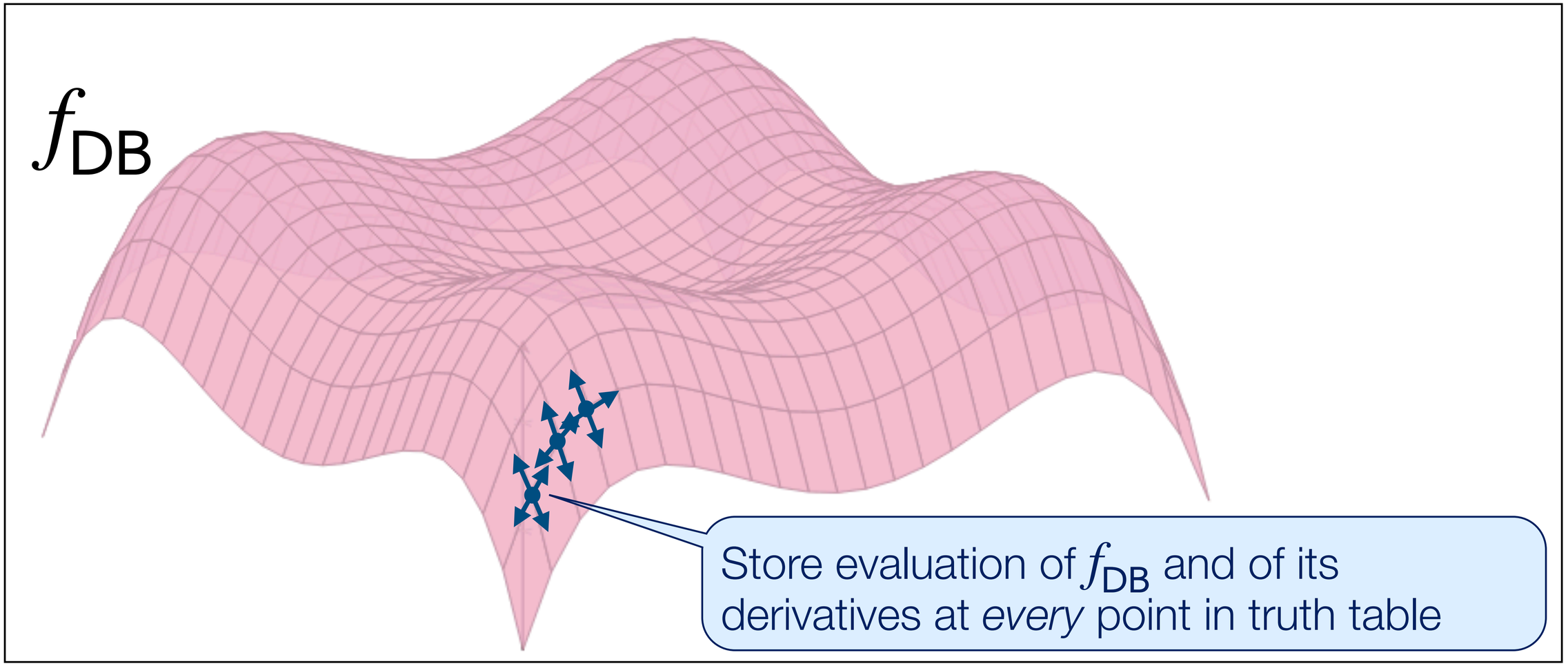

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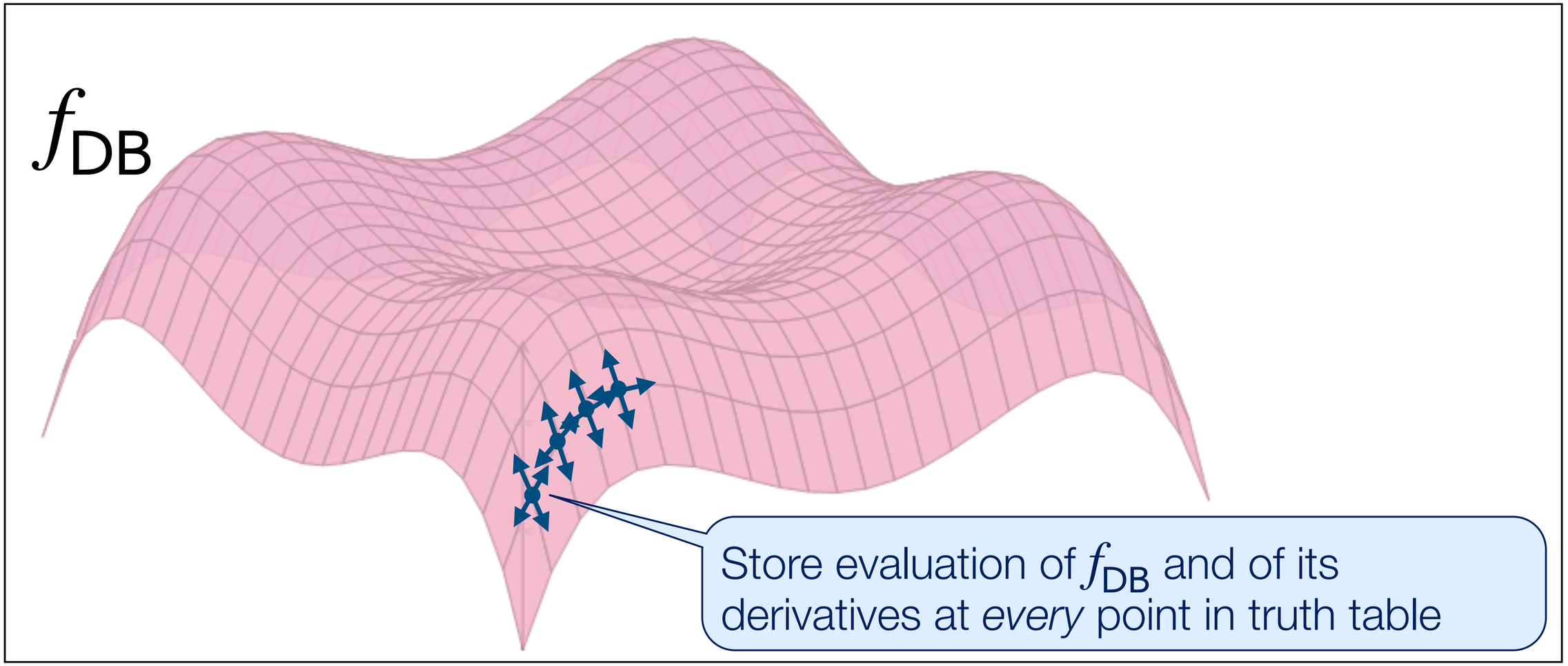

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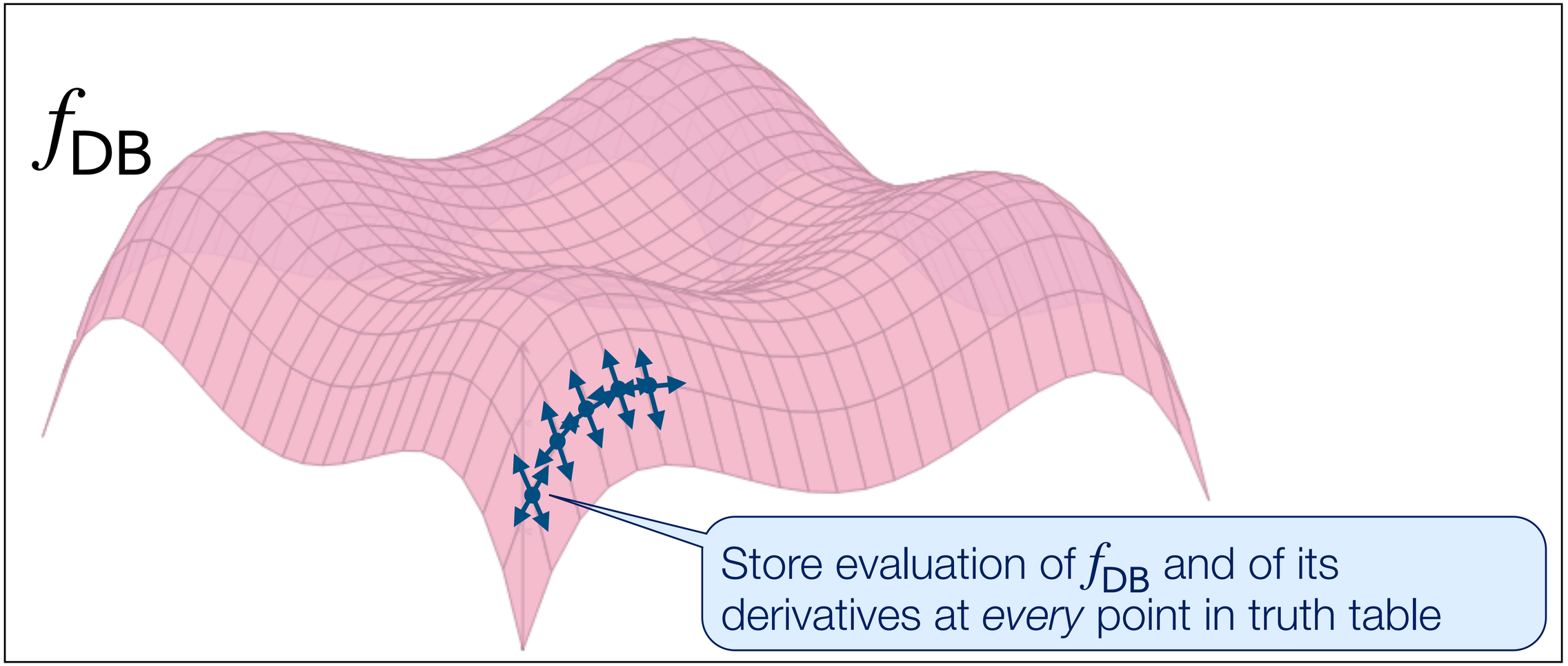

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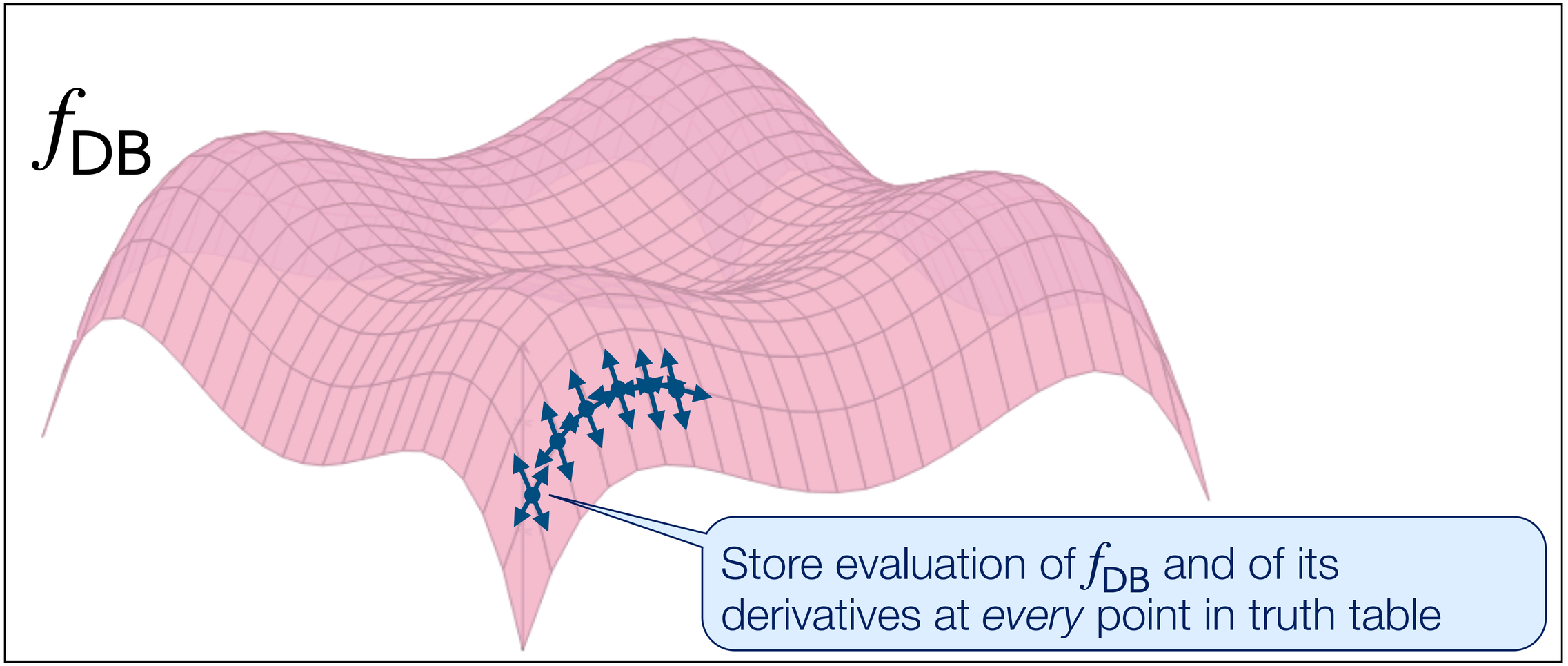

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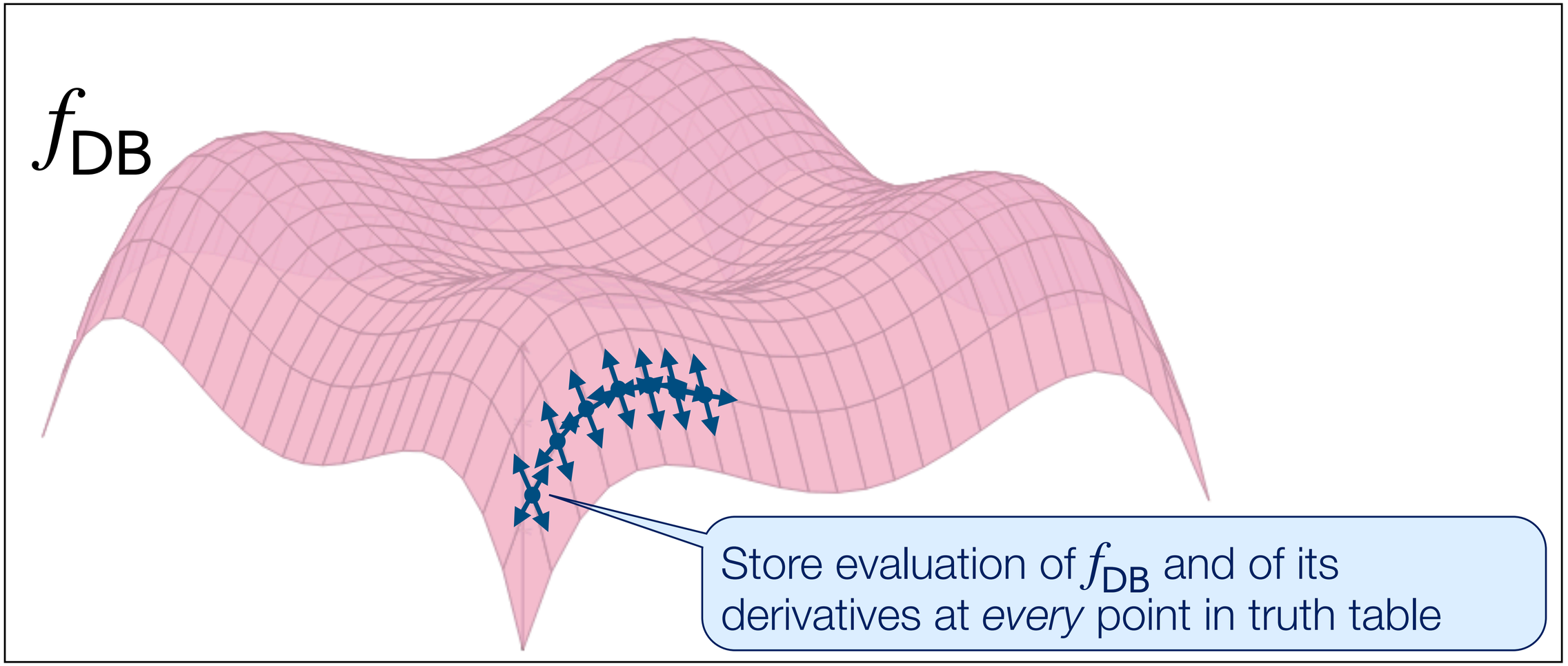

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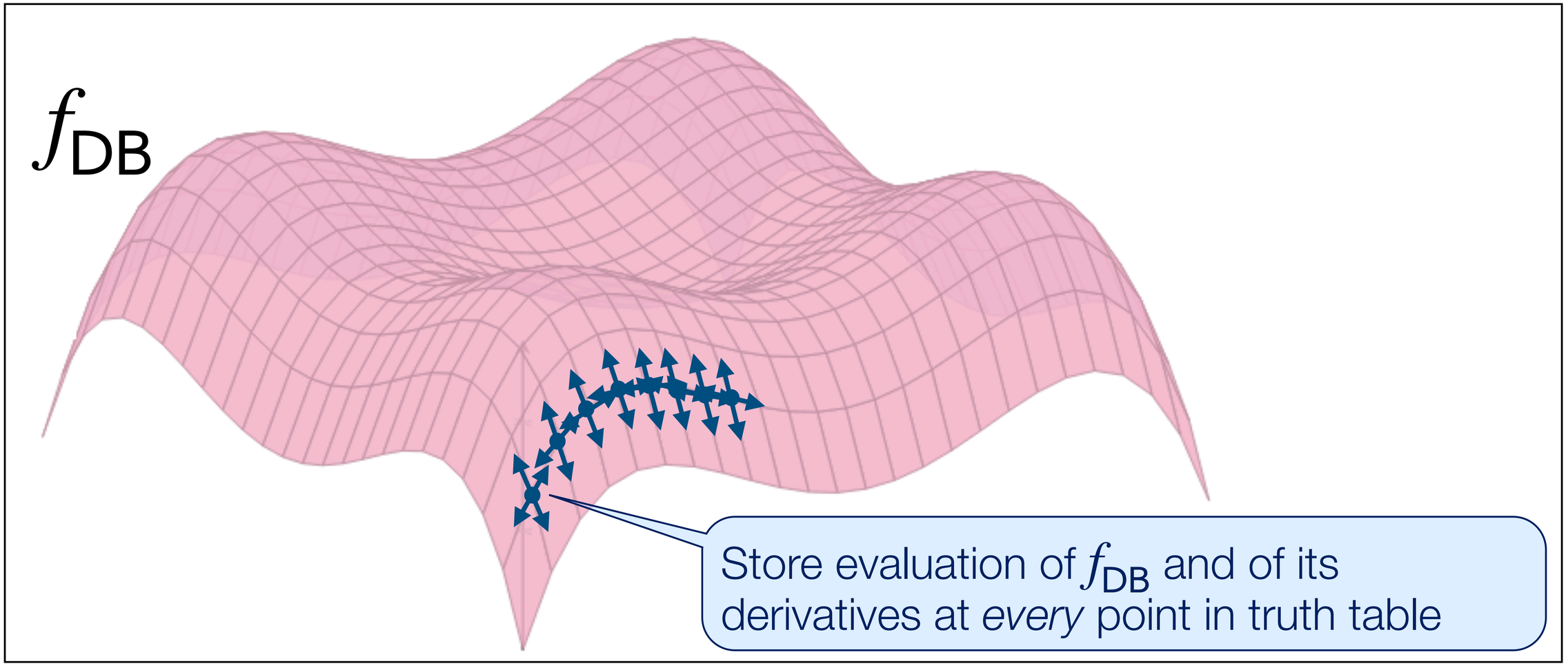

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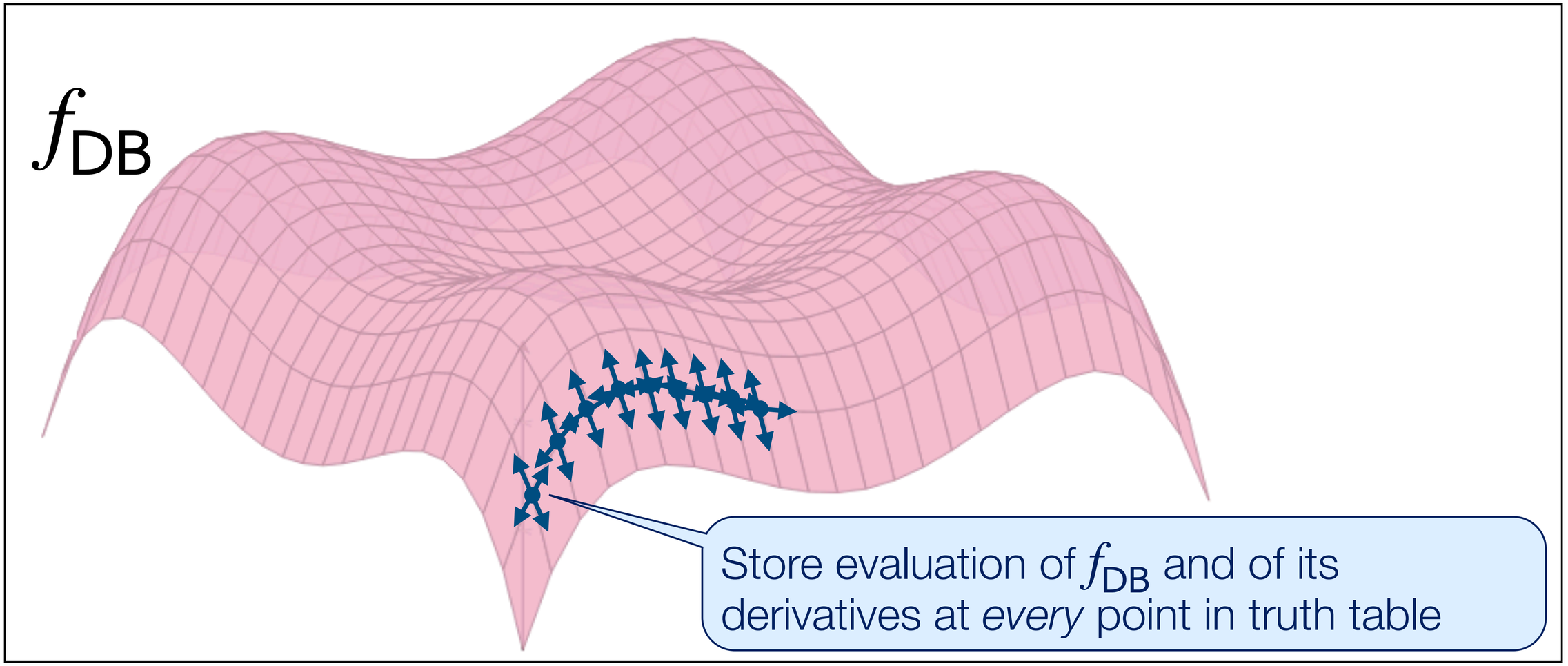

