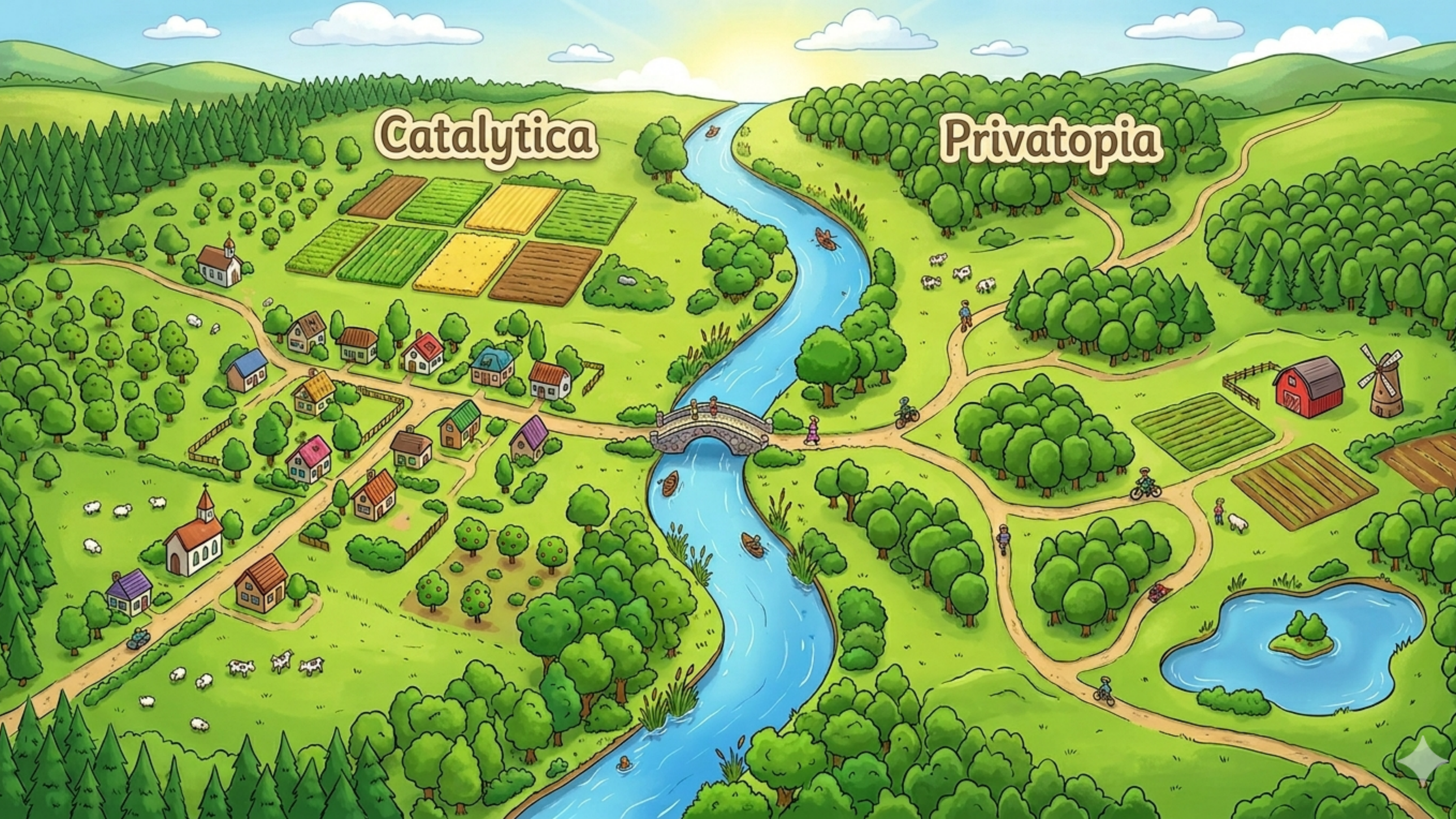
Catalytic Tree Evaluation from Matching Vectors

Seyoon Ragavan (MIT)

