Parallel Spooky Pebbling Makes Regev Factoring More Practical

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Quantum integer factorization

Shor ’94: Factoring n-bit integers in poly(n) time on a quantum computer

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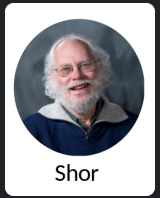
- What is the precise complexity of quantum factoring?
- What are the time/space tradeoffs?
- **What does it take in practice?**

Circuits for factoring general-form numbers



Core of Shor's algorithm

To factor an n-bit number N, run quantum period finding on

$$f(x) = a^x \bmod N$$

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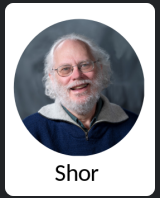

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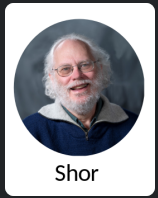

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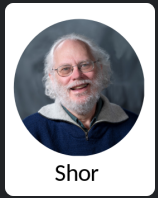
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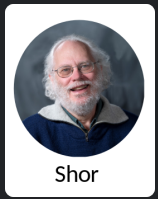
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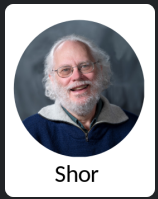
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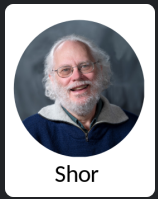
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↳ Gidney '25	[concrete improvements]	

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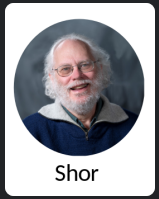

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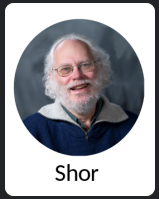

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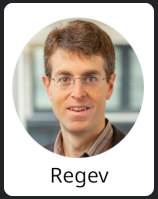

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Roadmap

Previous approaches to Regev

Pebble games

Our work: parallel spooky pebble games

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Cost of computing $a^x \bmod N$

Example: how to compute $a^{11} \bmod N$

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Cost: $O(\log x)$ multiplications of n-bit integers

Regev's gate savings

→ Computing $a^x \bmod N$ costs $O(\log x)$ multiplications

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Making repeated squaring quantum

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$$a \to a^2 \to a^4 \to a^5 \to a^{10} \to a^{11} \pmod{N}$$

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Quantum repeated squaring?

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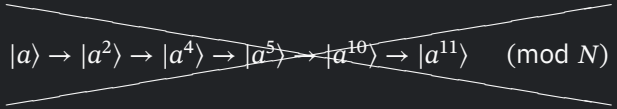

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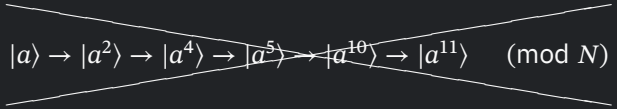

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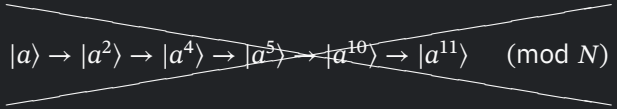

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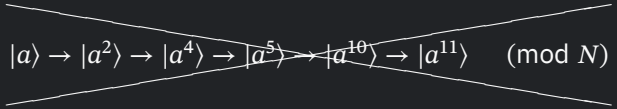

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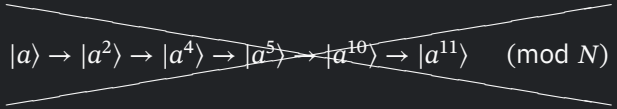

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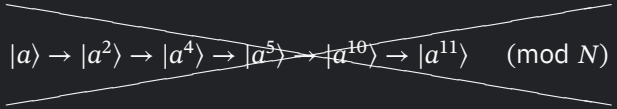

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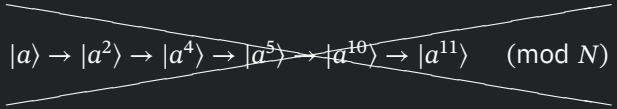

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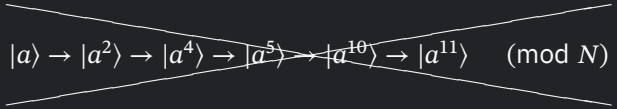

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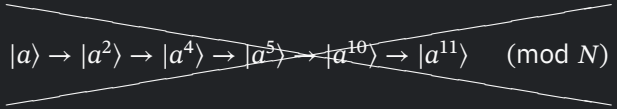

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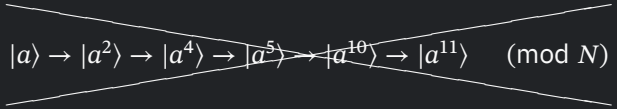

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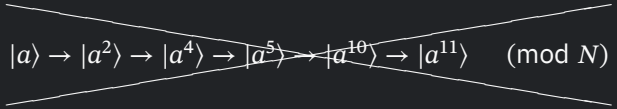