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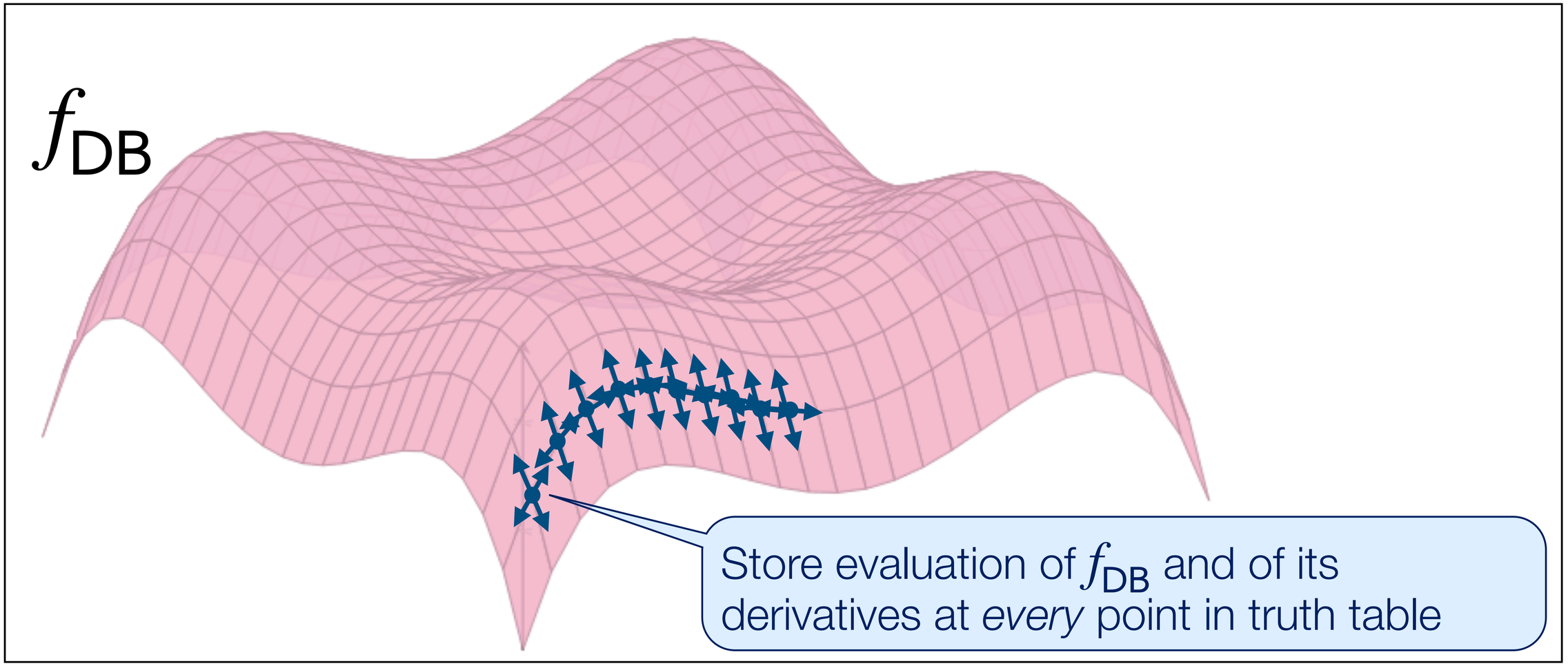

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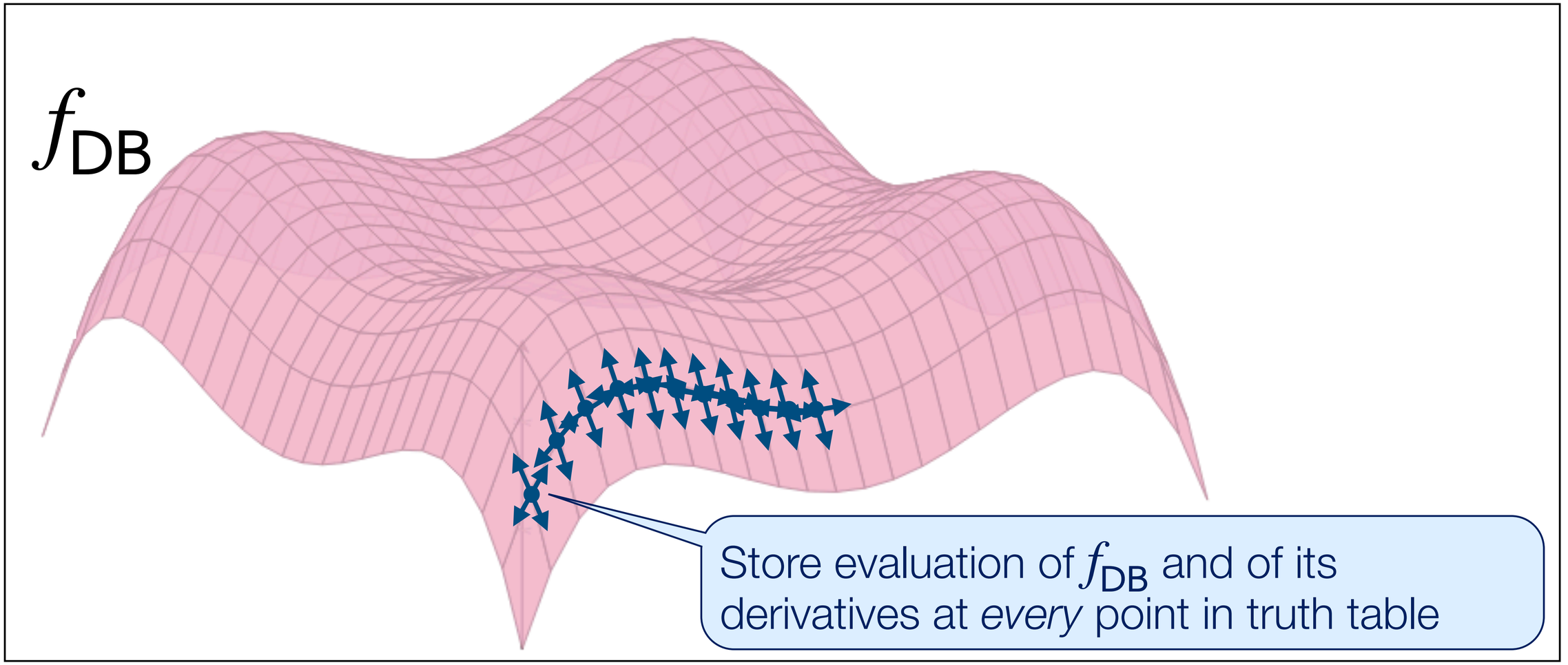

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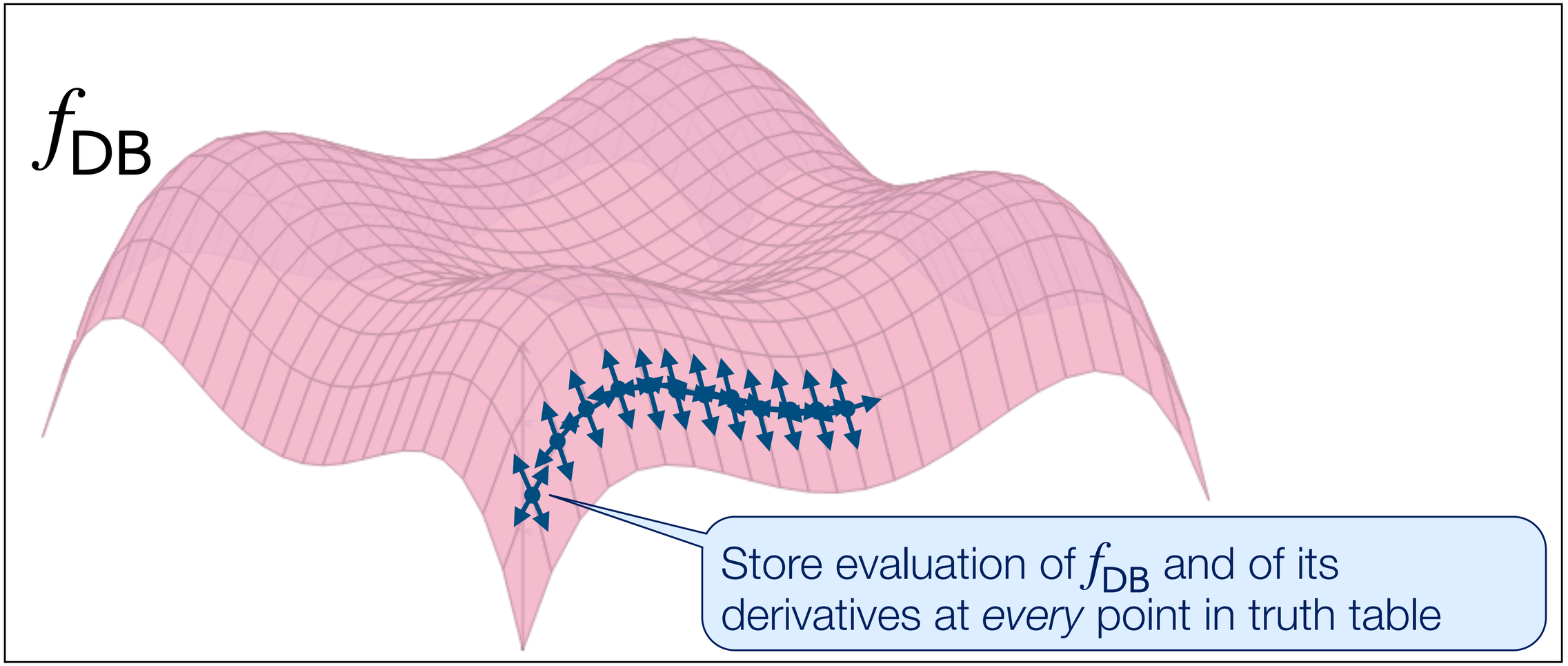

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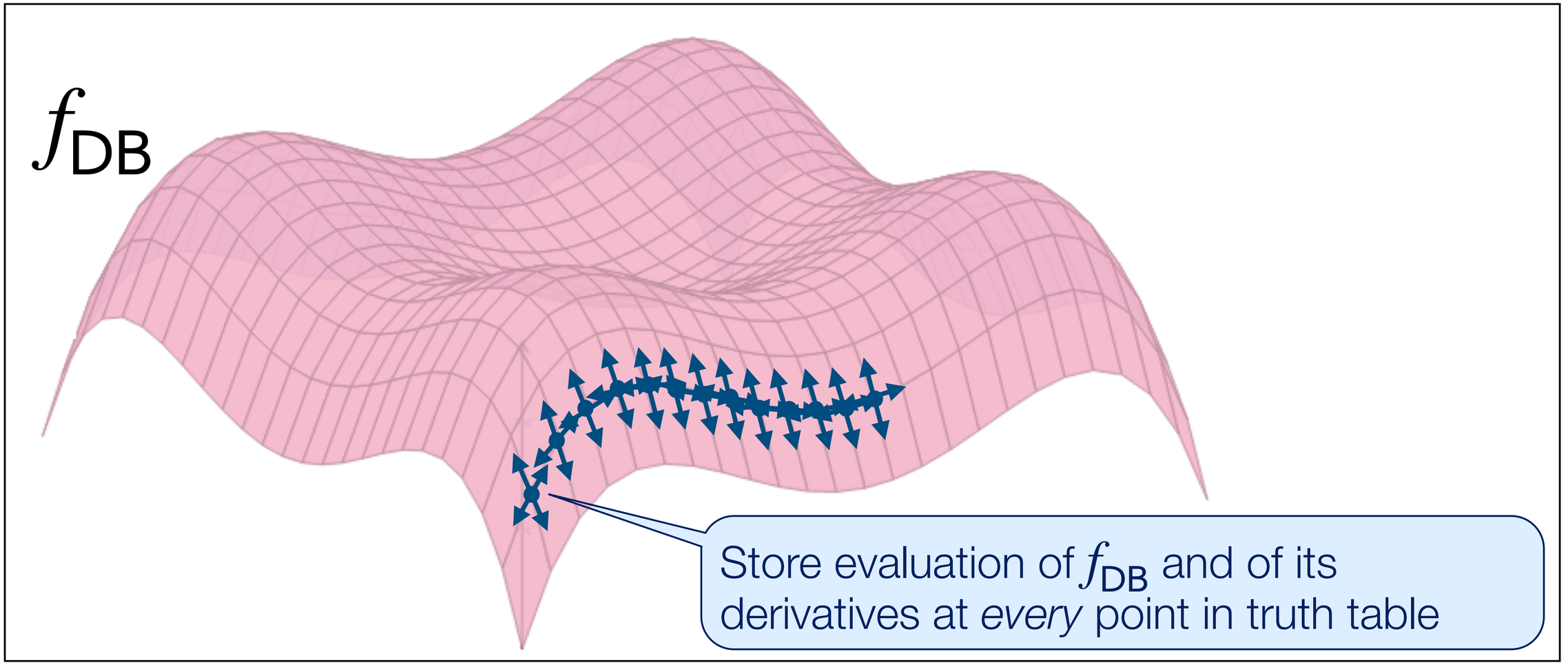

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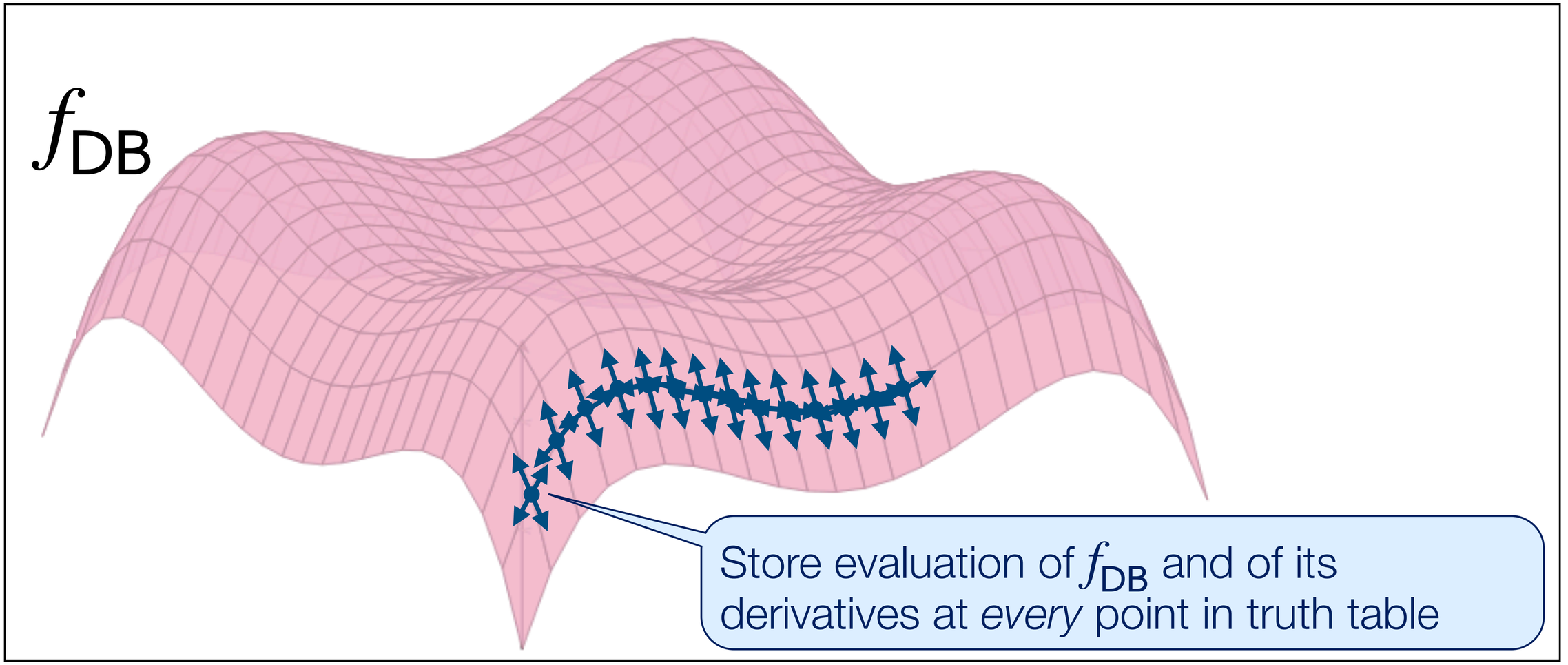

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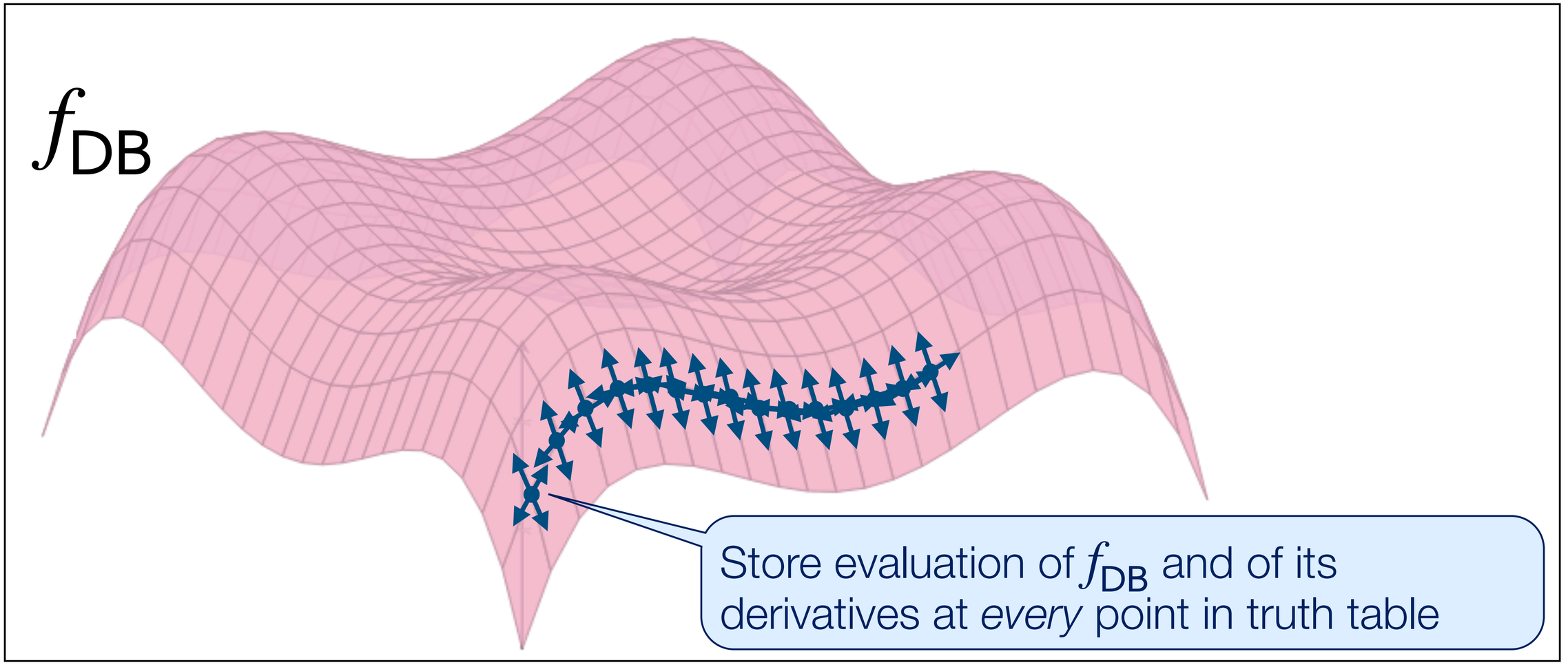

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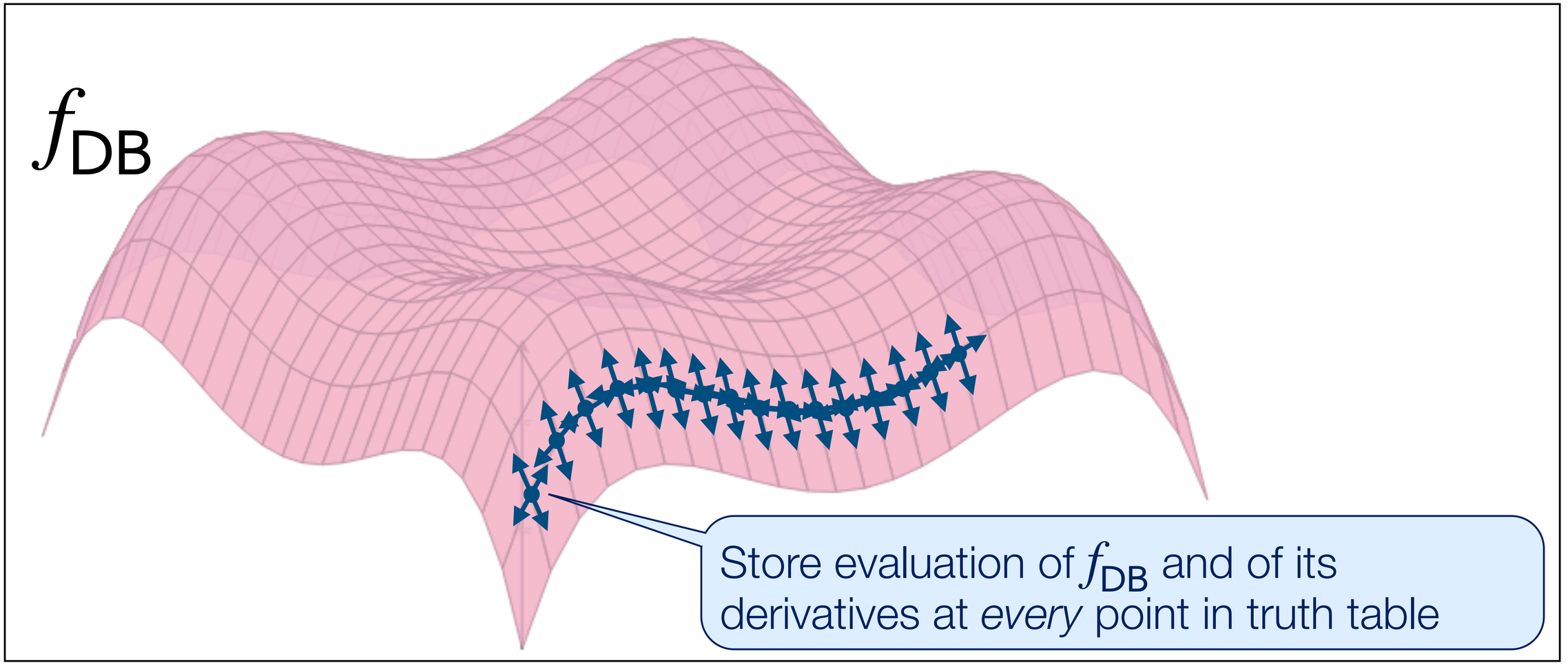

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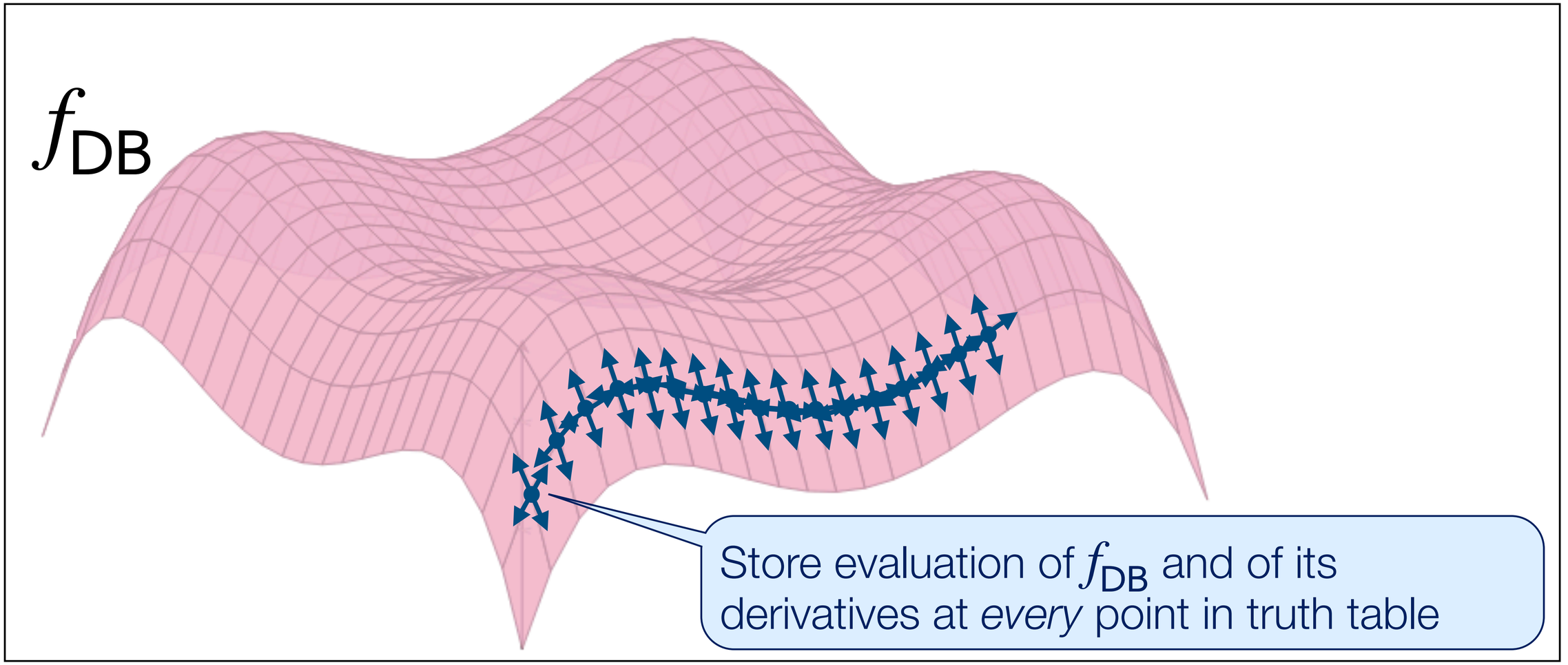