Joint work with Alexandra Henzinger and Ted Pyne (MIT)

arXiv:2602.14320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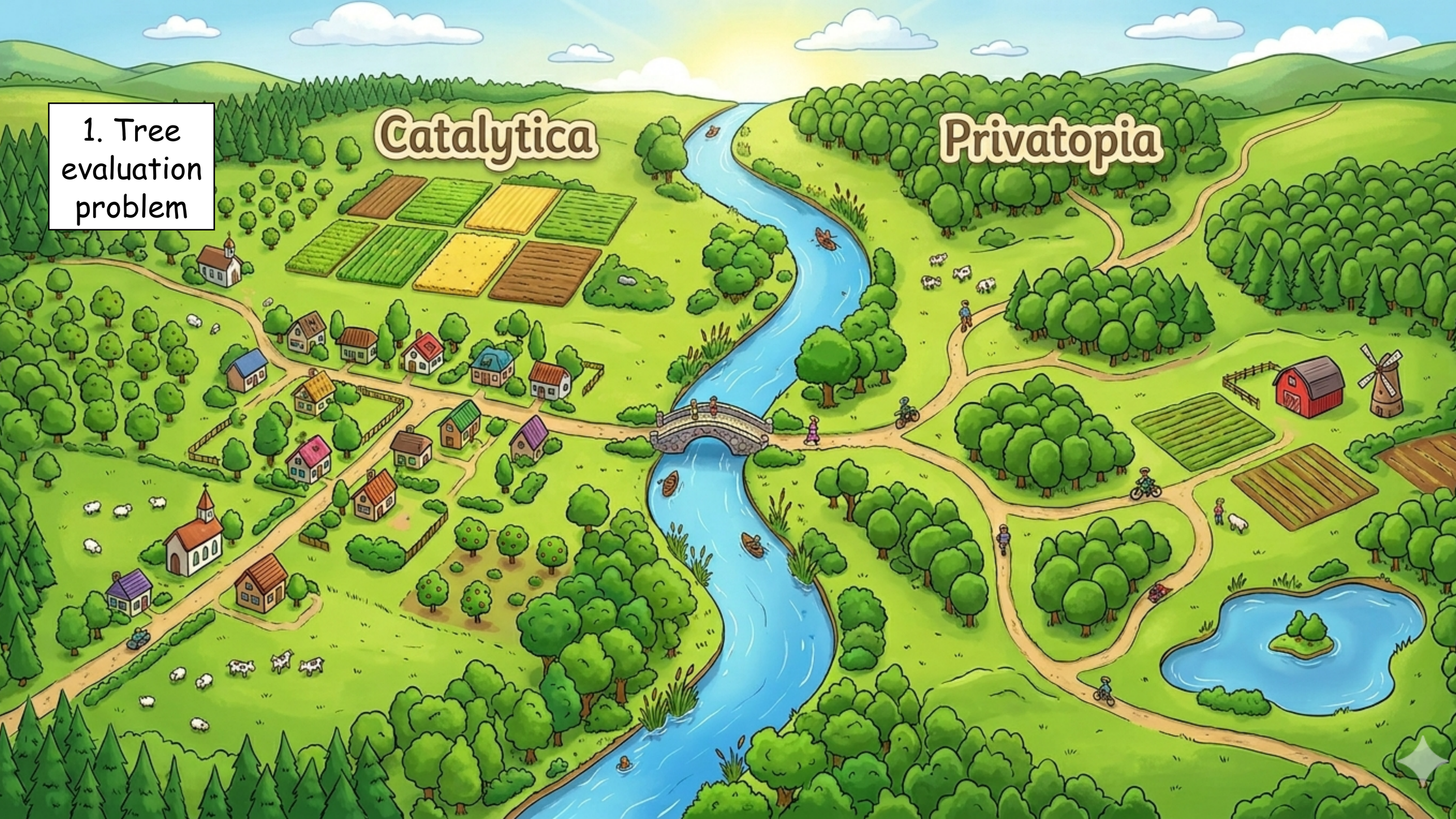