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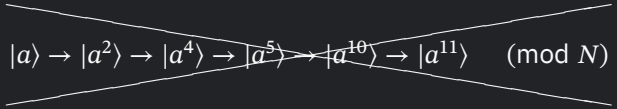

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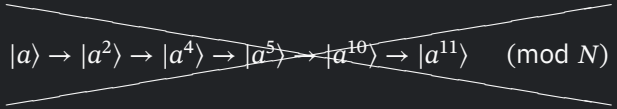

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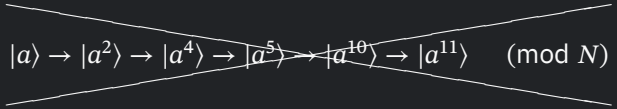

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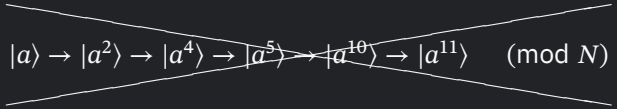

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Steps are ❌ *out-of-place* but ✅ *cheap*

Irreversible sequential algorithms

General problem: How to do $|x\rangle|0\rangle \to |x\rangle|f(x)\rangle$ efficiently when computing f uses many irreversible steps?

Option 1

Find an algorithm for f with *reversible steps*

Shor:
Steps are ✅ *in-place* and ✅ *cheap*

Regev + RV24:
Steps are ✅ *in-place* but ❌ *expensive*

Option 2

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This talk:
Better manage Regev's intermediate values!

Roadmap

Previous approaches to Regev

Pebble games

Our work: parallel spooky pebble games

Pebble games for reversible computation

[Bennett ’89] introduced pebble games:

Let $f(x) = f_k(\cdots f_2(f_1(x))\cdots)$
and $x_i = f_i(\cdots)$

x x_1 x_2 x_3 $f(x)$

State: $|x\rangle\,|0\rangle\,|0\rangle\,|0\rangle\,|0\rangle$

Pebble games for reversible computation

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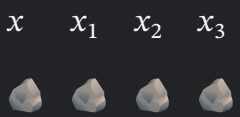

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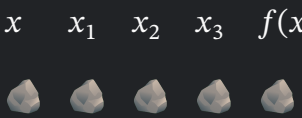

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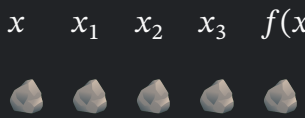

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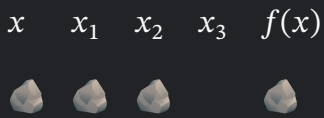

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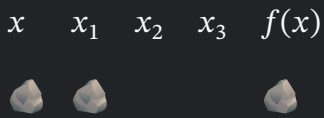

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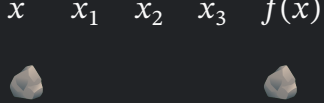

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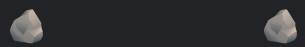

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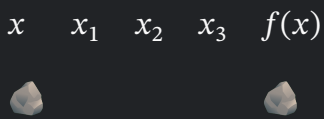

Naive strategy

Time cost (# steps): $2k - 1$ (optimal) 😃

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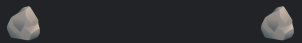

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Time cost (# steps): $2k - 1$ (optimal) 😃
Space (# pebbles): $O(k)$ 😭

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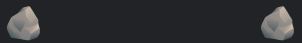

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→ Regev: $k = O(\sqrt{n})$ and n qubits per pebble ⇒ $O(n^{3/2})$ qubits

Using less space: a recursive strategy

1. Pebble game from start to $k/2$

x $\cdots$ $x_{k/2}$ $\cdots$ $f(x)$

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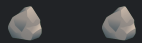

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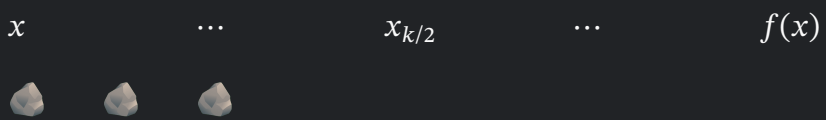

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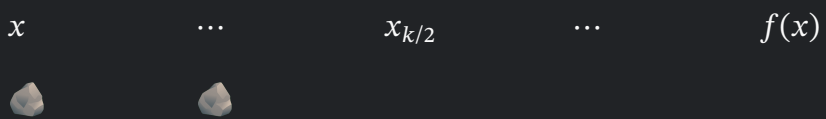

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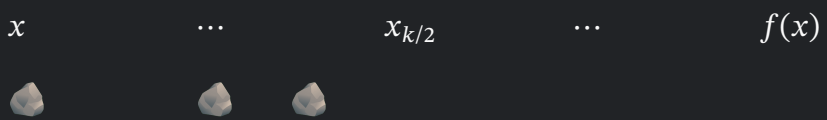

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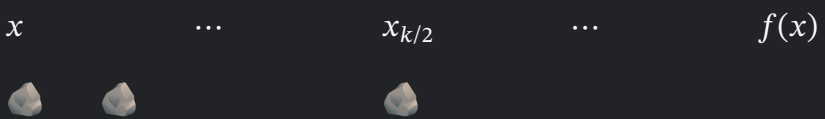