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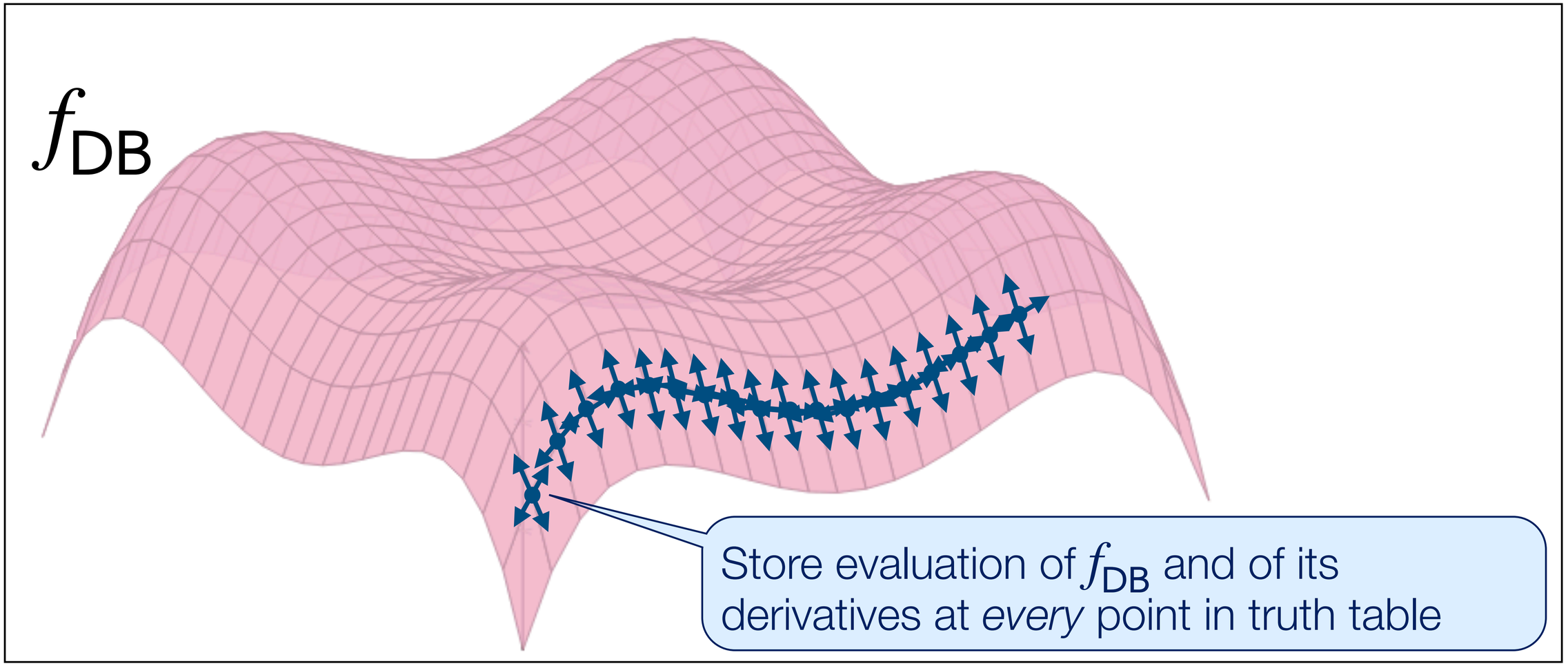

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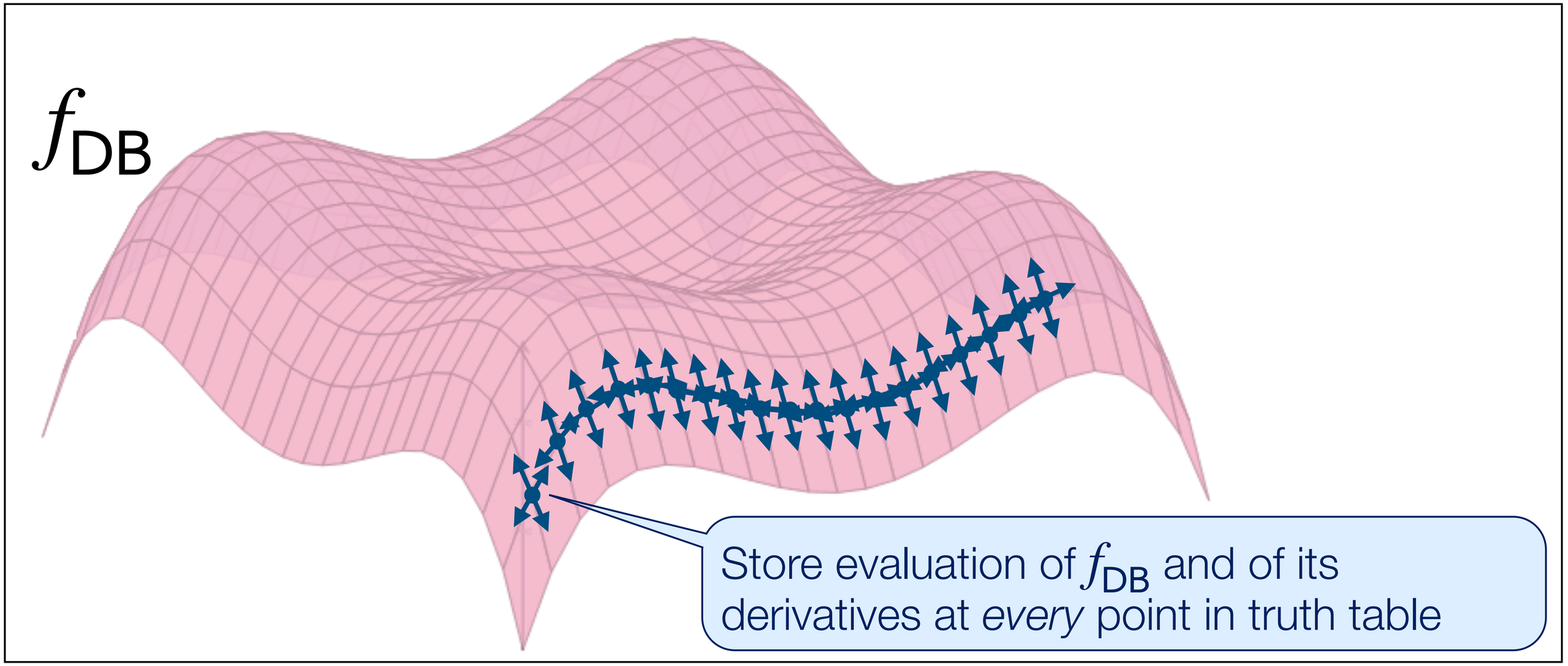

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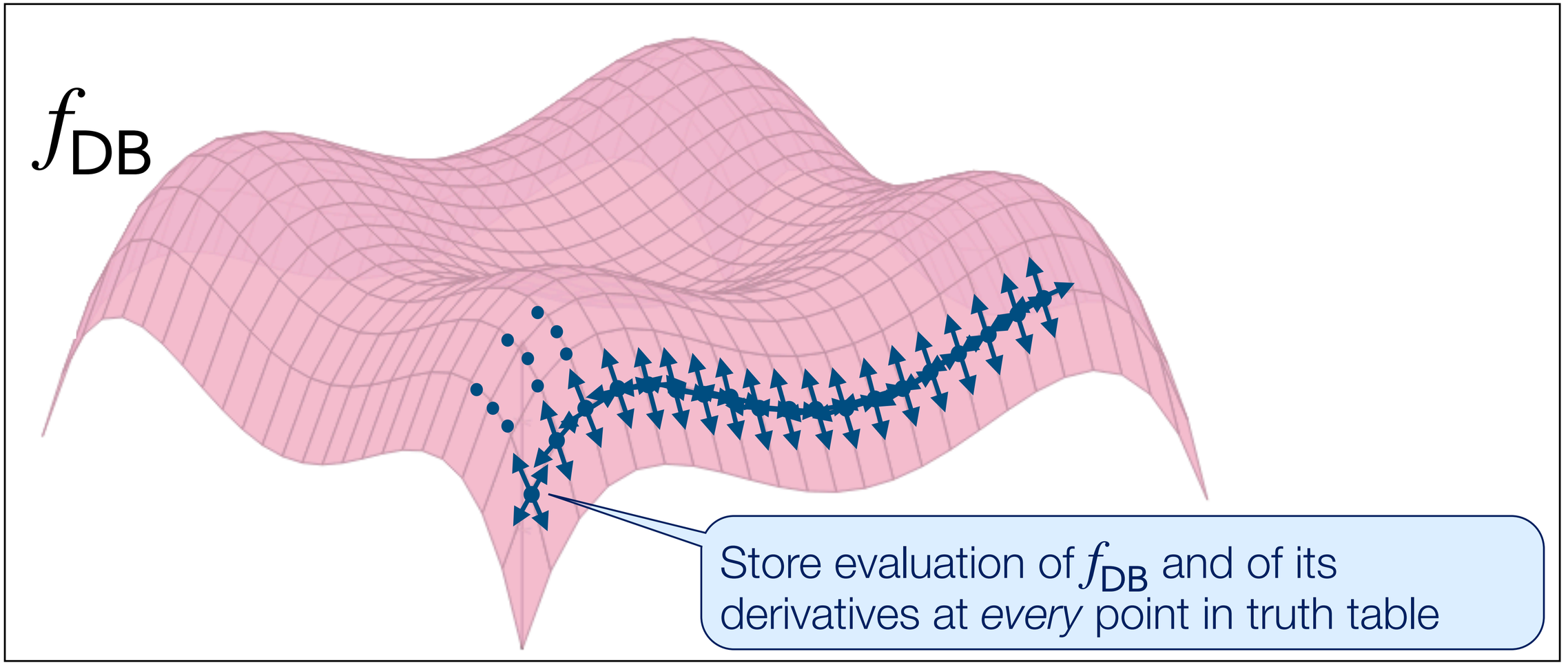

Fact 0. For a linear function $f : \mathbb{F} \to \mathbb{F}$, we have

$$f'(x) = f(x+1) - f(x)\,.$$

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Fact 1. Since f_{DB} is multilinear, for any evaluation point $\mathbf{x} \in \mathbb{F}_2^m$,

$$\nabla f_{\mathsf{DB}}(\mathbf{x}) = \begin{bmatrix} \vdots \\ f_{\mathsf{DB}}(\mathbf{x} + \mathbf{u}_i) - f_{\mathsf{DB}}(\mathbf{x}) \\ \vdots \end{bmatrix} \in \mathbb{F}_2^m$$

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In other words: anyone can deduce $\nabla f_{\mathsf{DB}}(\mathbf{x})$ from the evaluations of f_{DB} in a Hamming ball of radius 1 around point $\mathbf{x}$.

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Idea: Save storage by using evaluations in Hamming balls instead of derivatives.

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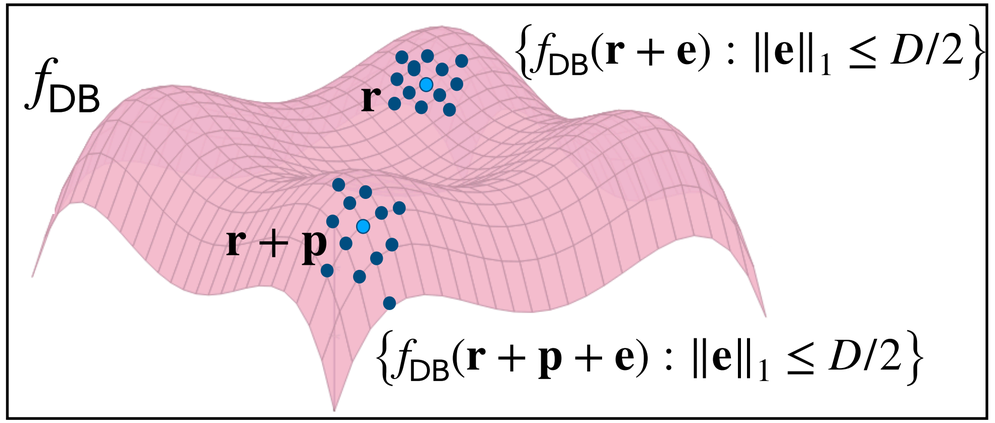
On query points $\mathbf{r}$ and $\mathbf{r}+\mathbf{p}$, the servers send back:



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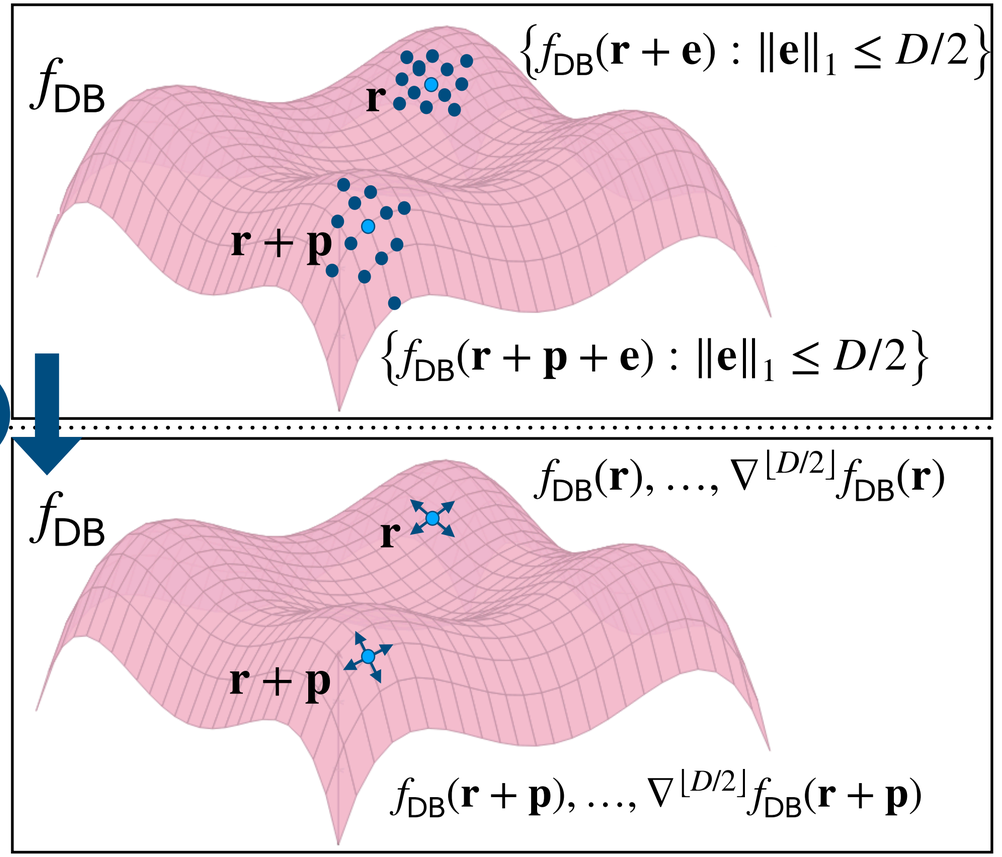

From these replies, the user computes:

1. Partial derivatives of f_{DB} at $\mathbf{r}, \mathbf{r}+\mathbf{p}$ (via finite differences)

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On query points $\mathbf{r}$ and $\mathbf{r}+\mathbf{p}$, the servers send back:

f_{DB} $\{f_{\mathrm{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\|_1 \le D/2\}$ $\mathbf{r}$ $\mathbf{r}+\mathbf{p}$ $\{f_{\mathrm{DB}}(\mathbf{r}+\mathbf{p}+\mathbf{e}) : \|\mathbf{e}\|_1 \le D/2\}$

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f_{DB} $f_{\mathrm{DB}}(\mathbf{r}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathrm{DB}}(\mathbf{r})$ $\mathbf{r}$ $\mathbf{r}+\mathbf{p}$ $f_{\mathrm{DB}}(\mathbf{r}+\mathbf{p}), \ldots, \nabla^{\lfloor D/2 \rfloor} f_{\mathrm{DB}}(\mathbf{r}+\mathbf{p})$

2.

2. $f_{\mathrm{DB}}(\mathbf{p})$ (via Hermite interpolation as before)

f_{DB} $f_{\mathrm{DB}}(\mathbf{p})$ $\mathbf{p}$

Our Resulting PIR with Preprocessing

1	$f_{\mathsf{DB}}(1)$
2	$f_{\mathsf{DB}}(2)$
2^m	$f_{\mathsf{DB}}(2^m)$

Our Resulting PIR with Preprocessing

$\mathbf{1}$	$f_{\mathrm{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\mathrm{DB}}(\mathbf{2})$
$\mathbf{2^m}$	$f_{\mathrm{DB}}(\mathbf{2^m})$

Query: $\mathbf{r}$

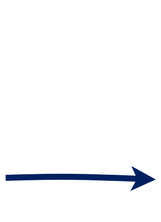
Ans:
$\{f_{\mathrm{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\| \leq D/2\}$

Query: $\mathbf{p}+\mathbf{r}$

Ans:
$\{f_{\mathrm{DB}}(\mathbf{r}+\mathbf{p}+\mathbf{e}) : \|\mathbf{e}\| \leq D/2\}$

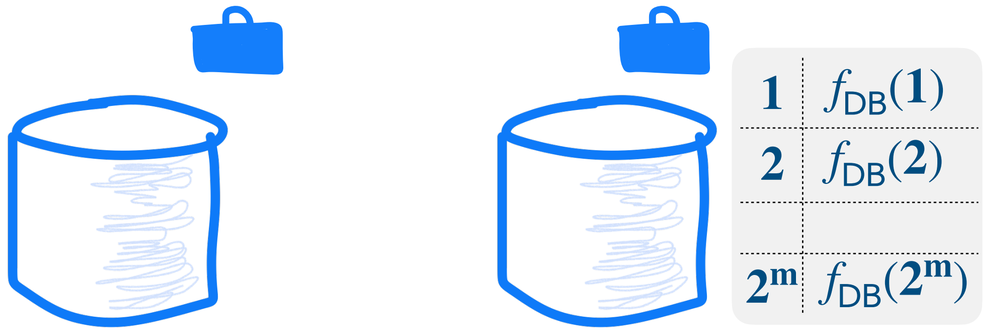
Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line
$L(t) = \mathbf{r} + t \cdot \mathbf{p}$



Recover $f_{\mathrm{DB}}(\mathbf{p})$ via finite differences and Hermite interpolation

Our Resulting PIR with Preprocessing



$\mathbf{1}$	$f_{\mathsf{DB}}(\mathbf{1})$
$\mathbf{2}$	$f_{\mathsf{DB}}(\mathbf{2})$
$\mathbf{2^m}$	$f_{\mathsf{DB}}(\mathbf{2^m})$

Query: $\mathbf{r}$

Ans: $\{f_{\mathsf{DB}}(\mathbf{r}+\mathbf{e}) : \|\mathbf{e}\| \le D/2\}$

Query: $\mathbf{p}+\mathbf{r}$

Ans: $\{f_{\mathsf{DB}}(\mathbf{r}+\mathbf{p}+\mathbf{e}) : \|\mathbf{e}\| \le D/2\}$

Point $\mathbf{p} \in \mathbb{F}_2^m$

Sample line $L(t) = \mathbf{r} + t \cdot \mathbf{p}$

Recover $f_{\mathsf{DB}}(\mathbf{p})$ via finite differences and Hermite interpolation

With 2 servers, gives preprocessing PIR with

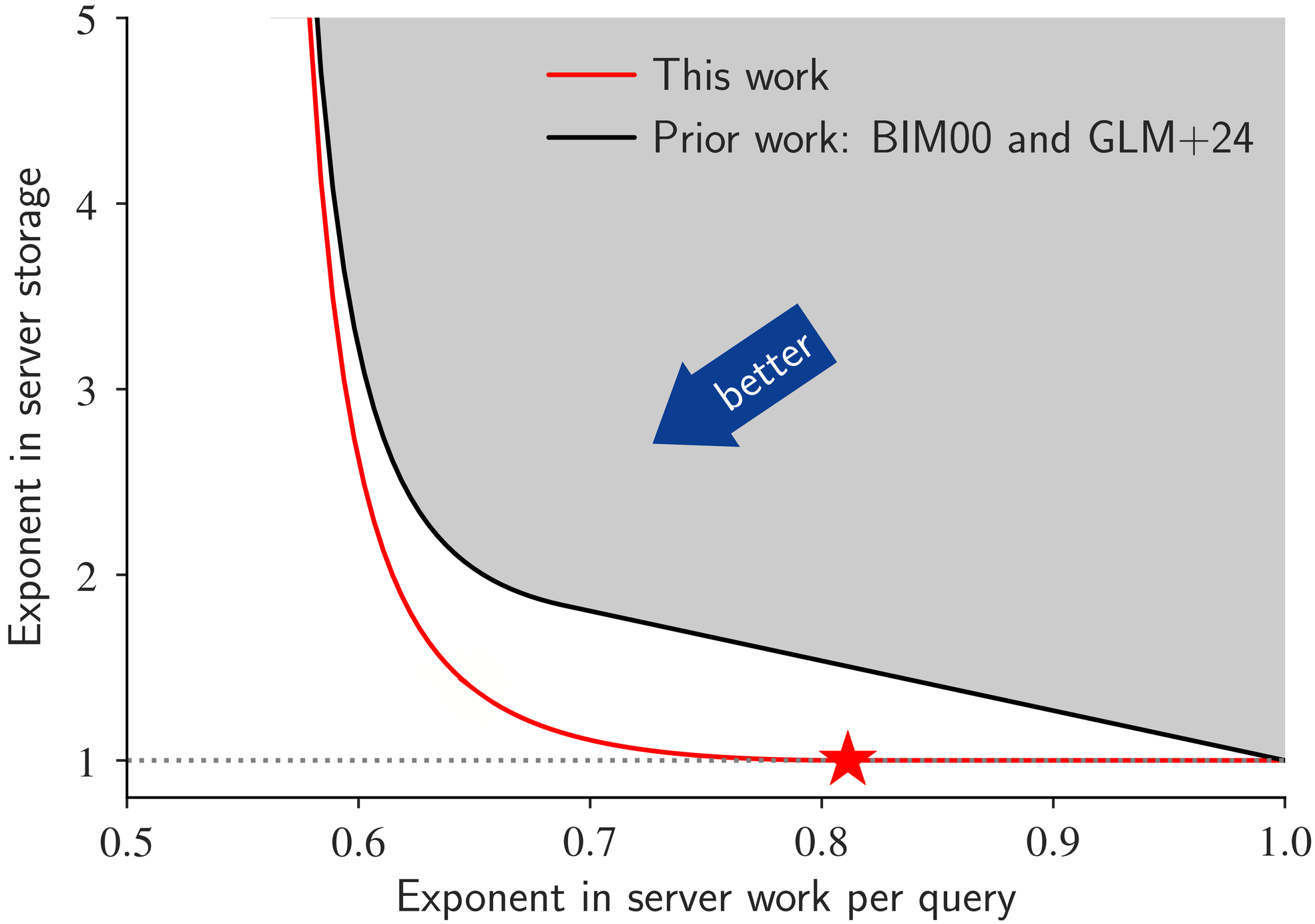
➡ Same comm. as [BIM00]:
query $O(\log n)$
answer $n^{0.82}$

➡ Same time as [BIM00]:
$O(n^{0.82})$ work

➡ Quasilinear space:
$2^m = n^{1+o(1)}$ bits

Theorem. On any database of $n > 10^6$ bits, there exists information-theoretic, two-server PIR with preprocessing with:

- $1.5 \cdot \sqrt{\log n} \cdot n$ bits of server storage,
- $12 \cdot n^{0.82}$ server RAM lookups per query, and
- $12 \cdot n^{0.82}$ bits of communication per query.