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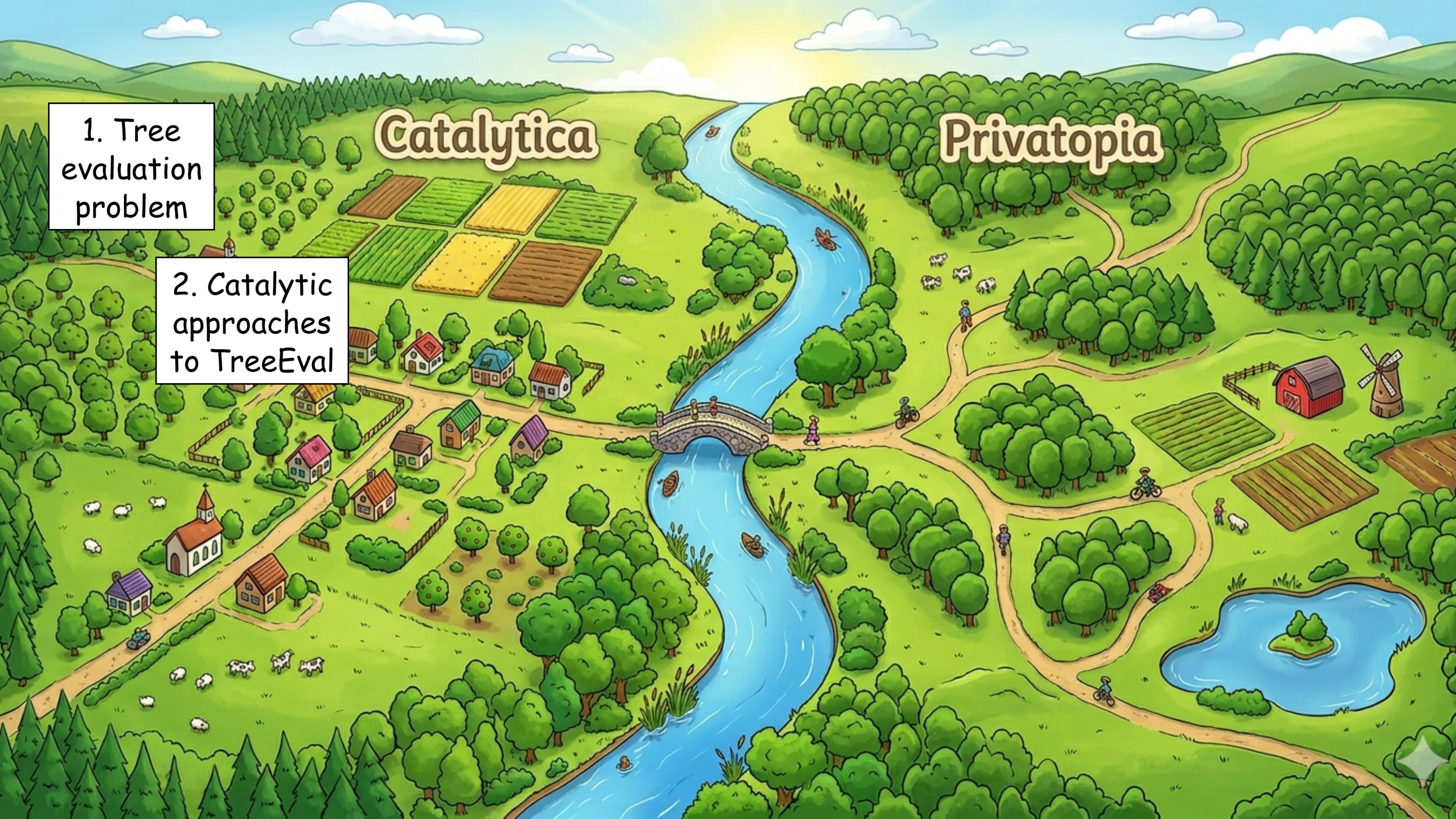