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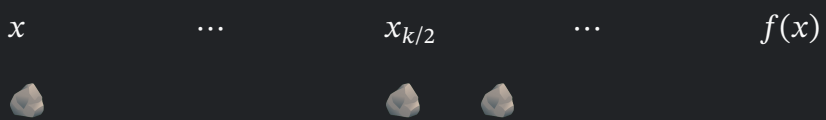

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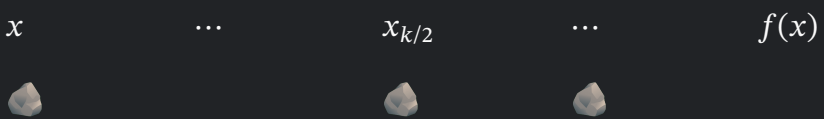

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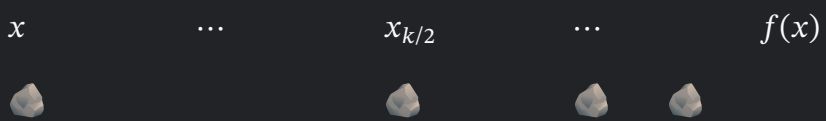

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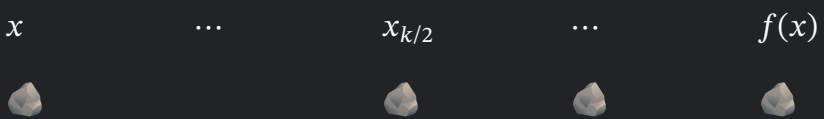

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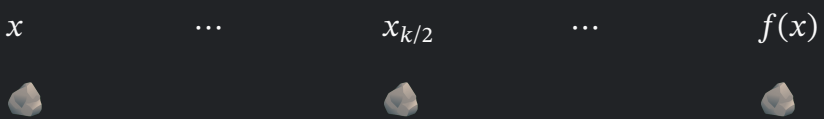

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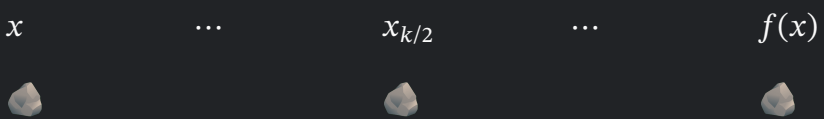

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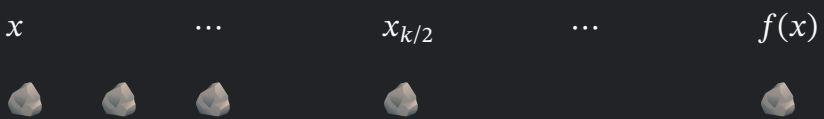

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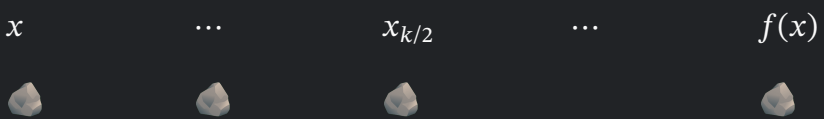

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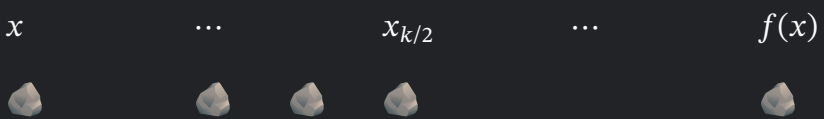

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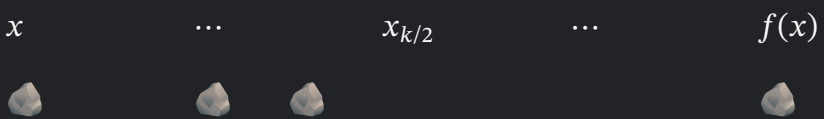

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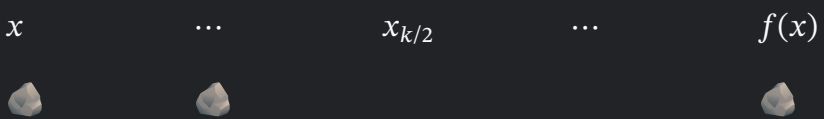

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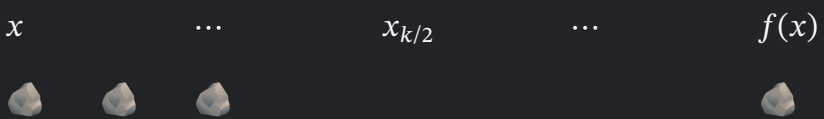

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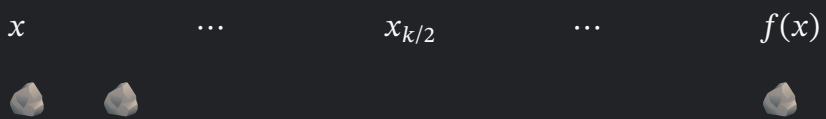