This work
Prior work: BIM00 and GLM+24
better
Exponent in server storage
Exponent in server work per query
5
4
3
2
1
0.5
0.6
0.7
0.8
0.9
1.0

Concrete Evaluation: 2GB Database

Construction	Storage	Memory accessed per query
Our work	1 TB	0.7 MB
[BIM00]	760000 TB	0.7 MB
XOR PIR [CGKS95]	2 GB (DB only)	2 GB (entire DB)

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Concrete Evaluation: 2GB Database

Construction	Storage	Memory accessed per query	Communication	Throughput (queries/s)
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[BIM00]	760000 TB	0.7 MB	0.7 MB	
XOR PIR [CGKS95]	2 GB (DB only)	2 GB (entire DB)	0.05 MB	374

Summary

- First information-theoretic PIR with $n^{1+o(1)}$ storage, $n^{1-\Omega(1)}$ server time, and $O(1)$ servers

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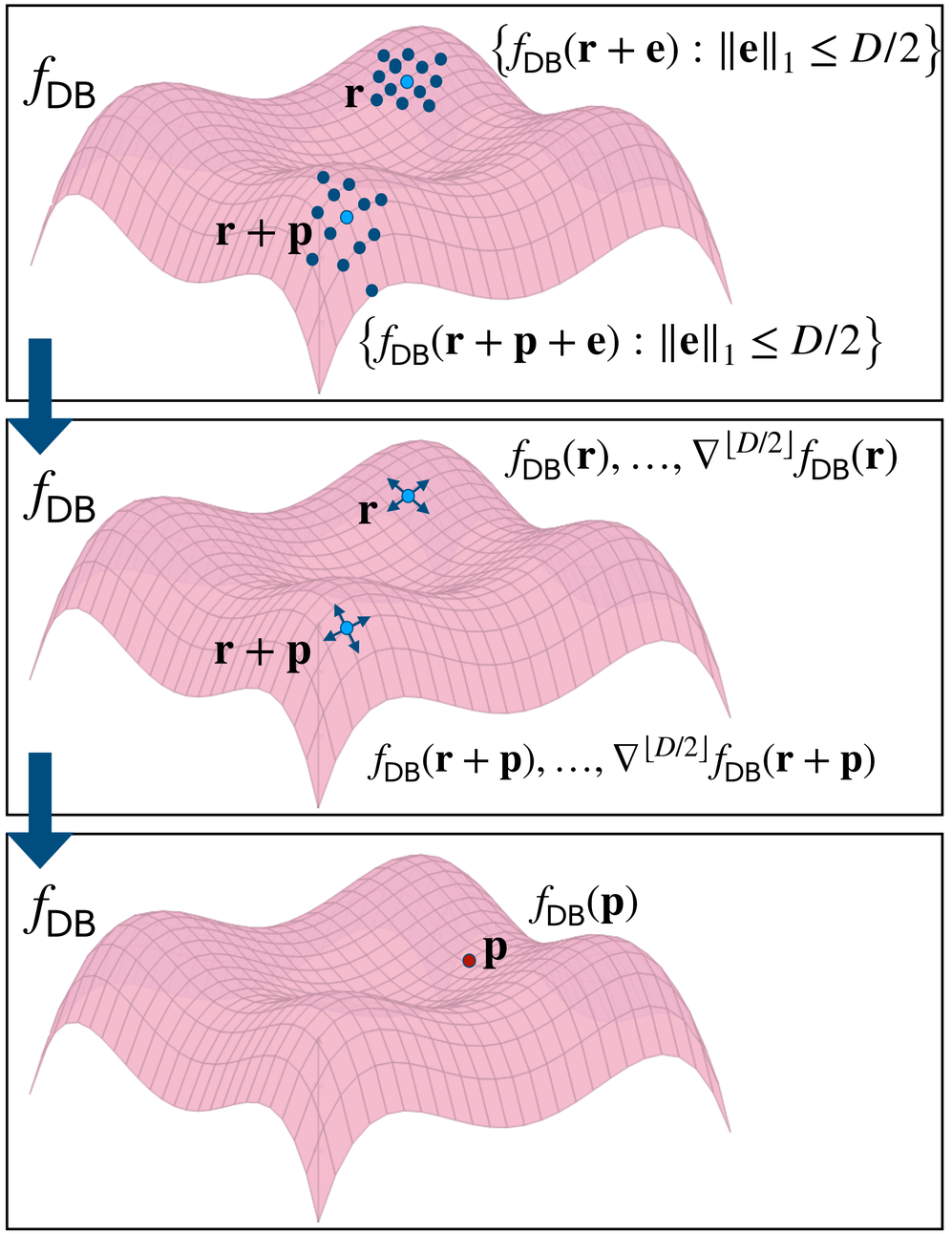
- Not covered in this talk:
  - Shrinking the communication with linearly or fully homomorphic encryption
  - Additional improvements for > 2 servers
- Seems like it could be the first practically feasible PIR with preprocessing!
- Open question: our result but with $n^{o(1)}$ communication (maybe with $\mathsf{poly}(n)$ storage)?
  - Also practically relevant; would reduce network-related expenses and latency

Thank you! Questions?

paper

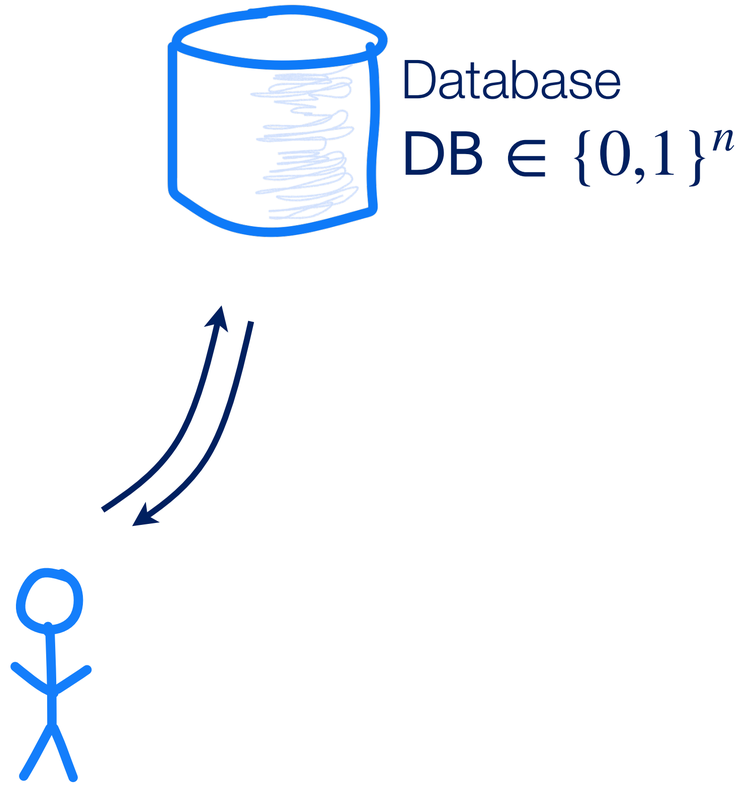
code

ePrint: 2025/2008



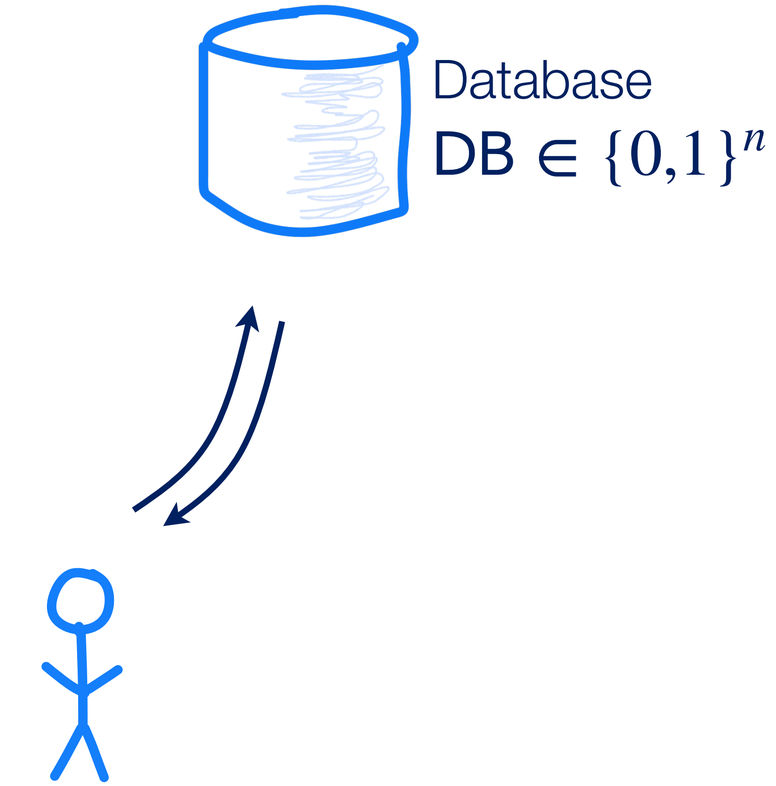
Bonus Slides on Reducing Communication Using Homomorphic Encryption

Homomorphic Encryption for Single-Server PIR



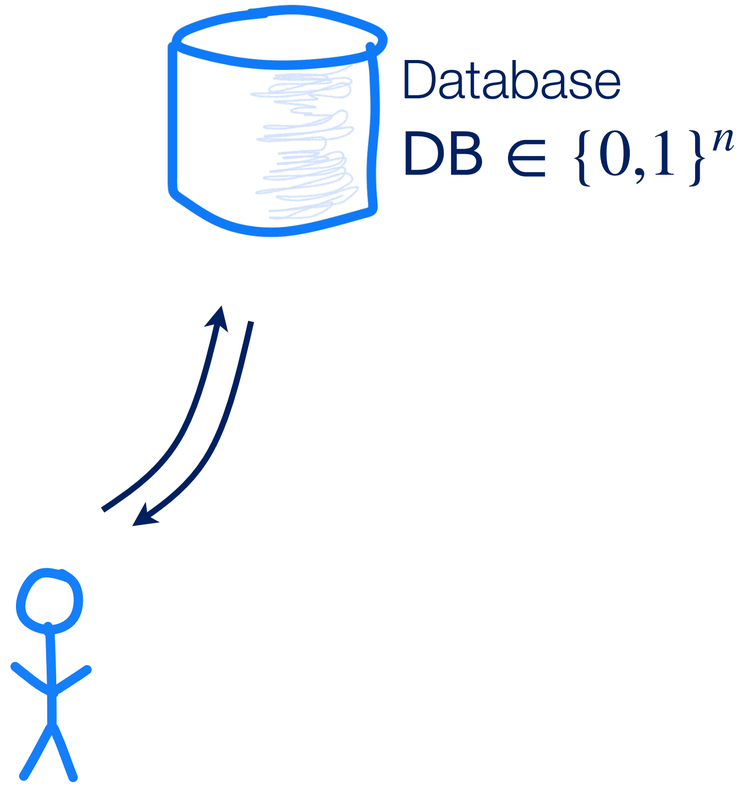
Homomorphic Encryption for Single-Server PIR

- PIR: computing $\mathsf{DB}[i]$ without revealing i



Homomorphic Encryption for Single-Server PIR

- PIR: computing $\mathsf{DB}[i]$ without revealing i
- Naive FHE solution:



Homomorphic Encryption for Single-Server PIR

- PIR: computing $\mathsf{DB}[i]$ without revealing i
- Naive FHE solution:
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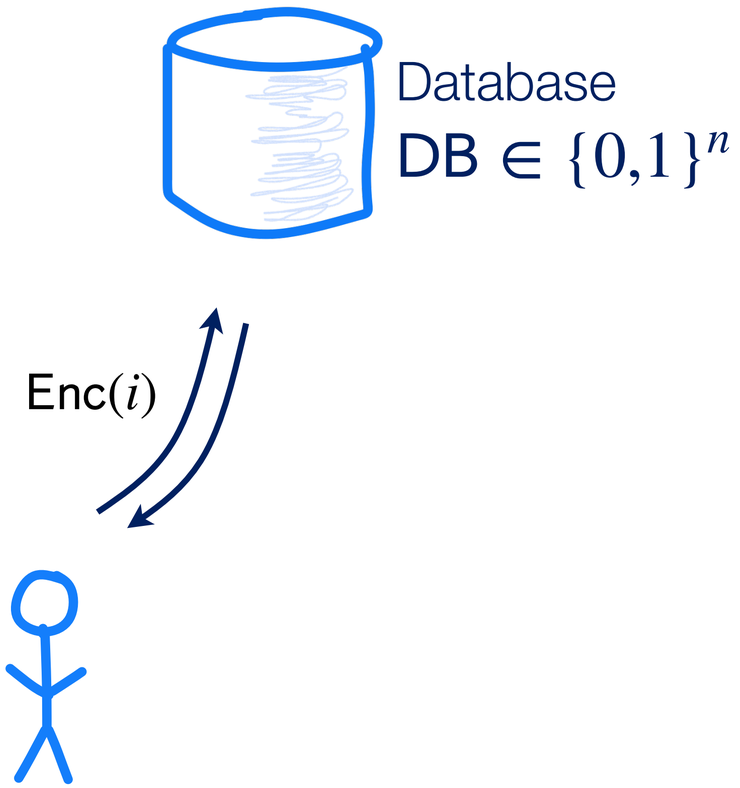

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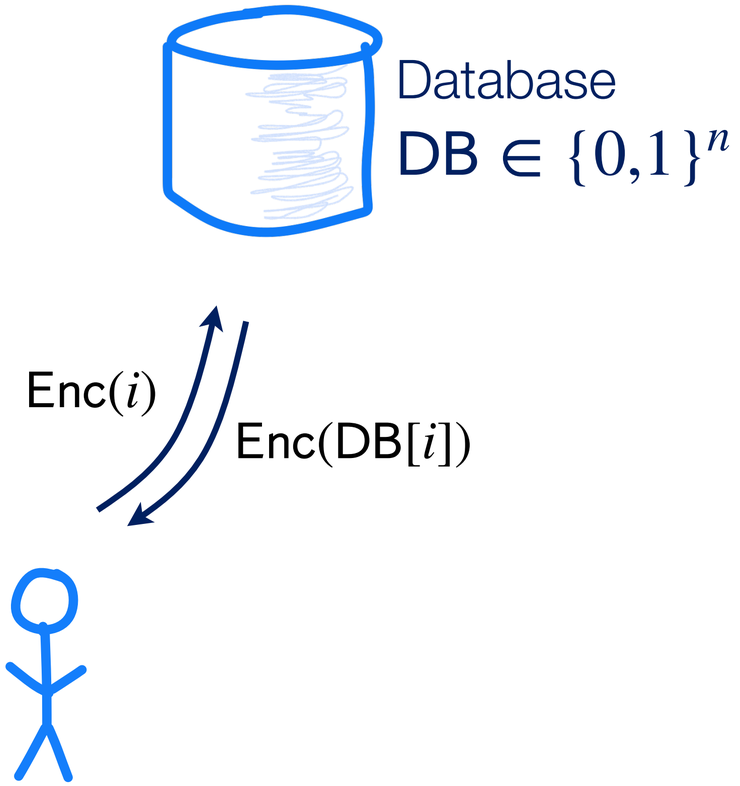

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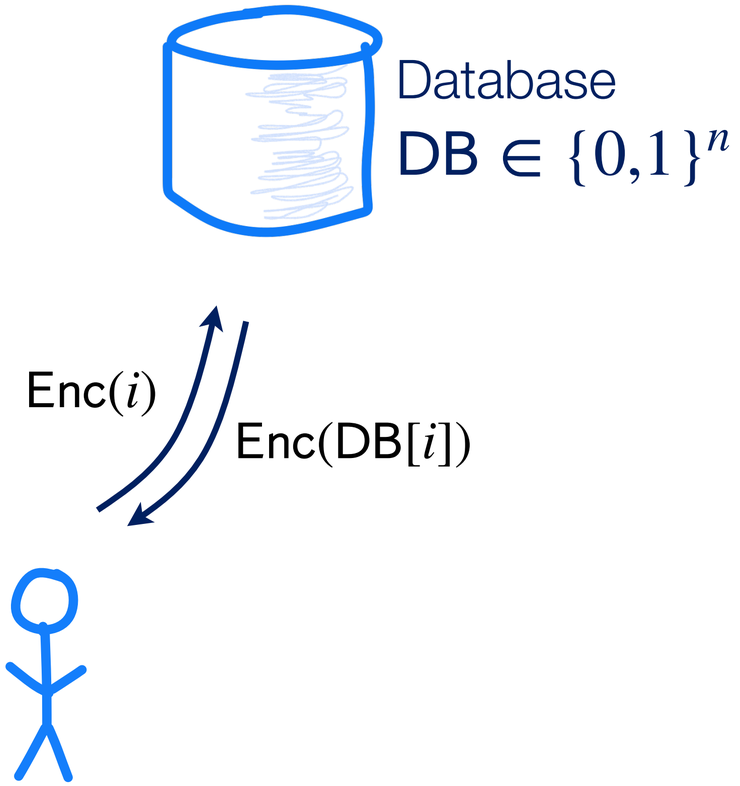

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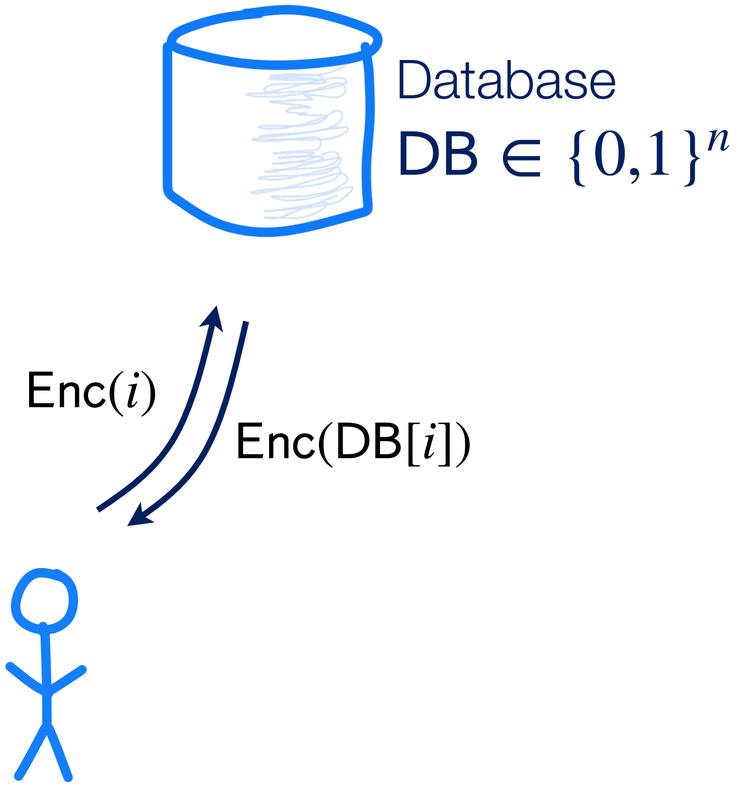

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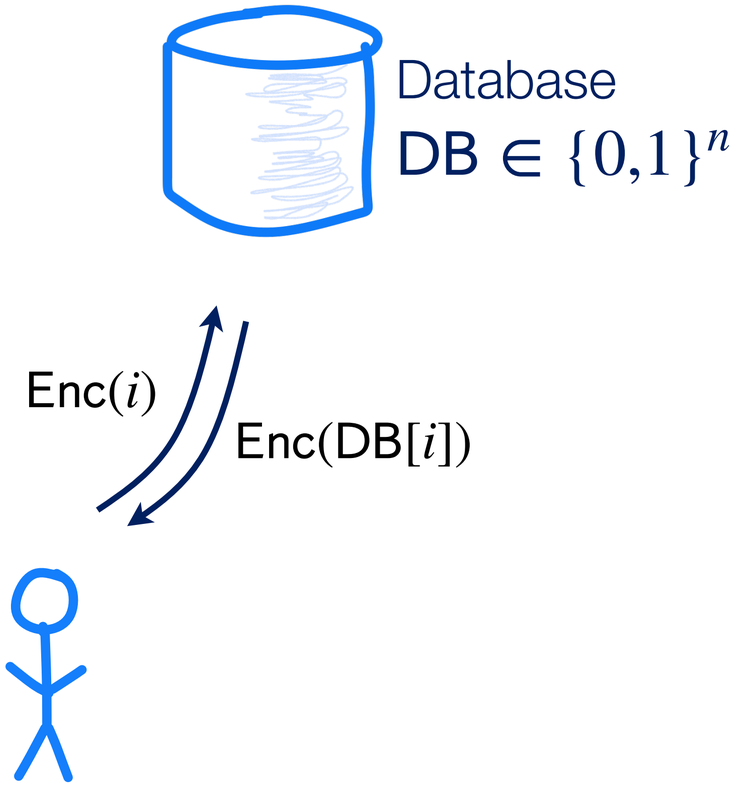
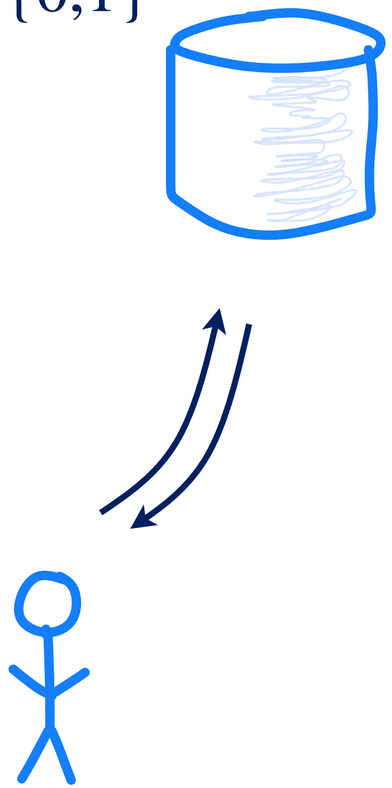
  - Server time per query: $O(n)$



Homomorphic Encryption for 2-Server PIR

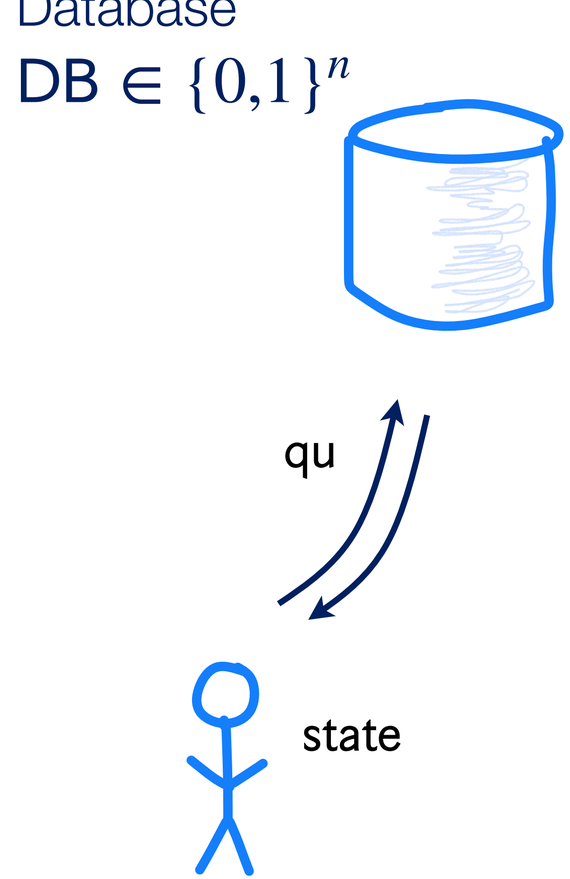
Database

DB $\in \{0,1\}^n$



Homomorphic Encryption for 2-Server PIR

- Three phases:
  - Query: $\mathsf{state}, \mathsf{qu}_1 \leftarrow \mathsf{Query}(i)$



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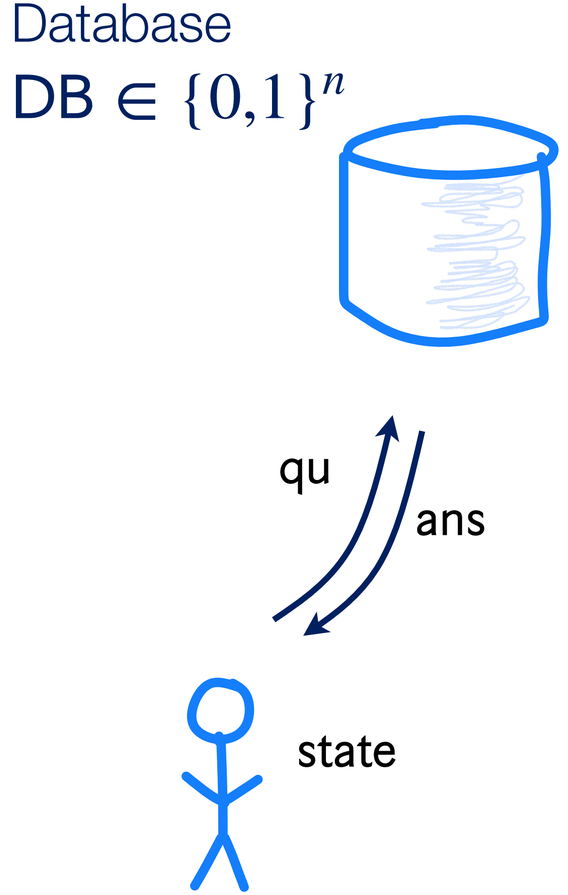

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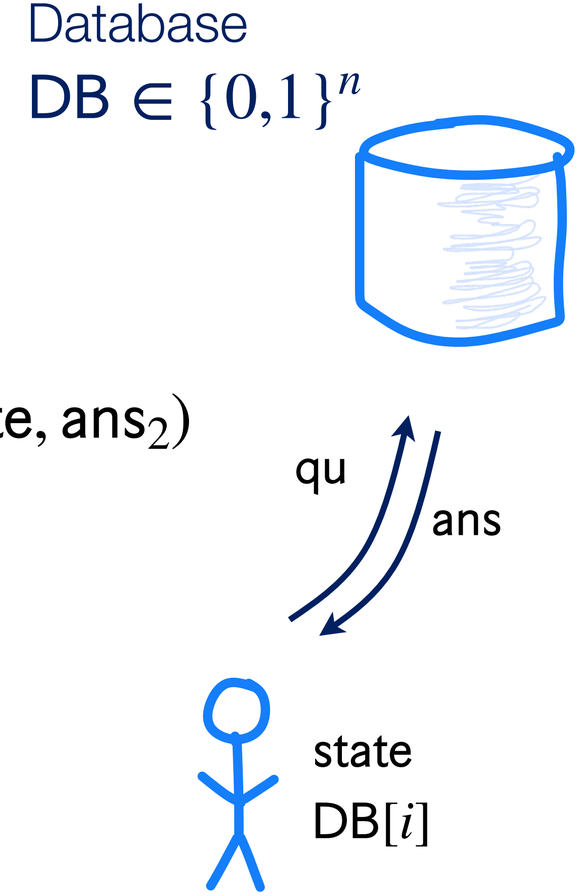

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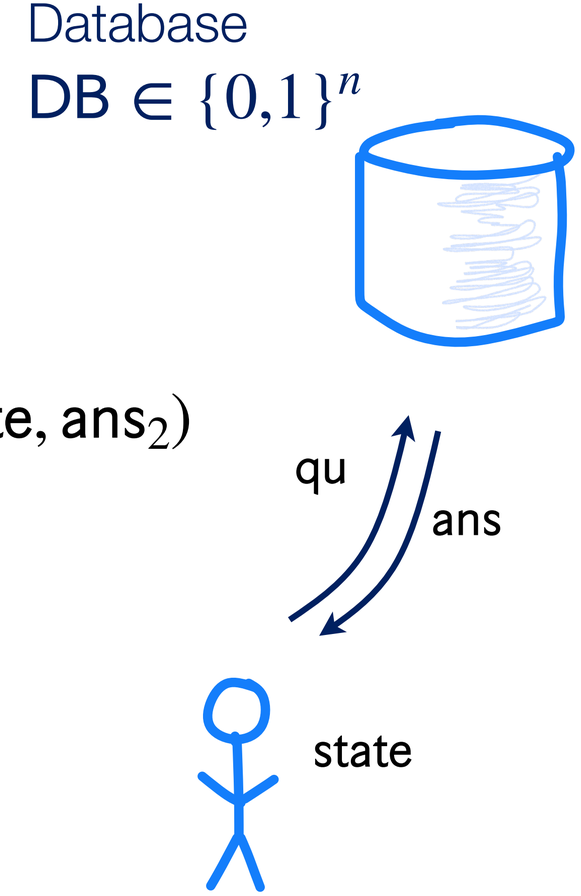

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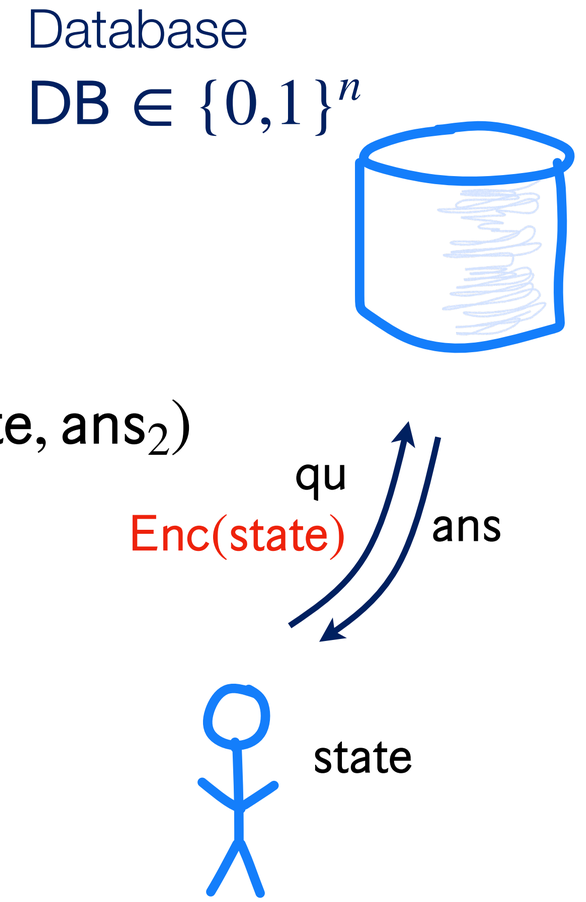

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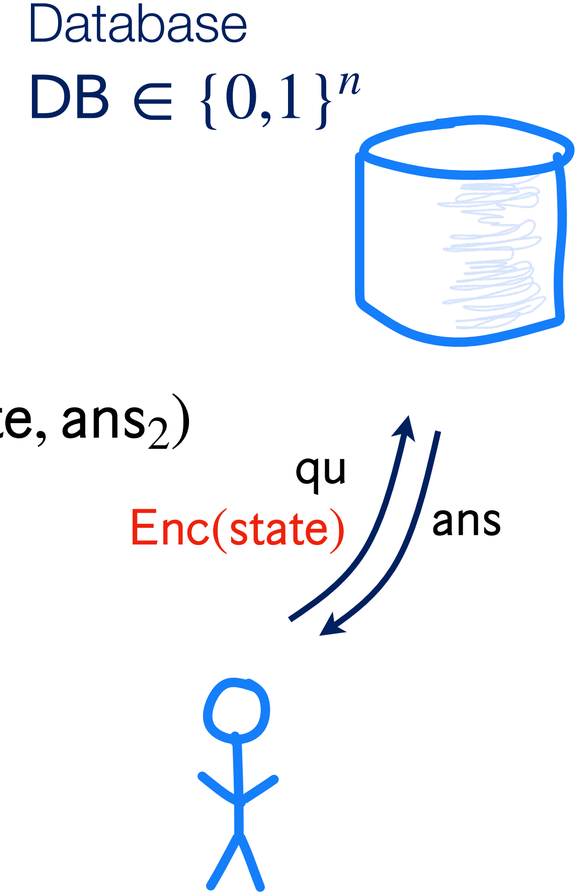

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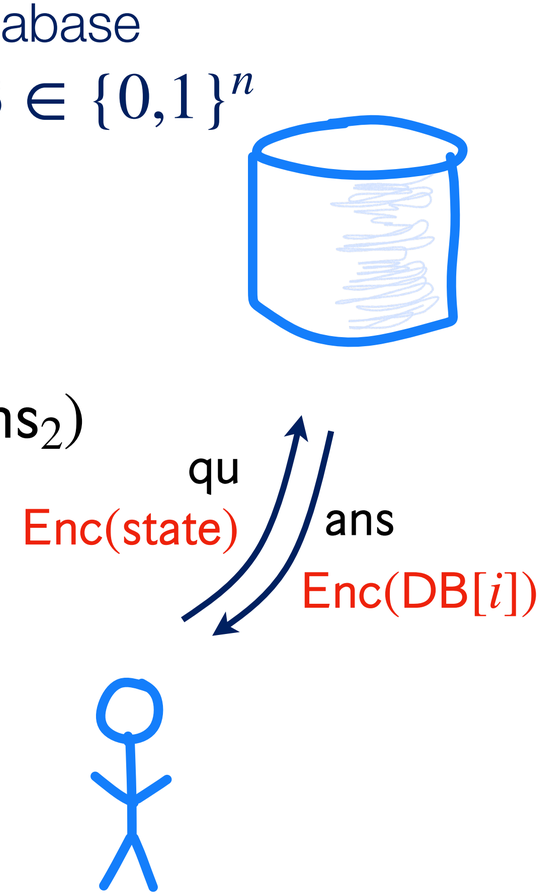

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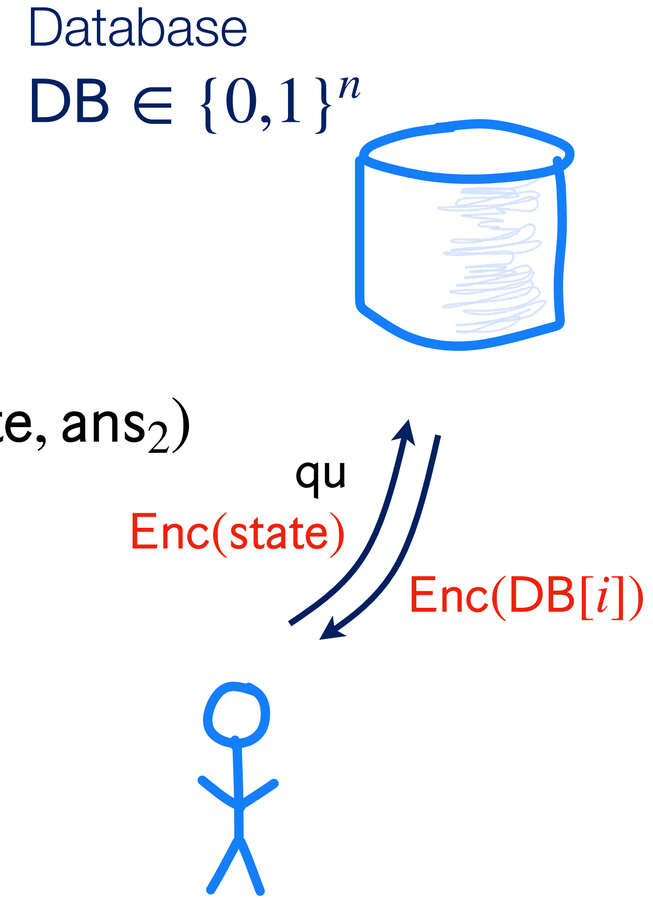

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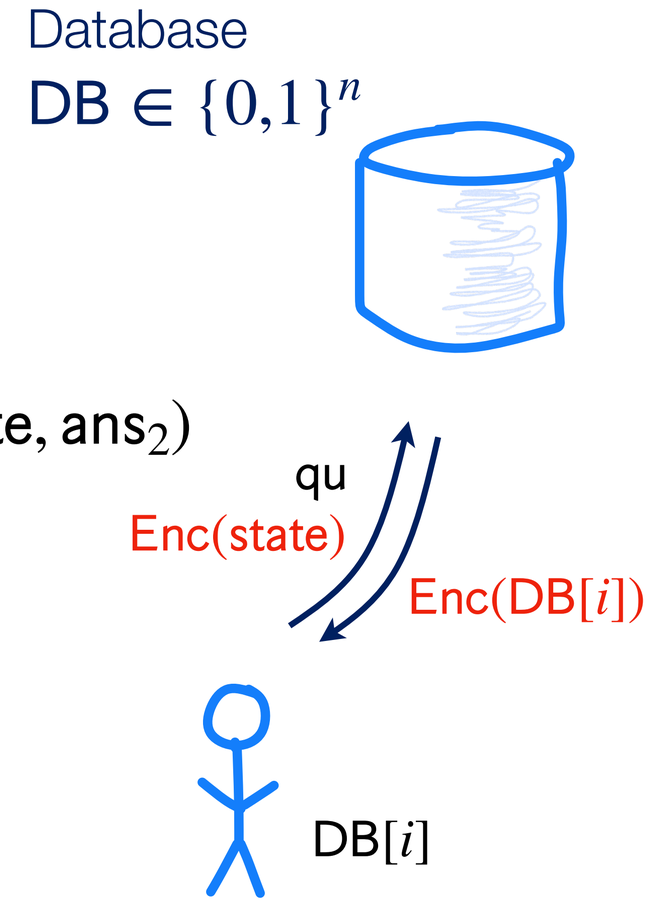

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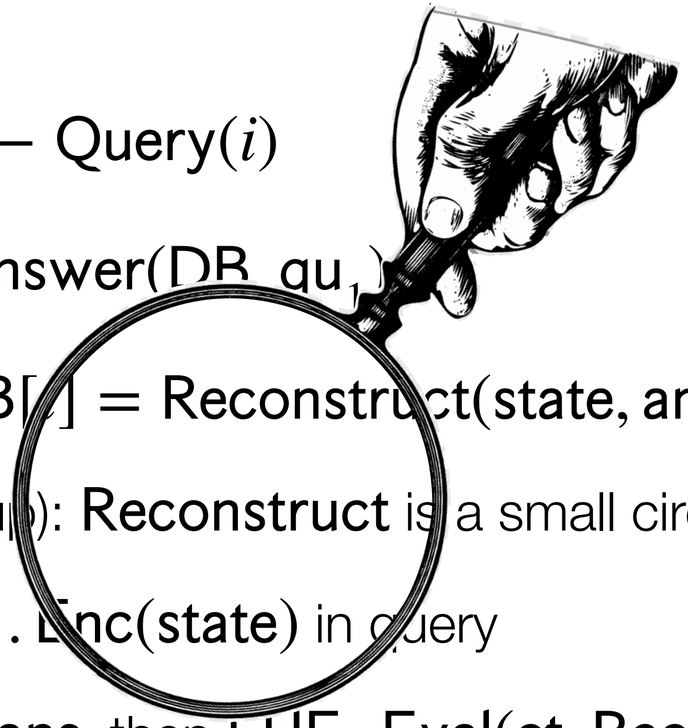

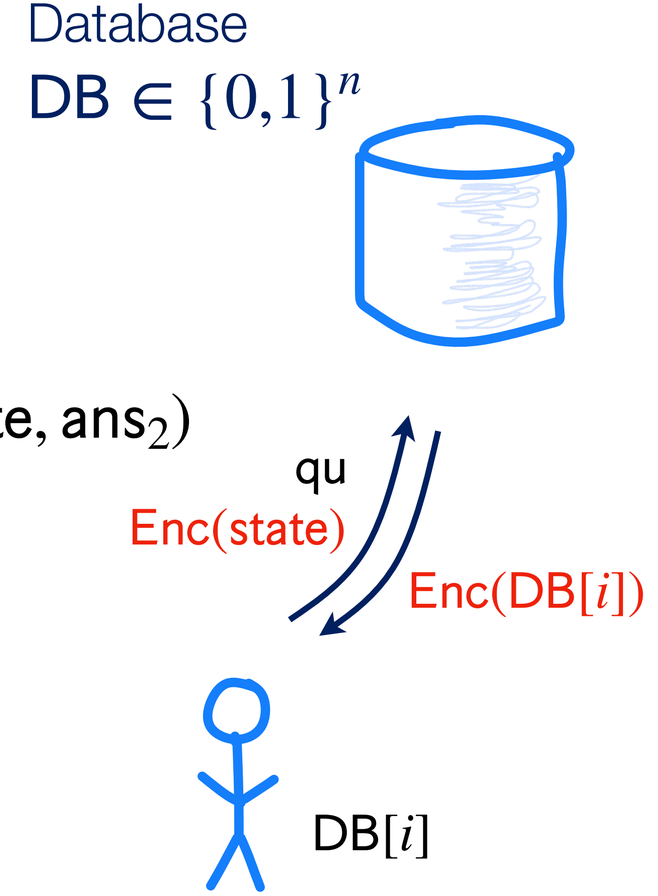

Our PIR Reconstruction

Cheatsheet

$m \approx \log n$

$D \approx m/2$

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$$\mathsf{DB}_i = f_{\mathsf{DB}}(\mathbf{p}) = \sum_{\substack{\|\mathbf{e}\| \leq \lfloor D/2 \rfloor \\ \mathbf{e} \leq \mathbf{p}}} \left(\underbrace{f(\mathbf{r} + \mathbf{e})}_{\text{from server 1's answer}} + \underbrace{f(\mathbf{p} + \mathbf{r} + \mathbf{e})}_{\text{from server 2's answer}} \right)$$

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- For each $\mathbf{e}$, $\mathbf{1}[\mathbf{e} \leq \mathbf{p}] = \mathbf{p}^{\mathbf{e}} = \prod_{i \in [m]} p_i^{e_i}$, which is degree $\leq D/2$ in $\mathbf{p}$

Abstract Setup

Cheatsheet

$m \approx \log n$

$D \approx m/2$

- Two polynomials $g_1, g_2 : \mathbb{F}_2^m \to \mathbb{F}_2$ of degree $D/2$ for servers 1 and 2 respectively

$$g_1(\mathbf{x}) = \sum_{\| \mathbf{e} \| \leq \lfloor D/2 \rfloor} \left(\underbrace{f(\mathbf{r}+\mathbf{e})}_{\text{from server 1's answer}} \right) \mathbf{x}^{\mathbf{e}}$$

$$g_2(\mathbf{x}) = \sum_{\| \mathbf{e} \| \leq \lfloor D/2 \rfloor} \left(\underbrace{f(\mathbf{p}+\mathbf{r}+\mathbf{e})}_{\text{from server 2's answer}} \right) \mathbf{x}^{\mathbf{e}}$$

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- All we need to do is get server 1 to help the user evaluate $g_1(\mathbf{p})$ and likewise for server 2

Succinct PIR from FHE

Cheatsheet

$$m \approx \log n$$

$$D \approx m/2$$

$$\binom{m}{\alpha m} \approx 2^{mH(\alpha)} \approx n^{H(\alpha)}$$

Succinct PIR from FHE

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$m \approx \log n$

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$\binom{m}{\alpha m} \approx 2^{mH(\alpha)} \approx n^{H(\alpha)}$

- Communication: upload $m \cdot \mathsf{poly}(\lambda) = \log n \cdot \mathsf{poly}(\lambda)$, download $\mathsf{poly}(\lambda)$

Succinct PIR from FHE

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  - Runtime $\approx \binom{m}{D/2} \cdot \mathsf{poly}(\lambda) \approx n^{H(1/4)} \cdot \mathsf{poly}(\lambda) \approx n^{0.82} \cdot \mathsf{poly}(\lambda)$

Succinct PIR from FHE

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- Communication: upload $m \cdot \mathsf{poly}(\lambda) = \log n \cdot \mathsf{poly}(\lambda)$, download $\mathsf{poly}(\lambda)$
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  - $H(\alpha) \in [0,1]$: binary entropy of $\alpha \in [0,1]$

Succinct PIR from FHE

Cheatsheet

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$D \approx m/2$

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Theorem: with compact **fully homomorphic encryption***, we get 2-server PIR with server storage $n^{1+o(1)}$, time per query $O(n^{0.82})$ and communication $O(\log n)$.

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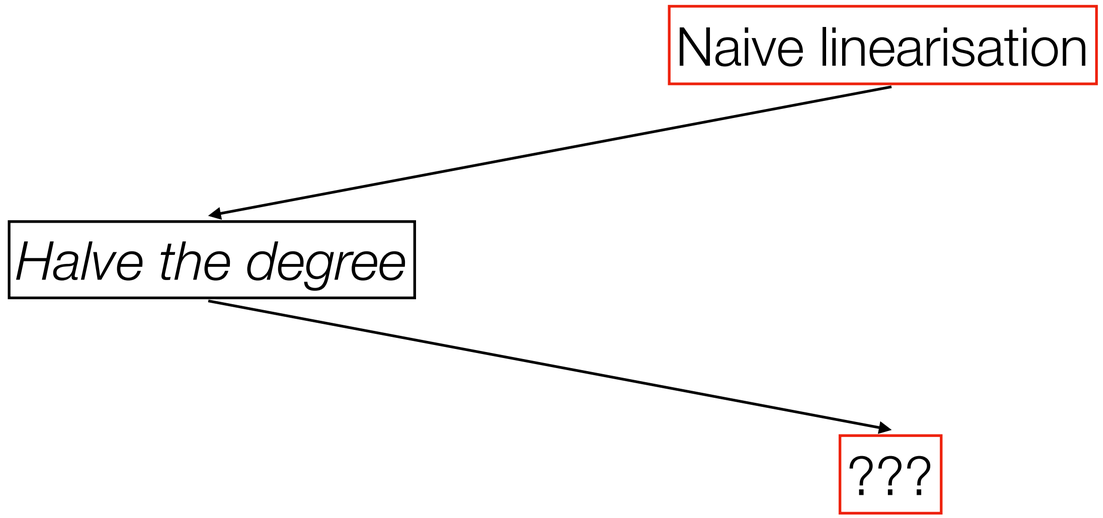

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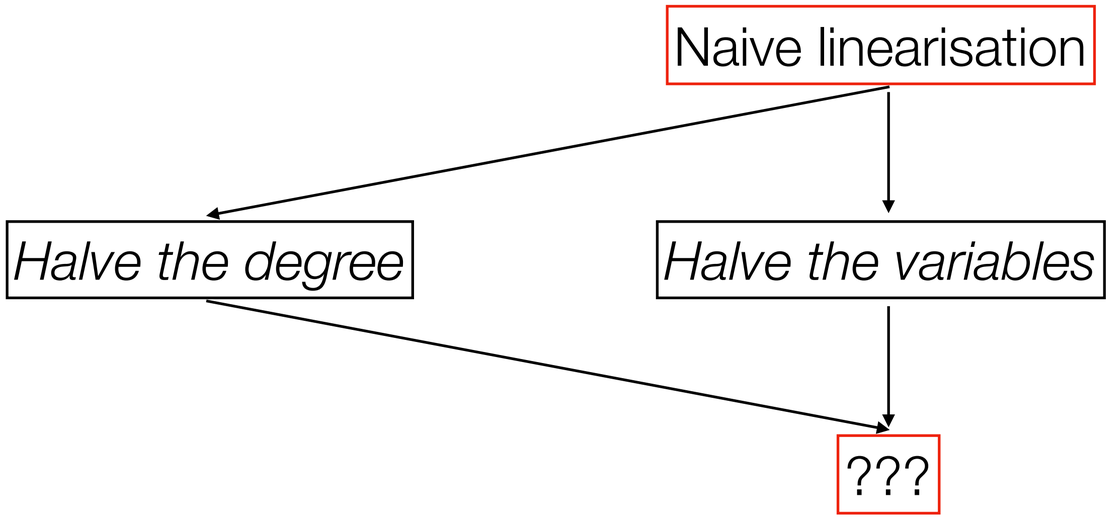

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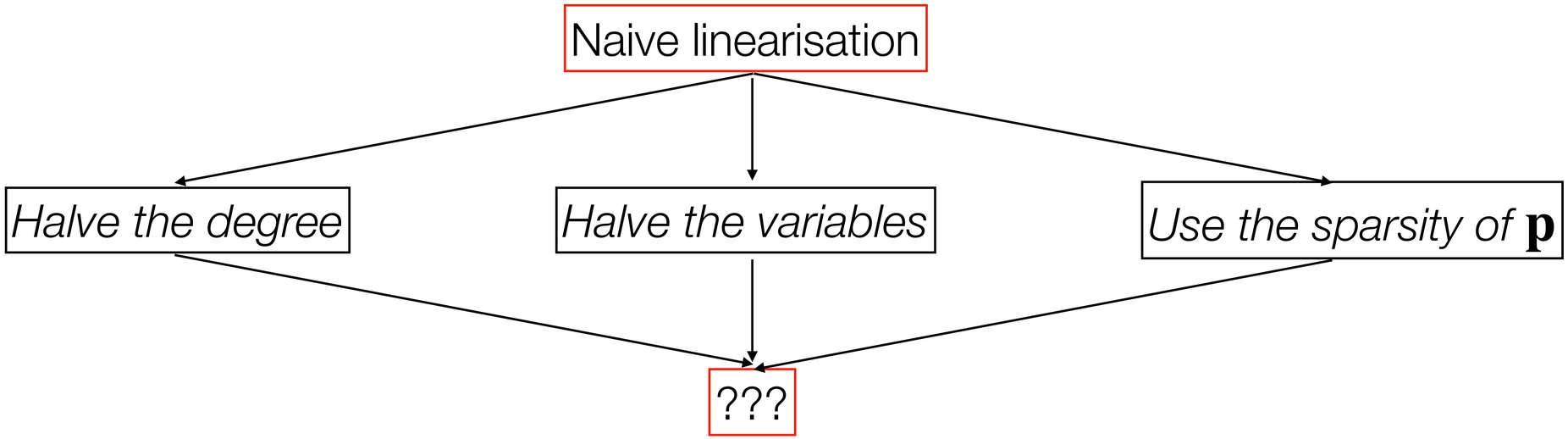

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    - Communication cost: $\binom{m}{D/2} \cdot \mathsf{poly}(\lambda) \approx n^{0.82} \cdot \mathsf{poly}(\lambda)$, same as before 😱
    - Ver... uploading $n^{0.82}$ ciphertexts and downloading just one → can rebalance!