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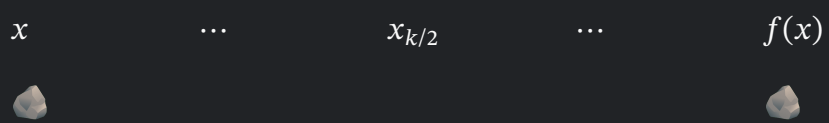

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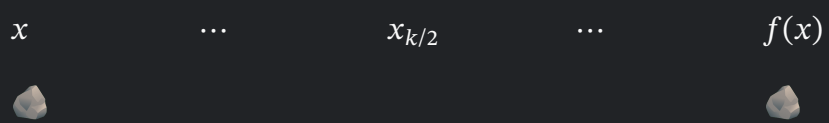

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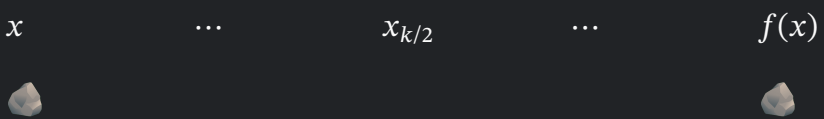

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Time cost (# steps): $T(k) = 3T(k/2)$

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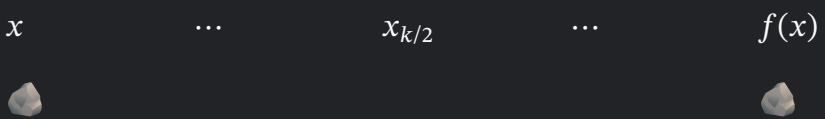

Space usage (# pebbles): $O(\log k)$ 😃

Time cost (# steps): $O(k^{\log_2 3}) \approx O(k^{1.58\cdots})$ steps 😢

Using less space: a recursive strategy

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2. Pebble game from $k/2$ to k
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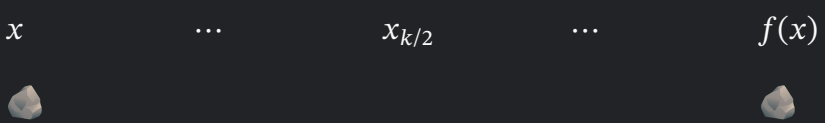


Space usage (# pebbles): $O(\log k)$ 😃

Time cost (# steps): $O(k^{\log_2 3}) \approx O(k^{1.58\cdots})$ steps 😢
Generalization: space $O(2^{1/\epsilon} \log k)$, time $O(k^{1+\epsilon})$ 😐

Changing the rules of the game

Resources at our disposal:

Changing the rules of the game

Resources at our disposal:

1. Parallelism

Changing the rules of the game

Resources at our disposal:

1. Parallelism
2. The spirit realm

Changing the rules of the game

Resources at our disposal:

1. Parallelism
2. ~~The spirit realm~~ Quantum!

Changing the rules of the game

Resources at our disposal:

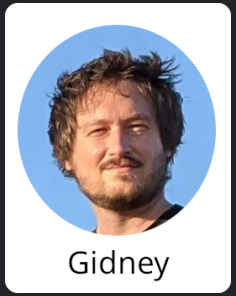
1. Parallelism
2. ~~The spirit realm~~ Quantum!

Pebble games using these resources *independently* have been explored—
what if we combine them?

Spookiness: a quantum resource

“Spooky Pebble Games and Irreversible Uncomputation”
algassert.com/post/1905

Spookiness: a quantum resource



"Spooky Pebble Games and Irreversible Uncomputation"
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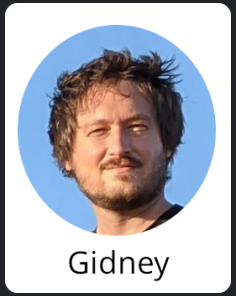
Using measurement to uncompute

Given an intermediate value

$$|x_i\rangle$$

Spookiness: a quantum resource



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what if we apply $H^{\otimes n}$ and then measure it?

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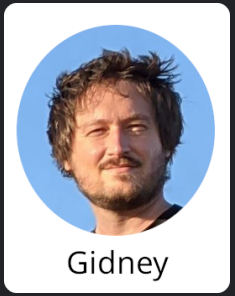

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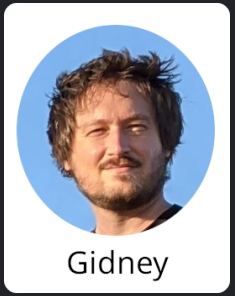
$$(-1)^{d \cdot x_i}$$

where d is *classically-known*.

Spookiness: a quantum resource



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where d is *classically-known*.

We've turned $|x_i\rangle$ into a ghost!