Natural target: reduce communication from $n^{0.82}$ to $\sqrt{n^{0.82}} = n^{0.41}$

Naive linearisation

Halve the degree | *Halve the variables* | *Use the sparsity of* $\mathbf{p}$

???

LHE → Computing Deg 2 Polynomials

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- **Claim** [KO97]: assume LHE. Then if the user has $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{\ell}$ and the server has $\mathbf{A} \in \mathbb{F}^{\ell \times \ell}$, the user can learn $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{y}$ without revealing $\mathbf{x}, \mathbf{y}$, with $\ell \cdot \mathsf{poly}(\lambda)$ communication.

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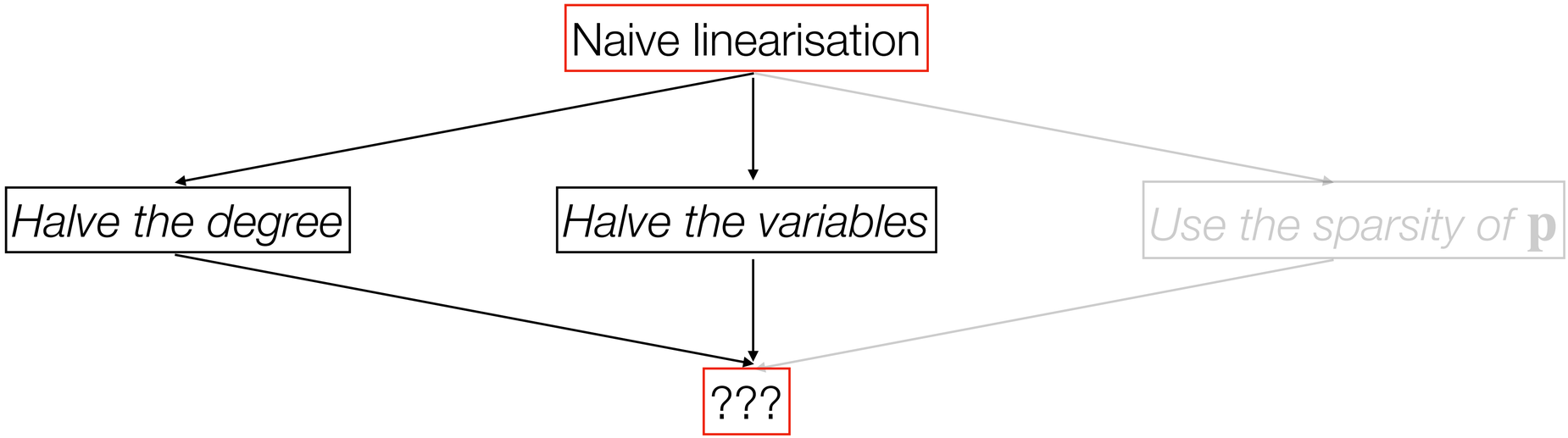
- **Proof:**
  - User sends $\mathsf{LHE.Enc}(\mathbf{y})$
  - Server replies with $\mathsf{LHE.Enc}(\mathbf{A}\mathbf{y})$
  - User decrypts and locally takes the inner product with $\mathbf{x}$

Rebalancing



Halving the Degree/Variables

Cheatsheet

$\mathbf{p} \in \mathbb{F}_2^m$

$D \approx m/2$

Evaluating $g(\mathbf{p})$

$\deg g \leq D/2$

$\binom{m}{\alpha m} \approx 2^{mH(\alpha)} \approx n^{H(\alpha)}$

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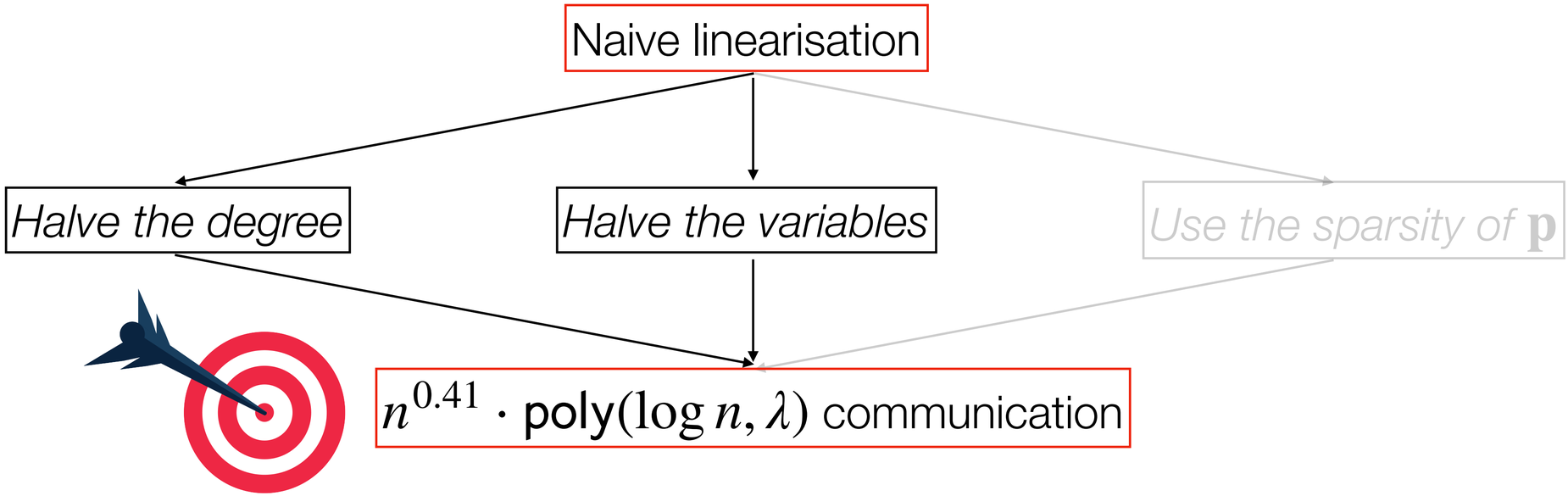
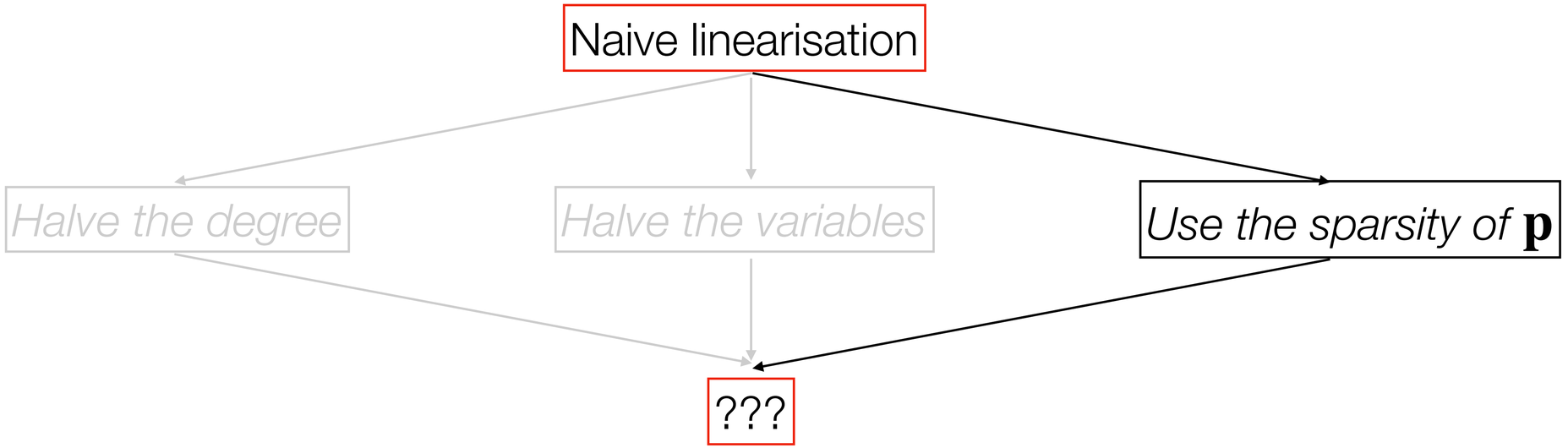
  - Number of variables: $\binom{m/2}{D/2} \approx 2^{mH(1/2)/2} \approx n^{0.5}$
- Careful combination of these ideas: $\approx \binom{m/2}{D/4} \approx 2^{mH(1/4)/2} \approx n^{0.41}$


Naive linearisation
Halve the degree
Halve the variables
Use the sparsity of $\mathbf{p}$
$n^{0.41} \cdot \mathsf{poly}(\log n, \lambda)$ communication


Naive linearisation
Halve the degree
Halve the variables
Use the sparsity of p
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Leveraging the Sparsity of $\mathbf{p}$ [BIM04]

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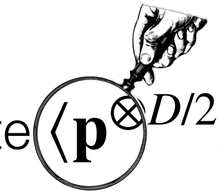
- Original naive idea: compute  under the hood

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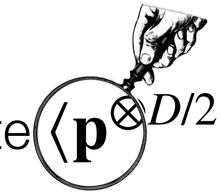
- Original naive idea: compute  $\langle \mathbf{p}^{\otimes D/2}, \mathsf{coefs}(g) \rangle$ under the hood
- Observation: $\mathbf{p}$ only has D nonzero entries → $\mathbf{p}^{\otimes D/2}$ has $\leq 2^D \ll \binom{m}{D/2}$ nonzero entries!

Leveraging the Sparsity of $\mathbf{p}$ [BIM04]

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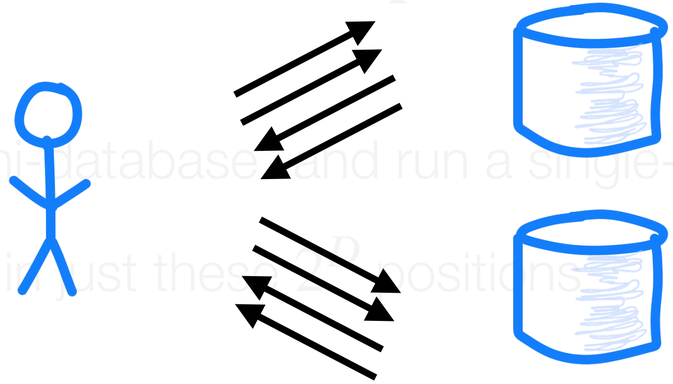
- Idea: view $\mathbf{coefs}(g)$ as a “mini-database” and run a single-server “mini-PIR” protocol (KO97, IKOS04) to retrieve $\mathbf{coefs}(g)$ in just these 2^D positions

Leveraging the Sparsity of $\mathbf{p}$ [BIM04]

Batch PIR with many, non-adaptive queries



[IKOS'04,HHG'13,GKL'10,AS'16,H'16,ACLS'18,CHLR'18]

Leveraging the Sparsity of $\mathbf{p}$ [BIM04]

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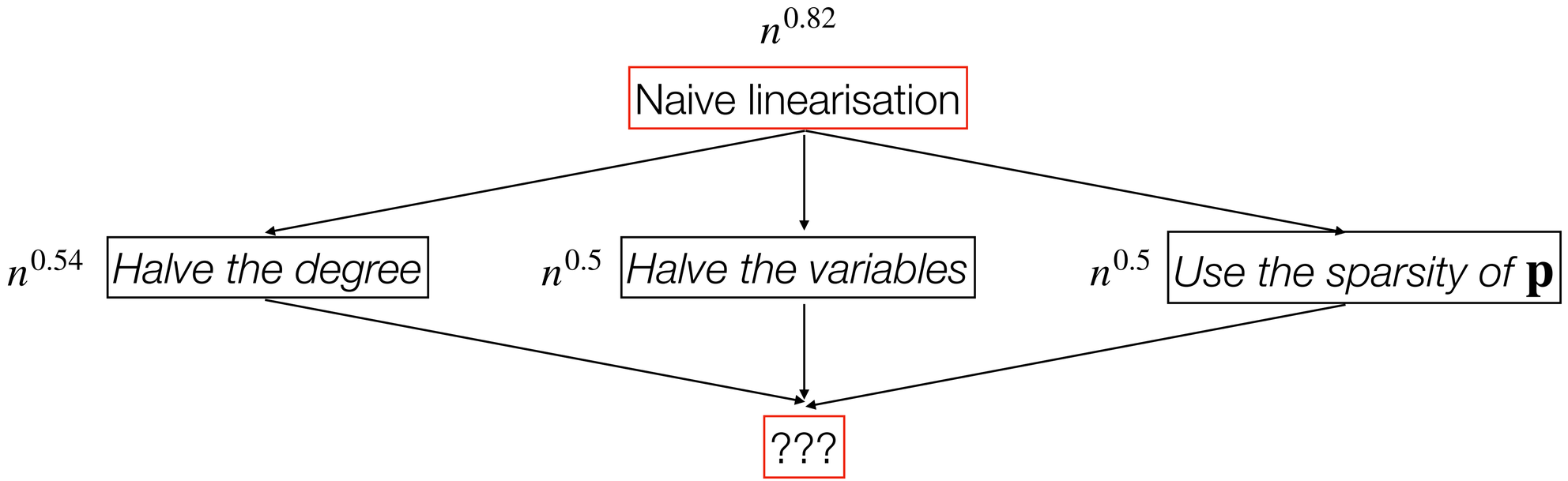
- Communication: $\approx 2^D \approx 2^{m/2} \approx n^{0.5}$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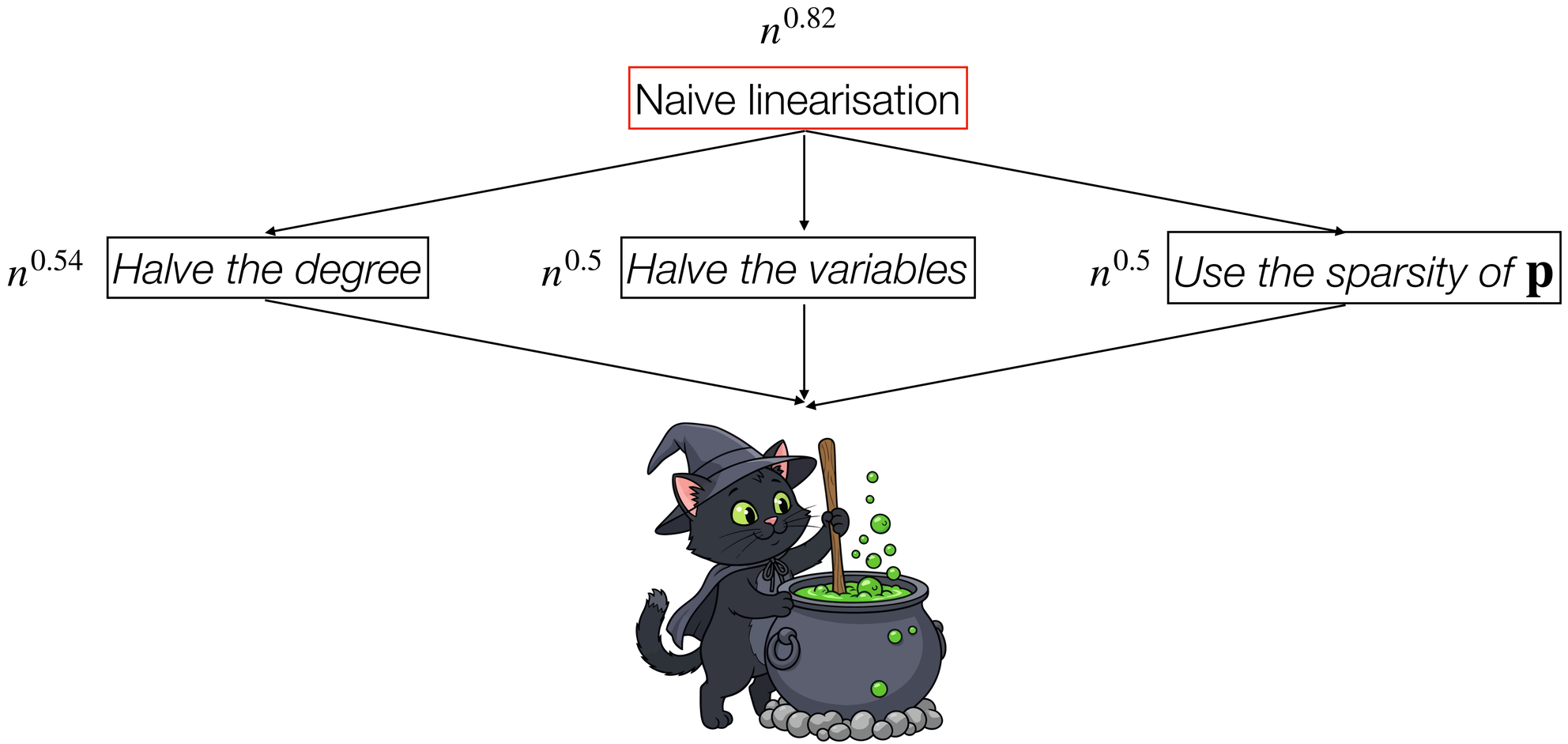
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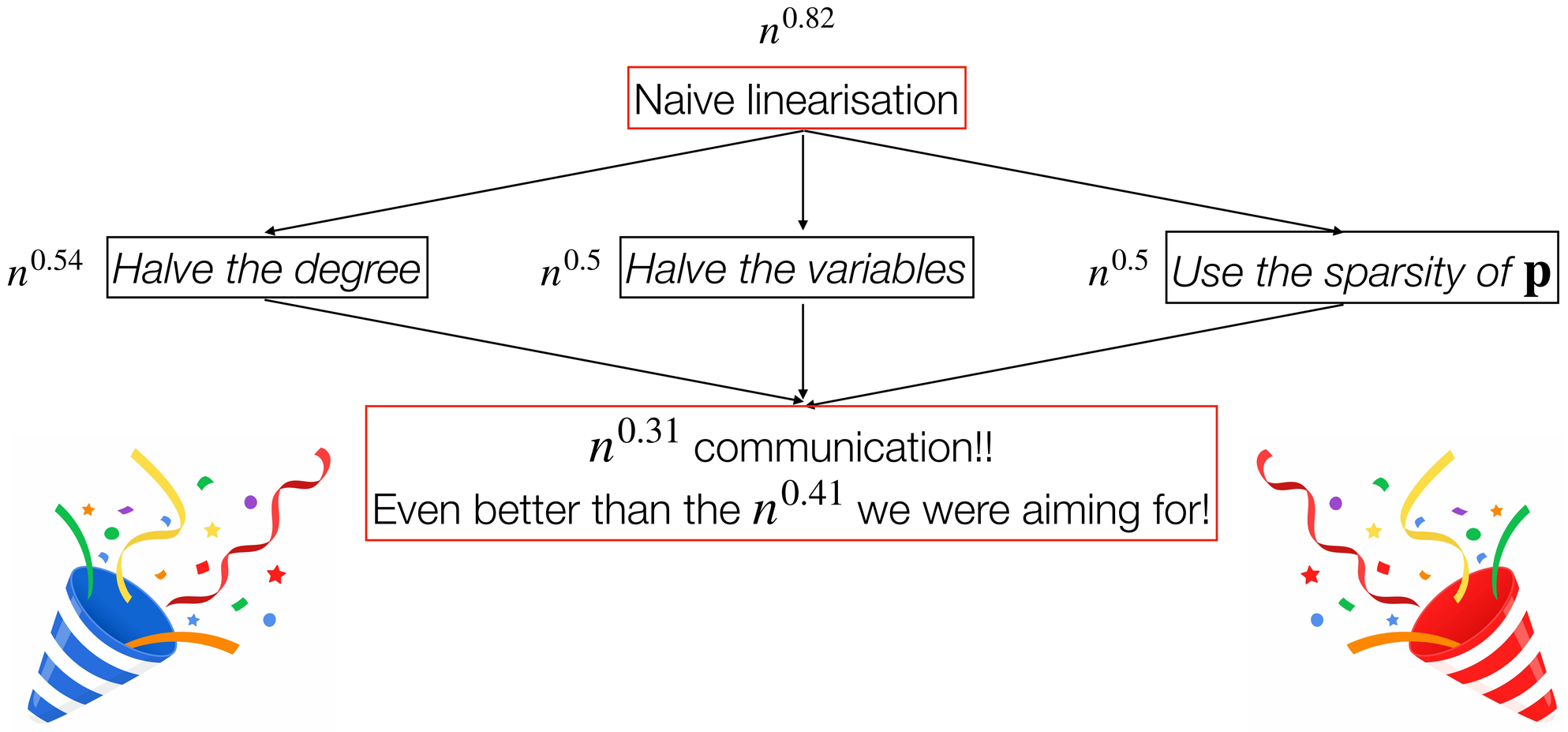
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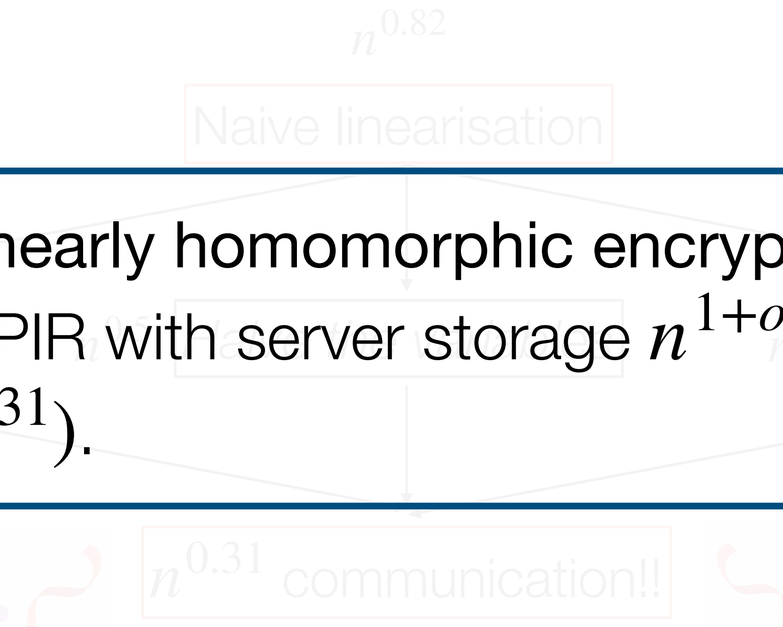
Use the sparsity of $\mathbf{p}$
$n^{0.31}$ communication!!
Even better than the $n^{0.41}$ we were aiming for!

Theorem: with compact **linearly homomorphic encryption** [known from DDH, DCR, QR, LWE], we get 2-server PIR with server storage $n^{1+o(1)}$, time per query $O(n^{0.82})$ and communication $O(n^{0.31})$.