Spookiness: a quantum resource

Ghosting: replacing $|x_i\rangle$ with phase error $(-1)^{d \cdot x_i}$ for *classically-known* d

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- Need x_i again eventually to fix phase

Rules:

- can only place or remove a pebble if there is a pebble to its left
- can ghost a pebble at any time—but must exorcise later
- exorcism is free when x_i has a pebble

The power of spooky pebbling

Recursive strategy

1. Pebble game from start to $k/2$
2. Pebble game from $k/2$ to k
3. Pebble game from start to $k/2$

x $\cdots$ $x_{k/2}$ $\cdots$ $f(x)$

The power of spooky pebbling

Spooky recursive strategy

1. Spooky pebble game from start to $k/2$
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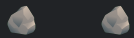

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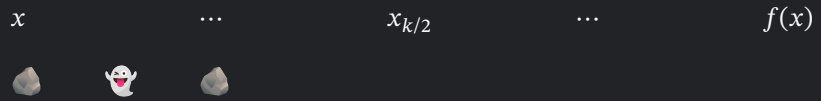

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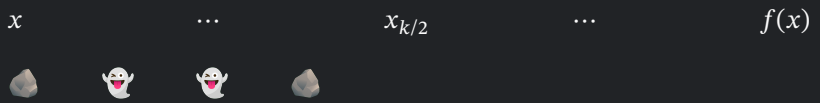

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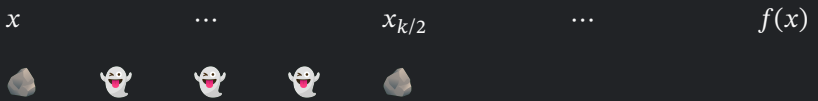

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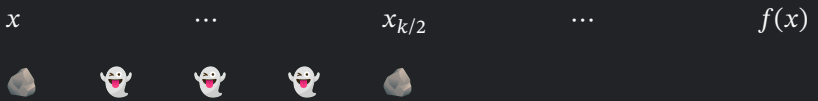

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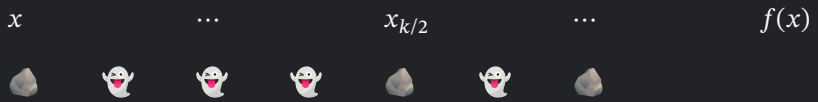

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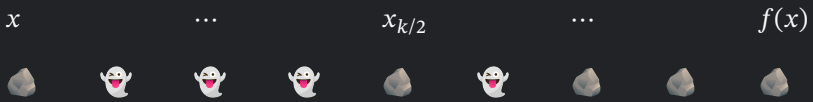

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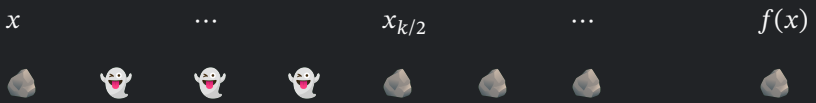

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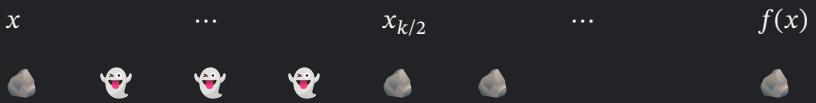

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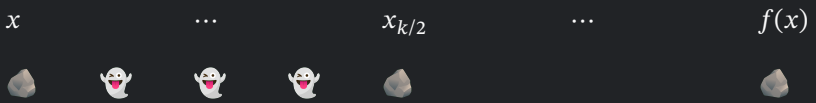

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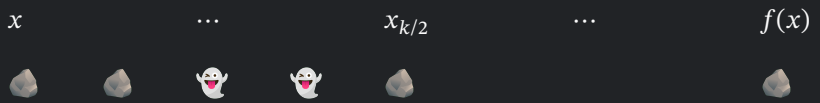

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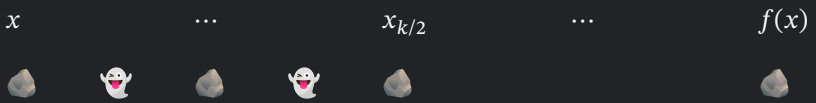

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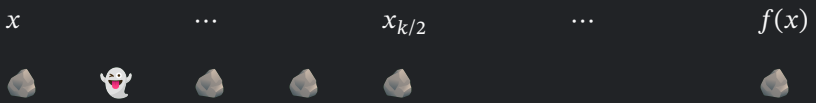

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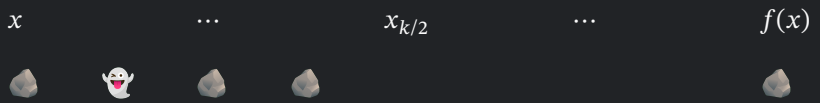

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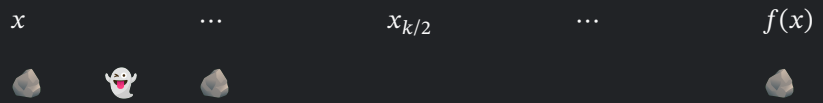

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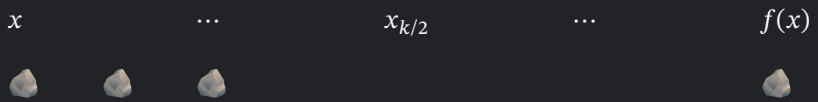