Bonus Slides on ≥ 3 Servers

$s = 2$ Servers: A Refresher

Query: $\mathbf{L}(0) = \mathbf{r}$

Query: $\mathbf{L}(1) = \mathbf{p} + \mathbf{r}$

Cheatsheet

Field: $\mathbb{F}_2$

f_{DB} multilinear

$\mathbf{L}(t) = \mathbf{r} + t \cdot \mathbf{p}$

$\binom{m}{D} \geq n$

$s = 2$ Servers: A Refresher

Query: $\mathbf{L}(0) = \mathbf{r}$

Ans: $\nabla^{\leq D/2} f_{\mathsf{DB}}(\mathbf{L}(0))$

Query: $\mathbf{L}(1) = \mathbf{p} + \mathbf{r}$

Ans: $\nabla^{\leq D/2} f_{\mathsf{DB}}(\mathbf{L}(1))$

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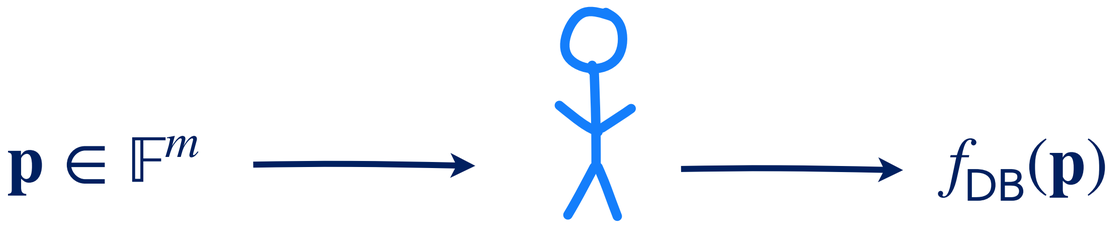

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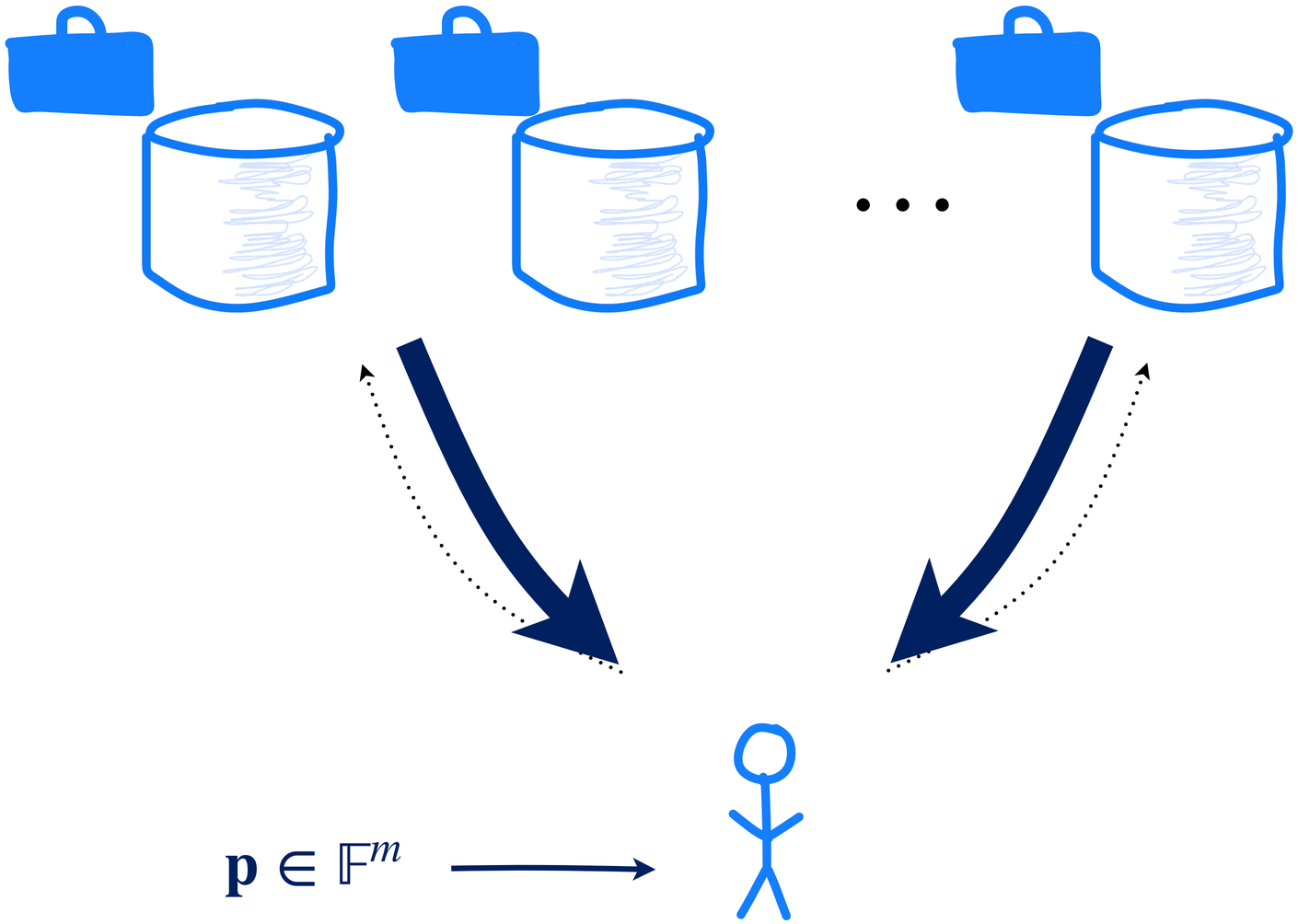
$s > 2$ Servers: What Changes? [GLMDS25]



Cheatsheet

Field: $\mathbb{F}_q$ (for $q \geq s$)

f_{DB} multilinear

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$\binom{m}{D} \geq n$

$s > 2$ Servers: What Changes? [GLMDS25]

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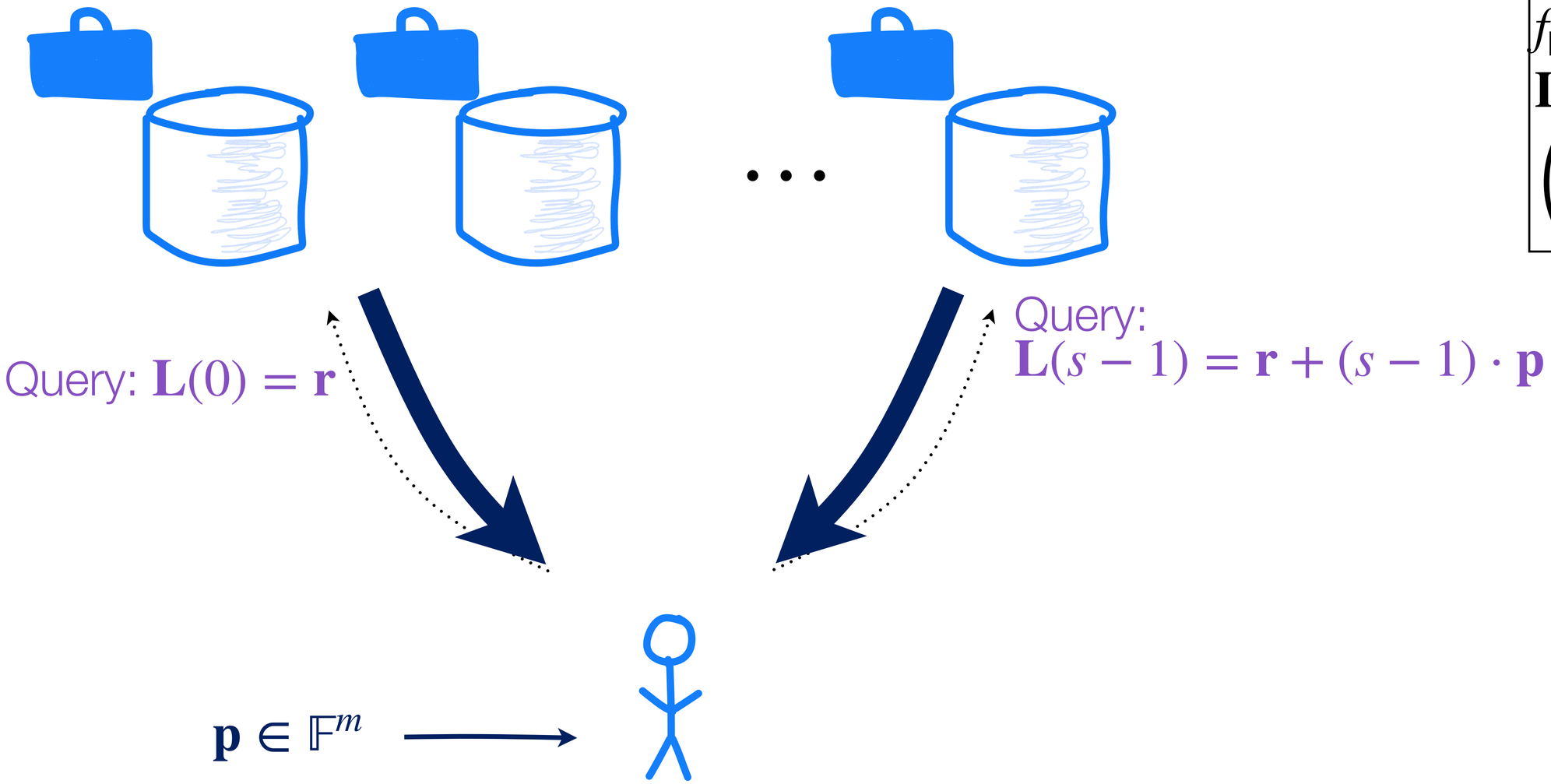

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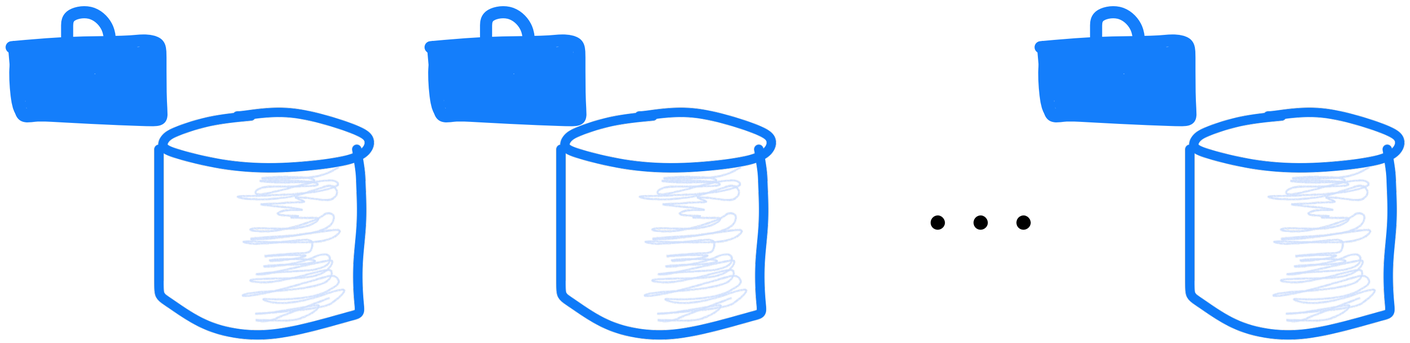

Query: $\mathbf{L}(0) = \mathbf{r}$

Ans: $\nabla^{\leq D/s} f_{\mathsf{DB}}(\mathbf{L}(0))$

Query: $\mathbf{L}(s-1) = \mathbf{r} + (s-1) \cdot \mathbf{p}$

Ans: $\nabla^{\leq D/s} f_{\mathsf{DB}}(\mathbf{L}(s-1))$

$\mathbf{p} \in \mathbb{F}^m$

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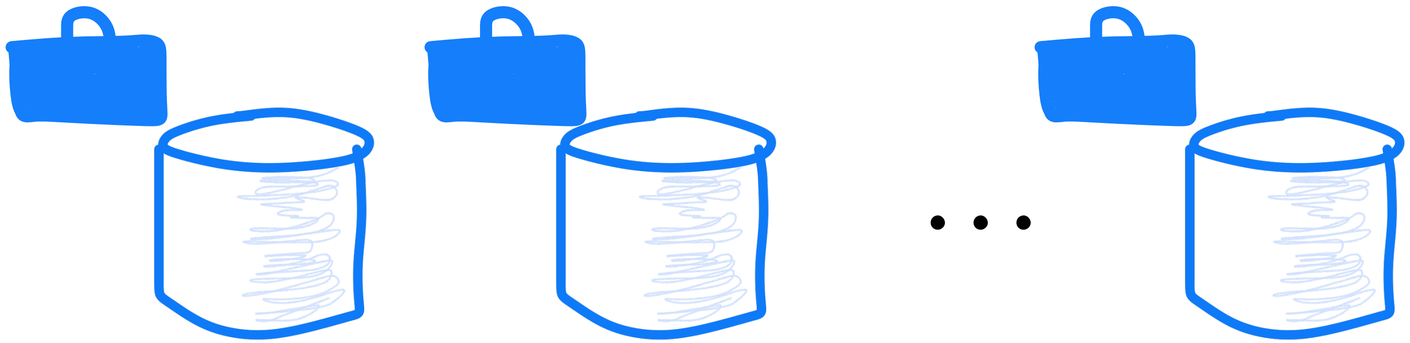

Query: $\mathbf{L}(0) = \mathbf{r}$

Ans: $\nabla^{\leq D/s} f_{\mathsf{DB}}(\mathbf{L}(0))$

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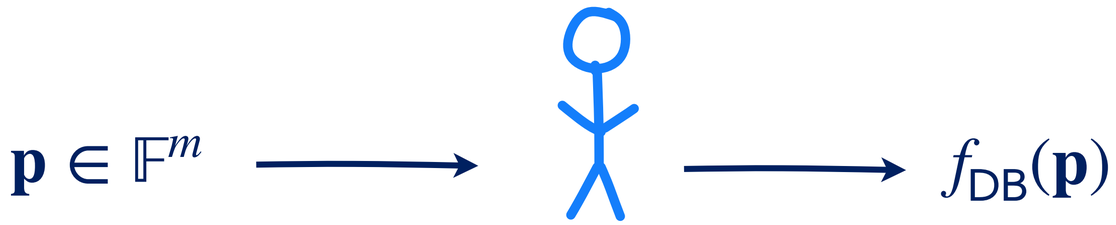

Collusion Resistance via Shamir Secret Sharing [BIK05]

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$\mathbf{p} \in \mathbb{F}^m$ → $f_{\mathsf{DB}}(\mathbf{p})$

Cheatsheet

Field: $\mathbb{F}_q$ (for $q \geq s$)

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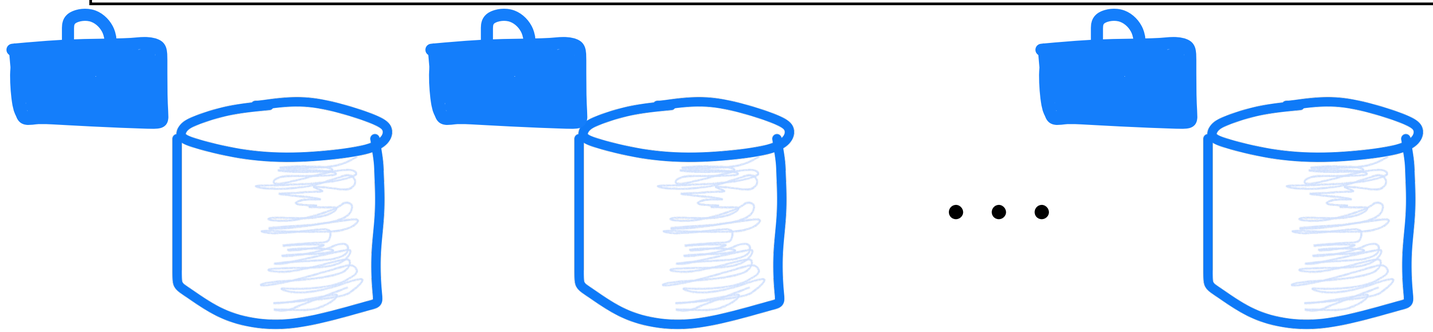

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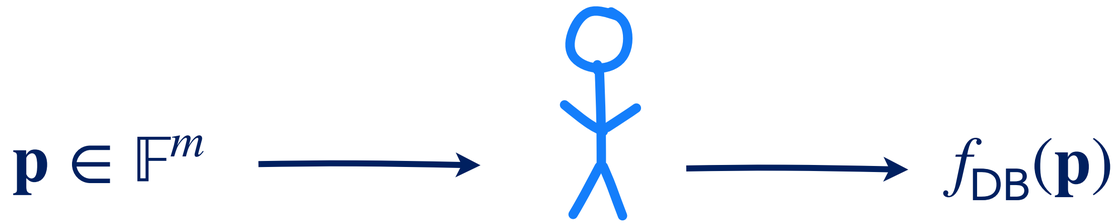

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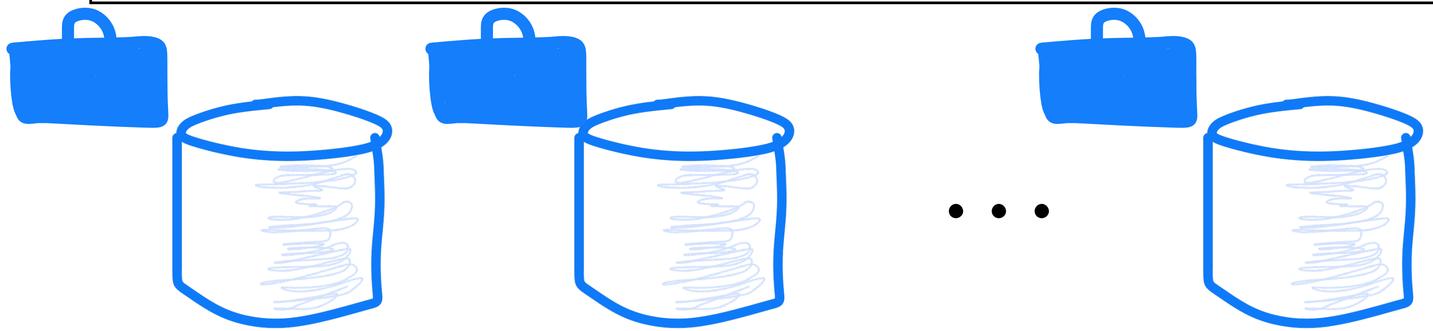

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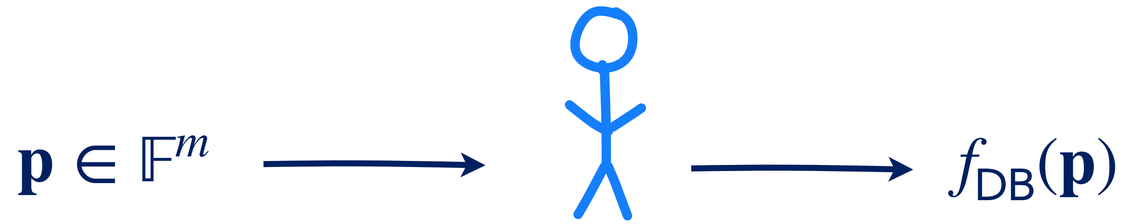

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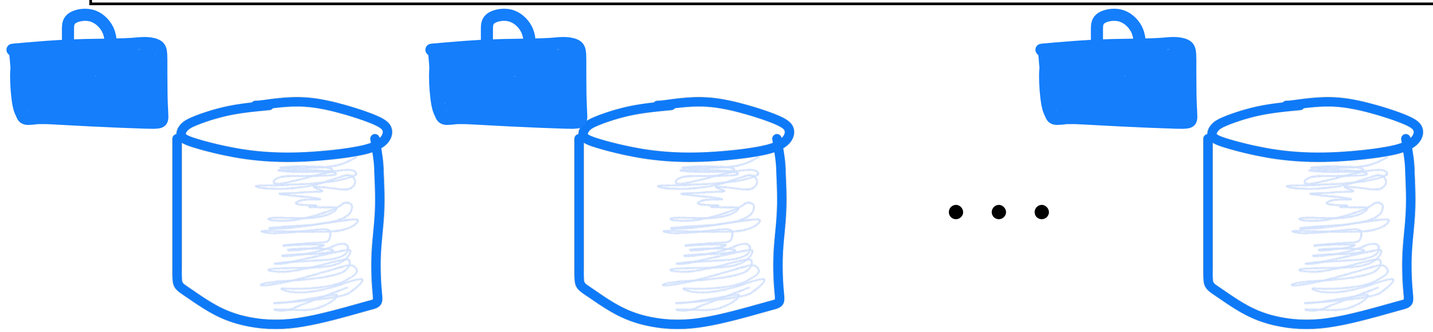

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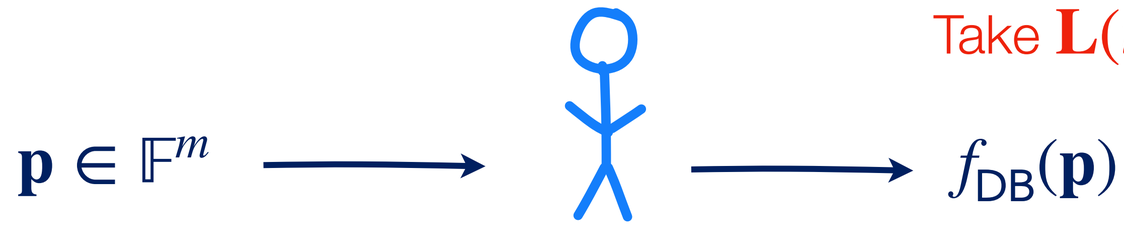

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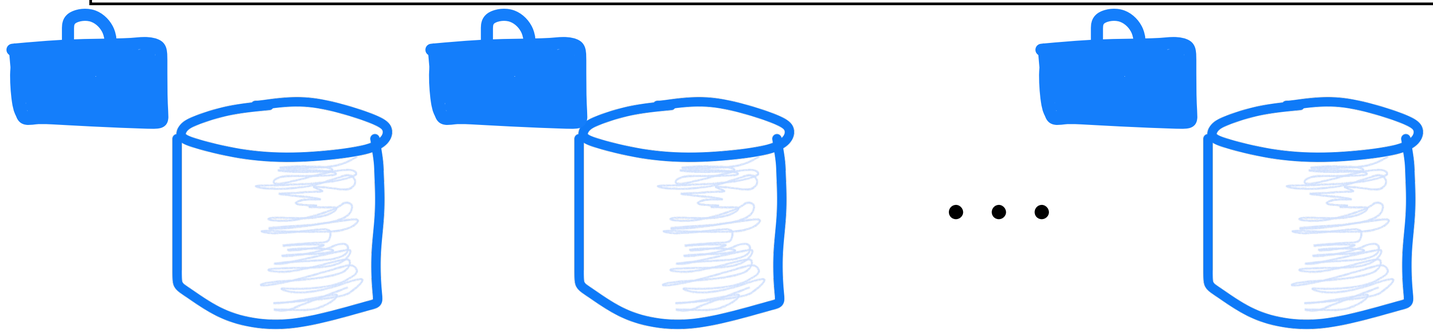

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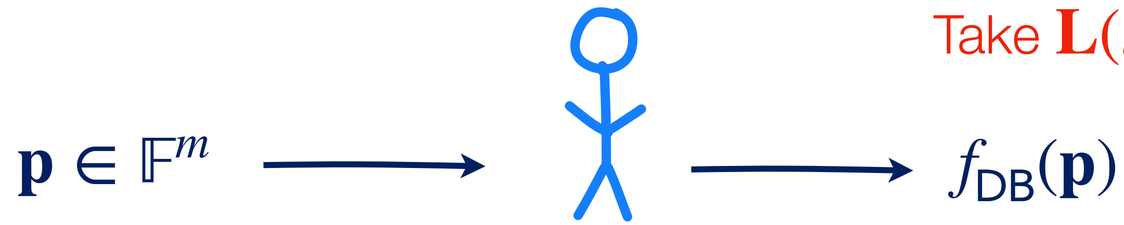

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- **TLDR: the sweet spot for d increases as the number of servers increases**