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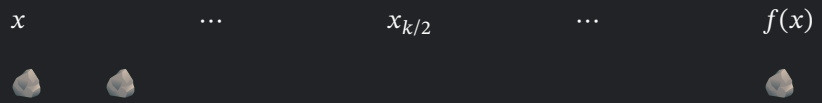

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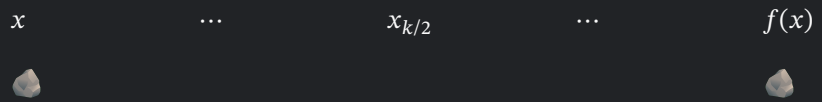

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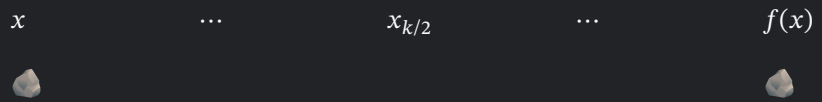

Space: $O(\log k)$ pebbles 😃 **Time cost (# steps):**

The power of spooky pebbling

Spooky recursive strategy

1. *Blast* from start to $k/2$
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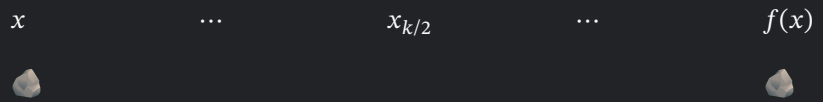

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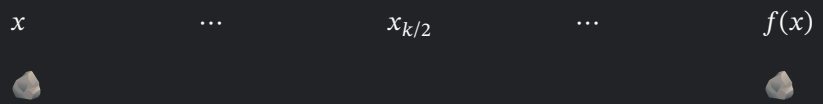

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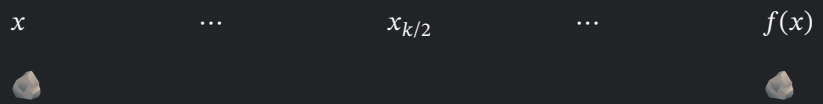

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $T(k) = O(k) + 2T(k/2)$

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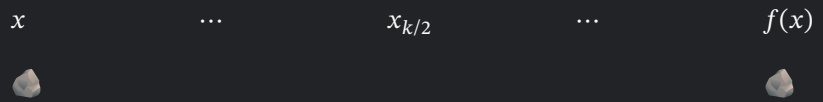

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $O(k \log k)$ steps 🤩

The power of spooky pebbling

Algorithm	Parallel	Spooky	Space	Depth
Trivial	N	N	k	$2k-1$
Bennett ’89	N	N	$O(2^{1/\epsilon} \cdot \log k)$	$O(k^{1+\epsilon})$
BHL ’22	Y	N	$O(4^{\sqrt{\log k}})$	$O(k)$
Gidney ’19 KSS ’25	N	Y	$O(\log k)$	$O(k \log k)$

The power of spooky pebbling

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Bennett ’89	N	N	$O(2^{1/\epsilon} \cdot \log k)$	$O(k^{1+\epsilon})$
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Is the best of both worlds possible?

Roadmap

Previous approaches to Regev

Pebble games

Our work: parallel spooky pebble games

Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 0

Max. pebble count: 1

Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 1

Max. pebble count: 2

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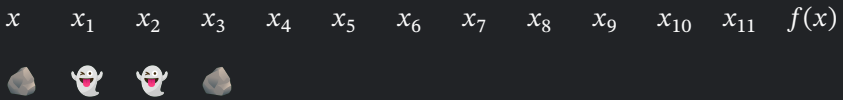
x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 2

Max. pebble count: 2

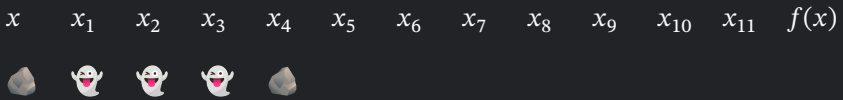
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 3

Max. pebble count: 2

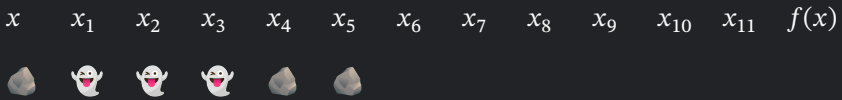
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 4

Max. pebble count: 2

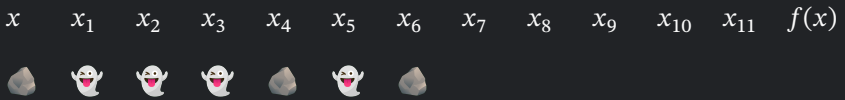
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 5

Max. pebble count: 3

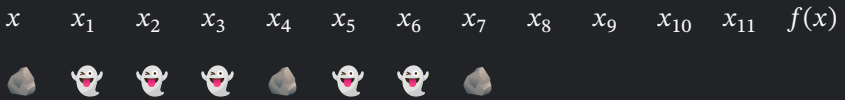
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 6

Max. pebble count: 3

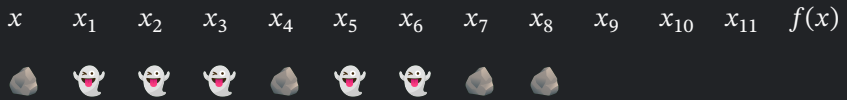
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 7

Max. pebble count: 3

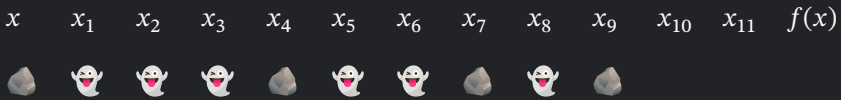
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 8

Max. pebble count: 4

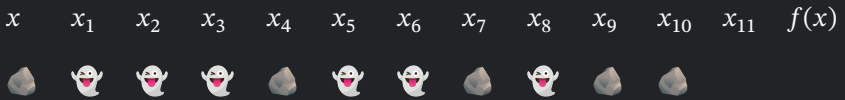
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 9

Max. pebble count: 4

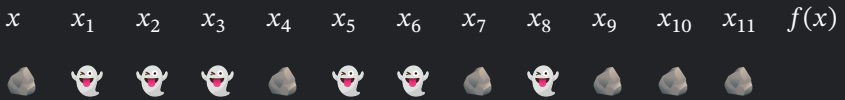
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 10

Max. pebble count: 5

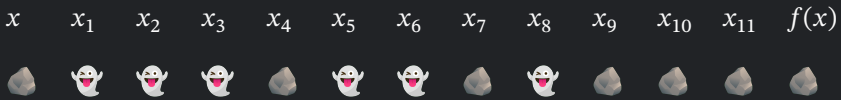
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 11

Max. pebble count: 6

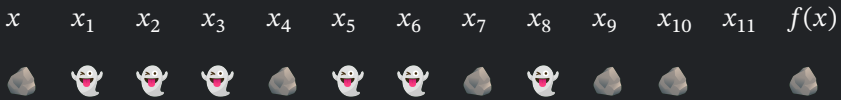
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 12

Max. pebble count: 7

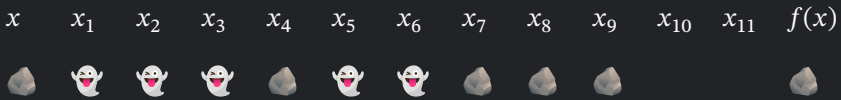
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 13

Max. pebble count: 7

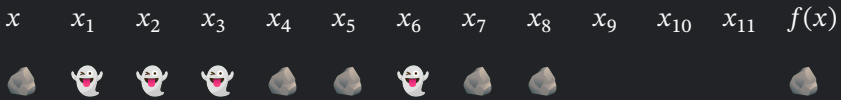
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 14

Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 15

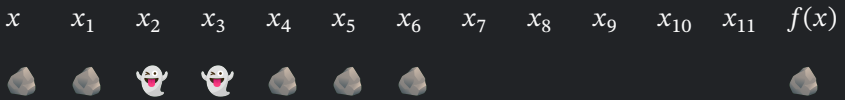
Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$

Step number: 16

Max. pebble count: 7

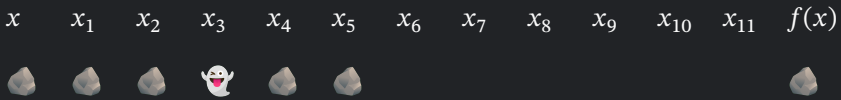
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 17

Max. pebble count: 7

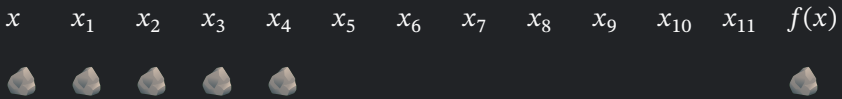
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 18

Max. pebble count: 7

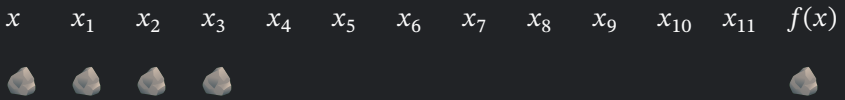
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 19

Max. pebble count: 7

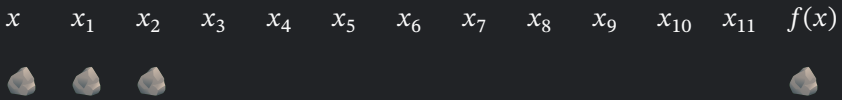
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 20

Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 21

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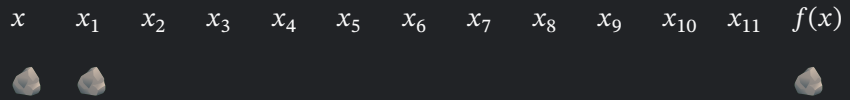
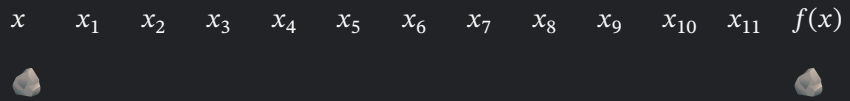
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 22

Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$



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x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: $23 = 2k - 1$, which is optimal 🤩

Max. pebble count: 7

Our results

Explicit algorithms for pebbling to length k:

Algorithm	Parallel	Spooky	Space	Depth
Trivial	N	N	k	$2k-1$
Bennett '89	N	N	$O(2^{1/\epsilon} \cdot \log k)$	$O(k^{1+\epsilon})$
BHL '22	Y	N	$O(4^{\sqrt{\log k}})$	$O(k)$
Gidney '19 KSS '25	N	Y	$O(\log k)$	$O(k \log k)$
Our work	**Y**	**Y**	$\approx 2.47 \log k$	$2k-1$

Automated search

What if we have too few pebbles for depth $2k - 1$?