Special Case: Storage $n^{1+o(1)}$, Individual Degree $s-1$

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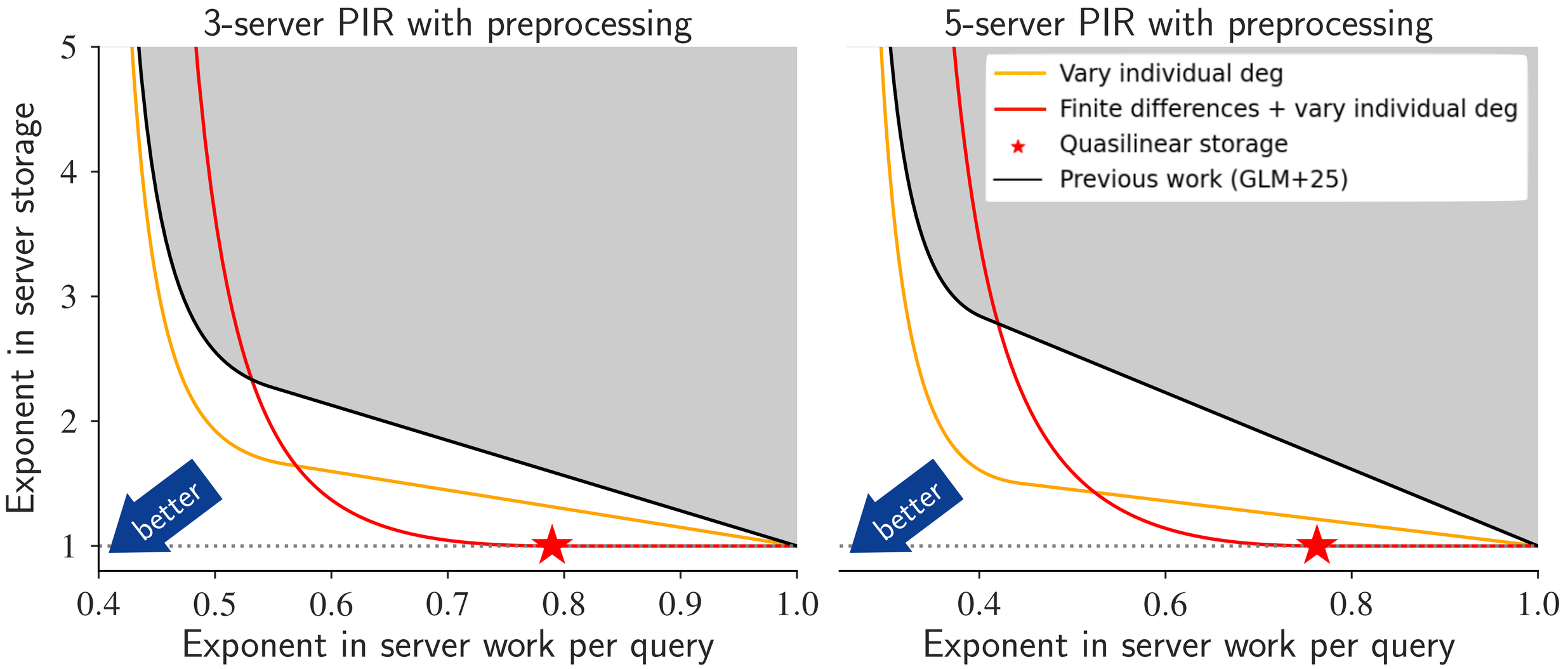
3-server PIR with preprocessing
5-server PIR with preprocessing
Exponent in server storage
Exponent in server work per query
Vary individual deg
Finite differences + vary individual deg
Quasilinear storage
Previous work (GLM+25)
better
better
5
4
3
2
1
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Bonus Slides on Locally Decodable Codes

Smooth Locally Decodable Codes [KT00]

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Cheatsheet

n: message length

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Smooth Locally Decodable Codes [KT00]

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  - Randomised $\mathsf{Query}(i \in [n]) \to \mathsf{qu} \in [\ell]^q$
  - $\mathsf{Decode}(i \in [n], \mathsf{qu} \in [\ell]^q, c[\mathsf{qu}]) \to \mathsf{msg}[i]$
- **Smoothness:** marginal distribution of qu_j is uniform over $[\ell]$ for all j

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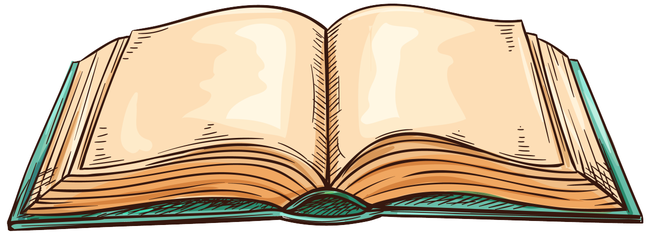
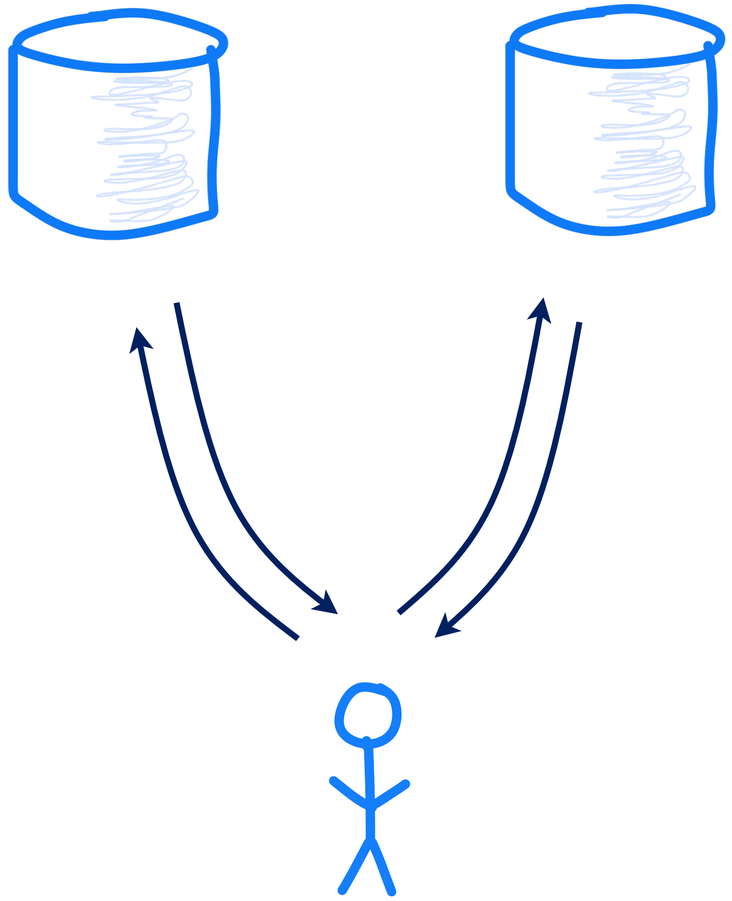
2-query LDCs → 2-server PIR





LDCs	PIR
Message	DB
Encoding	Preprocessing
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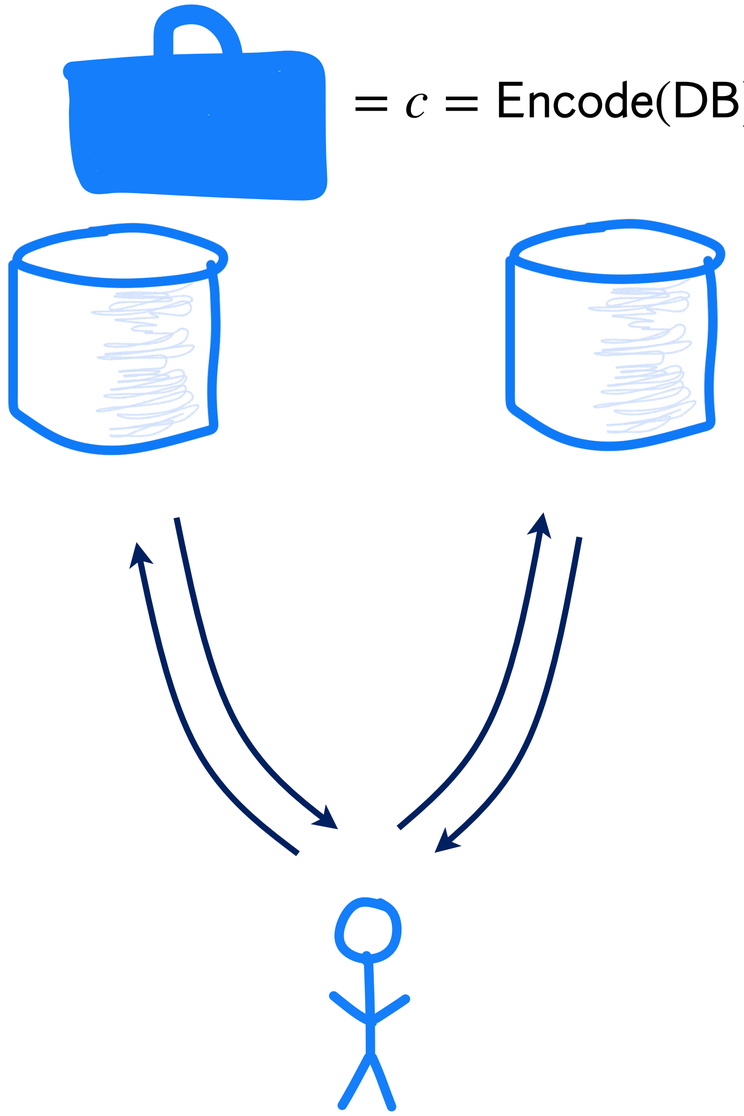

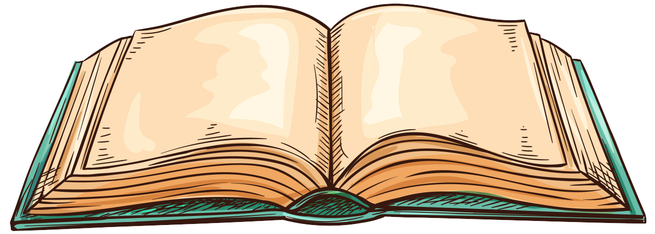

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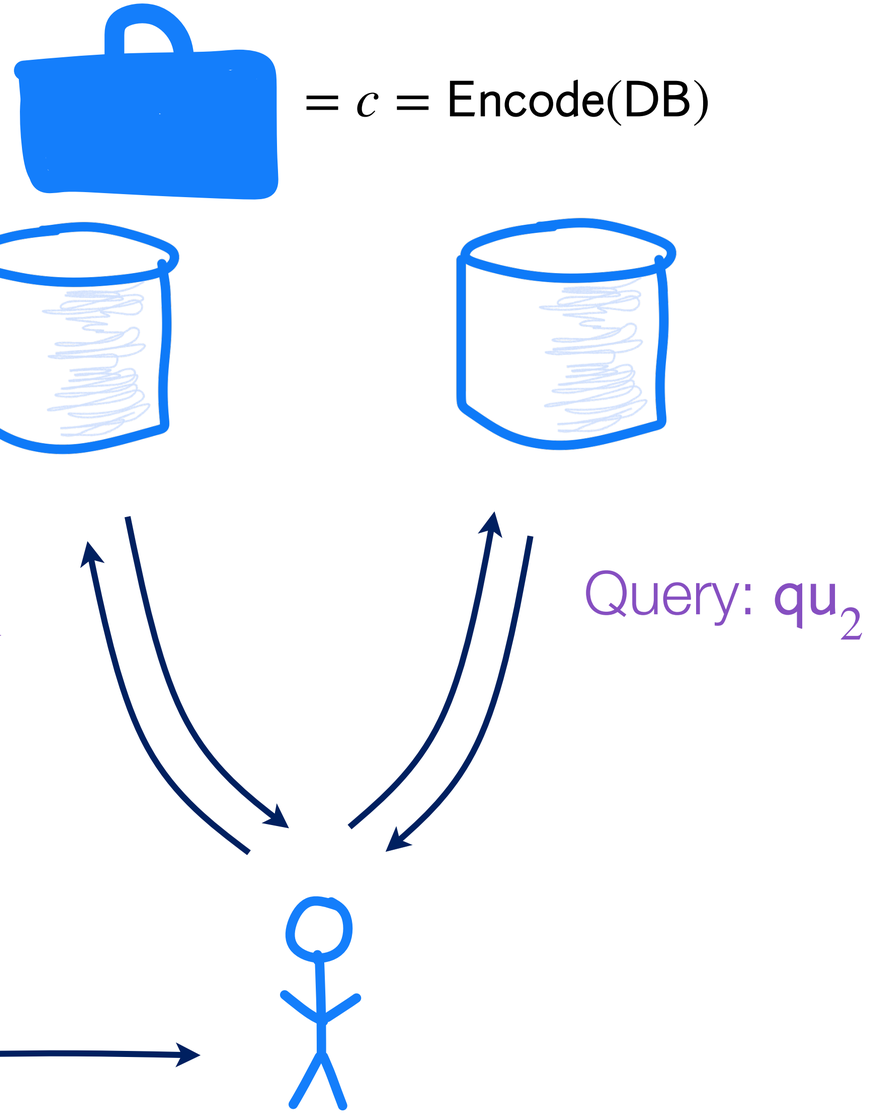

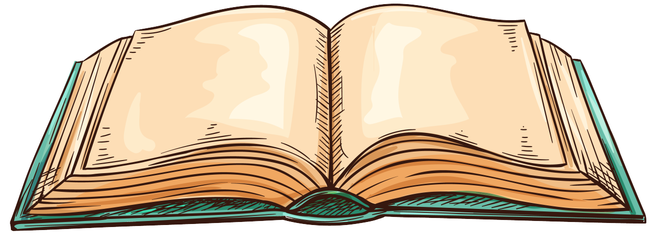

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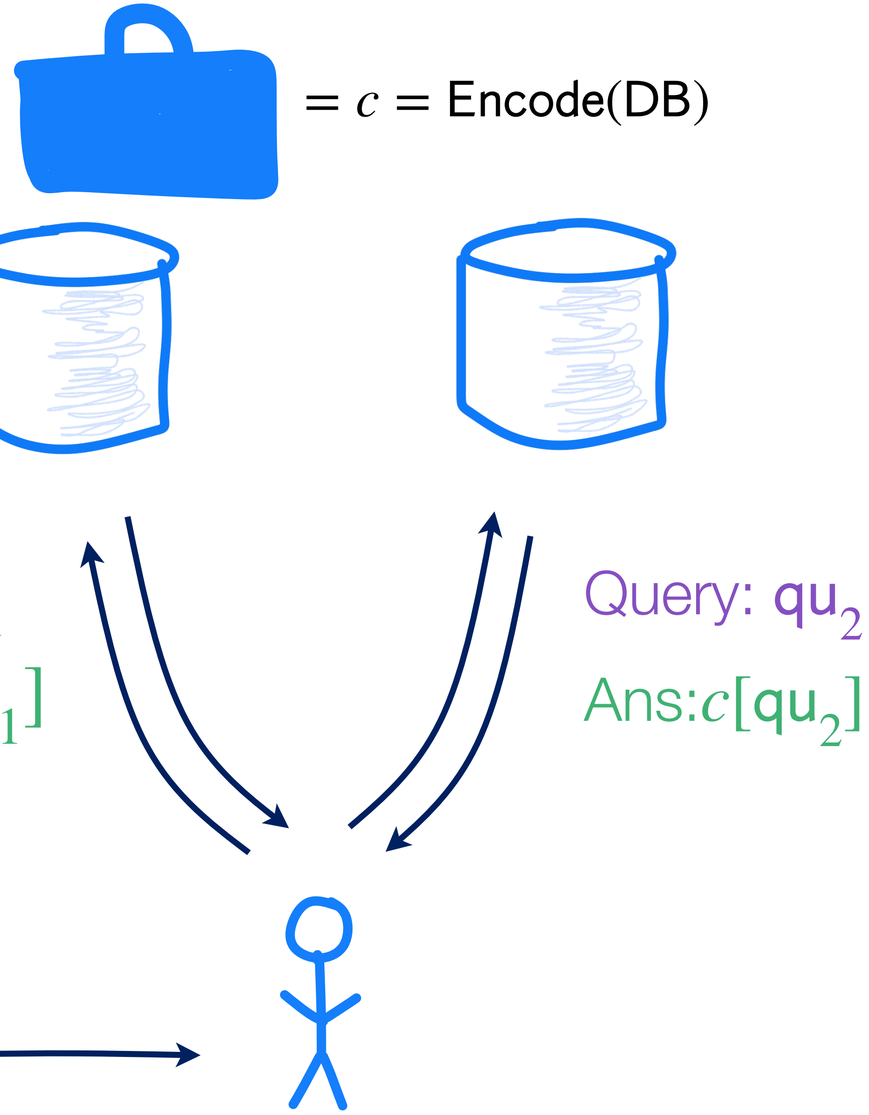

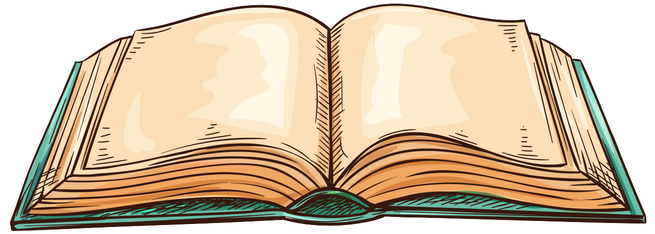

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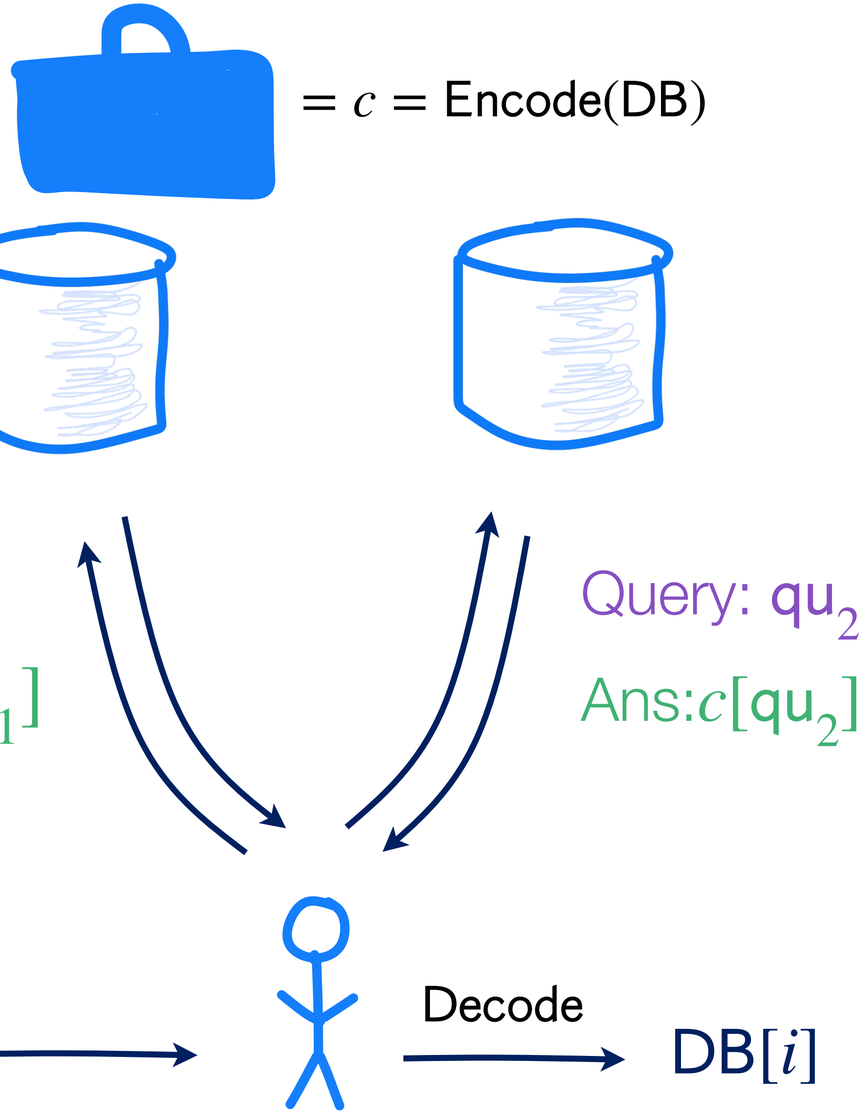

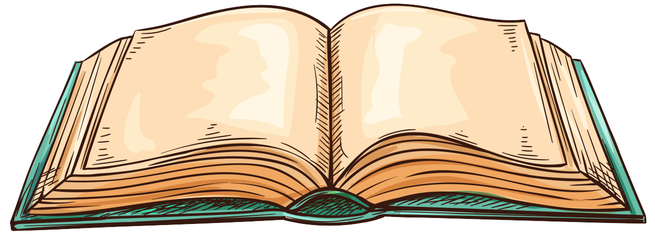

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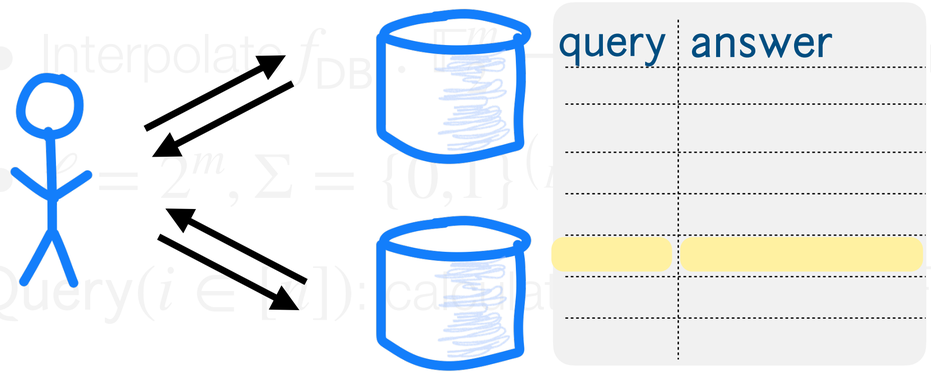
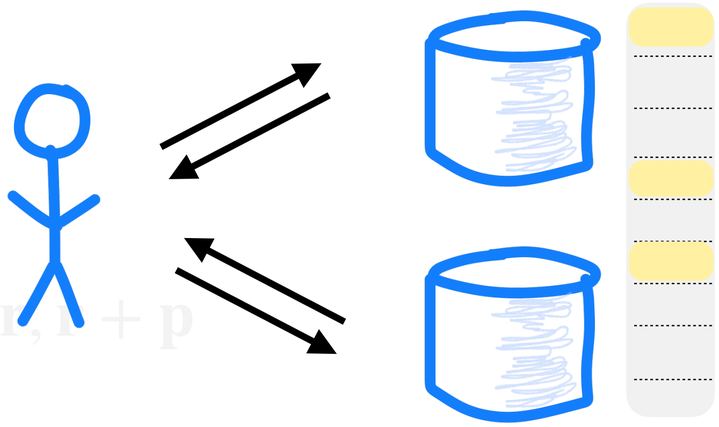
Casting BIM00 PIR as an LDC

LDCs and BIM00 PIR



Our PIR



Q: Does our finite differences technique imply a new 2-query LDC?
A: Actually no! LDCs are rigid: they require you to separately write out the answer for every query

Defining Batch-Smooth LDCs

Cheatsheet

n: message length

ℓ: codeword length

Σ: alphabet

q: locality

b: number of batches

Defining Batch-Smooth LDCs

- Three functions:
  - $\mathsf{Encode}(\mathsf{msg} \in \{0,1\}^n) \to c \in \Sigma^\ell$
  - Randomised $\mathsf{Query}(i \in [n]) \to \mathsf{qu} \in [\ell]^q$
  - $\mathsf{Decode}(i \in [n], \mathsf{qu} \in [\ell]^q, c[\mathsf{qu}]) \to \mathsf{msg}[i]$
- **Smoothness:** marginal distribution of qu_j is uniform over $[\ell]$ for all j

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- **Batch-smoothness:** reorganise the queries into b batches:

1	$\mathsf{qu}_1, \ldots, \mathsf{qu}_{q/b}$
2	$\mathsf{qu}_{q/b+1}, \ldots, \mathsf{qu}_{2q/b}$
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Then each row's distribution should be independent of the index i being queried

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BIM00 PIR as a Batch-Smooth LDC

Cheatsheet

n: message length

$$\binom{m}{D} \geq n$$

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Our PIR as a Batch-Smooth LDC

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Consequence: the first batch-smooth LDC with constant alphabet size, constant number of batches b, codeword length $n^{1+o(1)}$, and polynomially sublinear number of queries $q = n^{1-\Omega(1)}$