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- Highly optimized **A* search** written in Julia

Automated search

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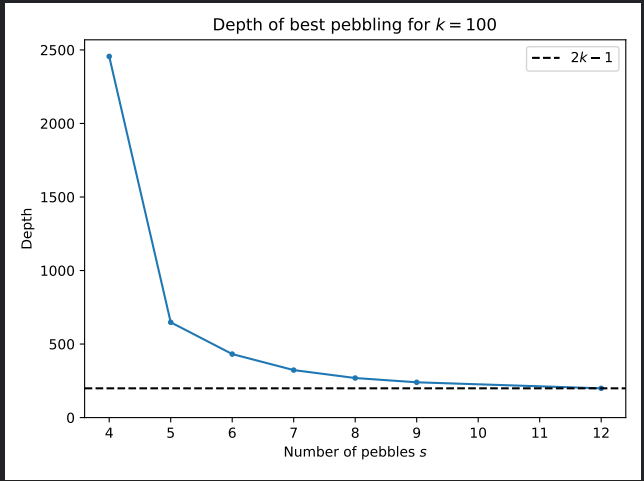
- Highly optimized **A* search** written in Julia
- On input number of pebbles s and length k, finds absolutely optimal solution



Factoring results

For factoring 4096-bit RSA:

- all depths counted in n-bit multiplications
- for previous estimates see: Ekerå + Gärtner, arXiv:2405.14381

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Note: this is shamelessly focusing on **depth** our best metric...

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- Concretely best Regev variant to date

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- Efficiently evaluating sequential algorithms is **widely applicable** beyond factoring
- **Generalization:** parallel spooky pebbling on arbitrary graphs

INTELLIGENCE COMMUNITY
POSTDOCTORAL RESEARCH
FELLOWSHIP PROGRAM

Jane Street

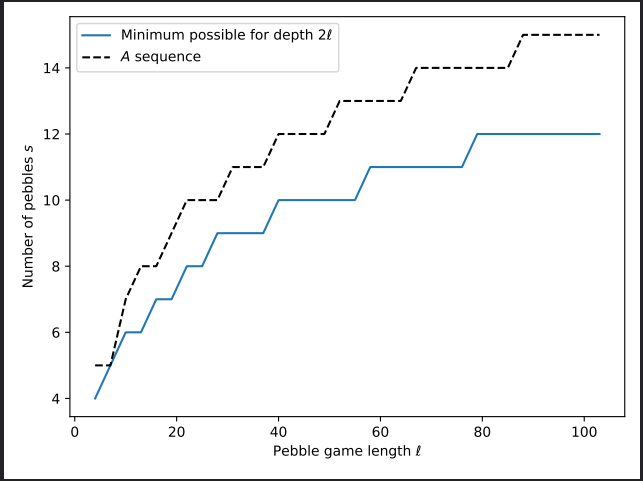
Thank you!

ePrint:2025/1887
arXiv:2510.08432

GDKM was supported by an appointment to the Intelligence Community Postdoctoral Research Fellowship Program at MIT administered by Oak Ridge Institute for Science and Education (ORISE) through an interagency agreement between the U.S. Department of Energy and the Office of the Director of National Intelligence (ODNI).

Backup

Numerical results



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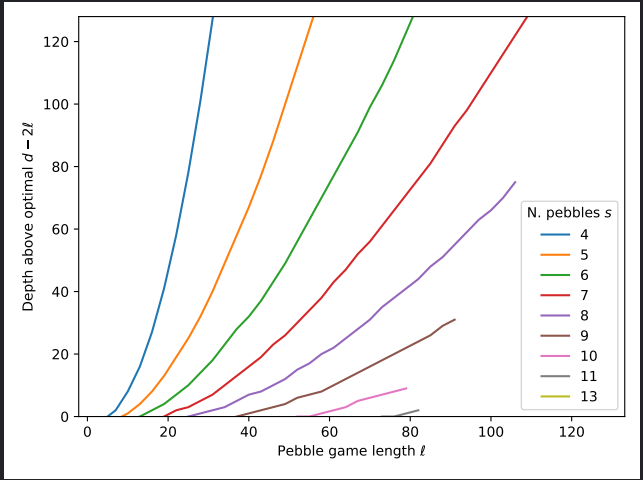

Making Shor reversible

Break up x into its individual bits x_i:

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Each iteration is a controlled multiplication by classical c_i—which is reversible!