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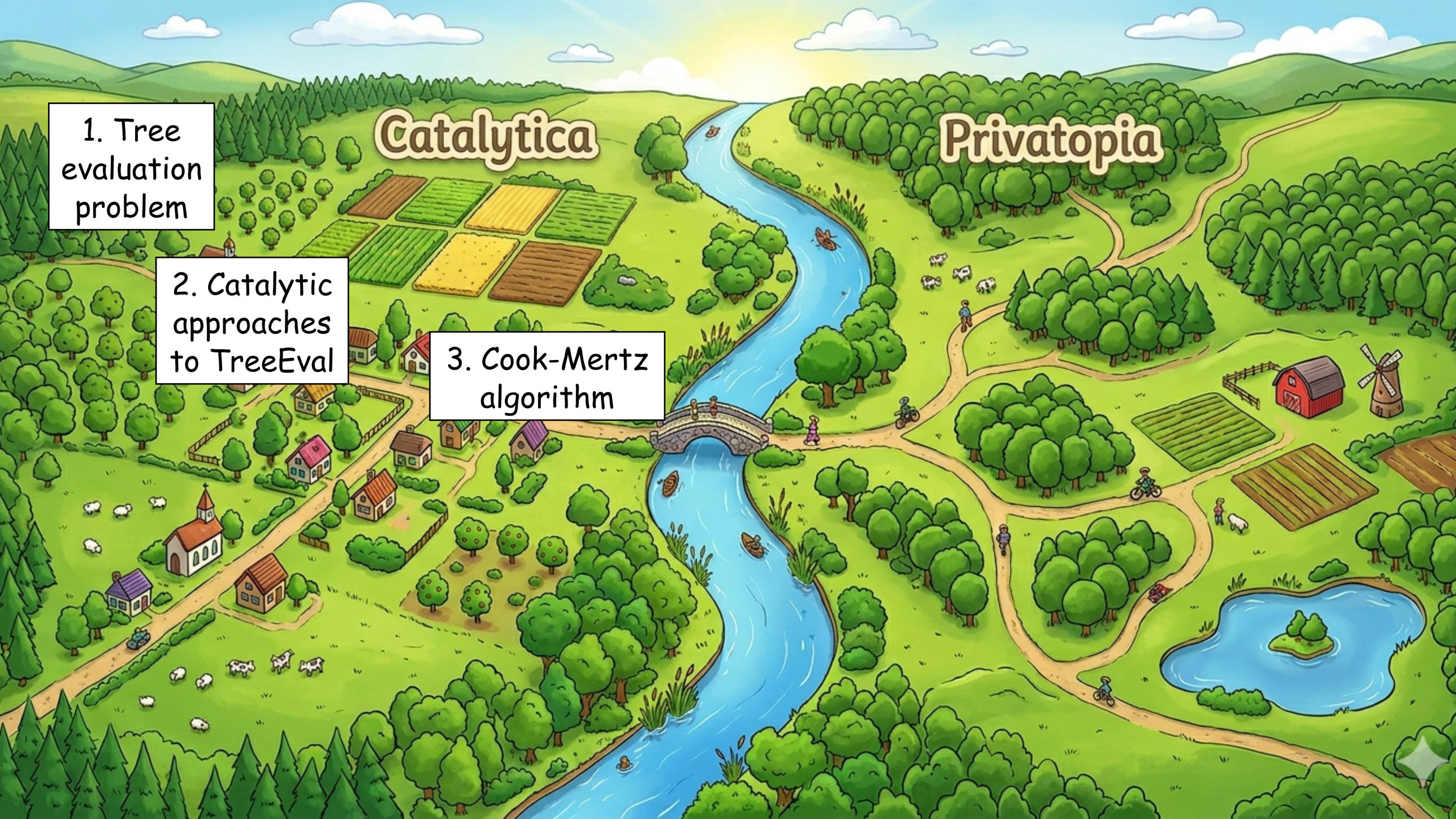
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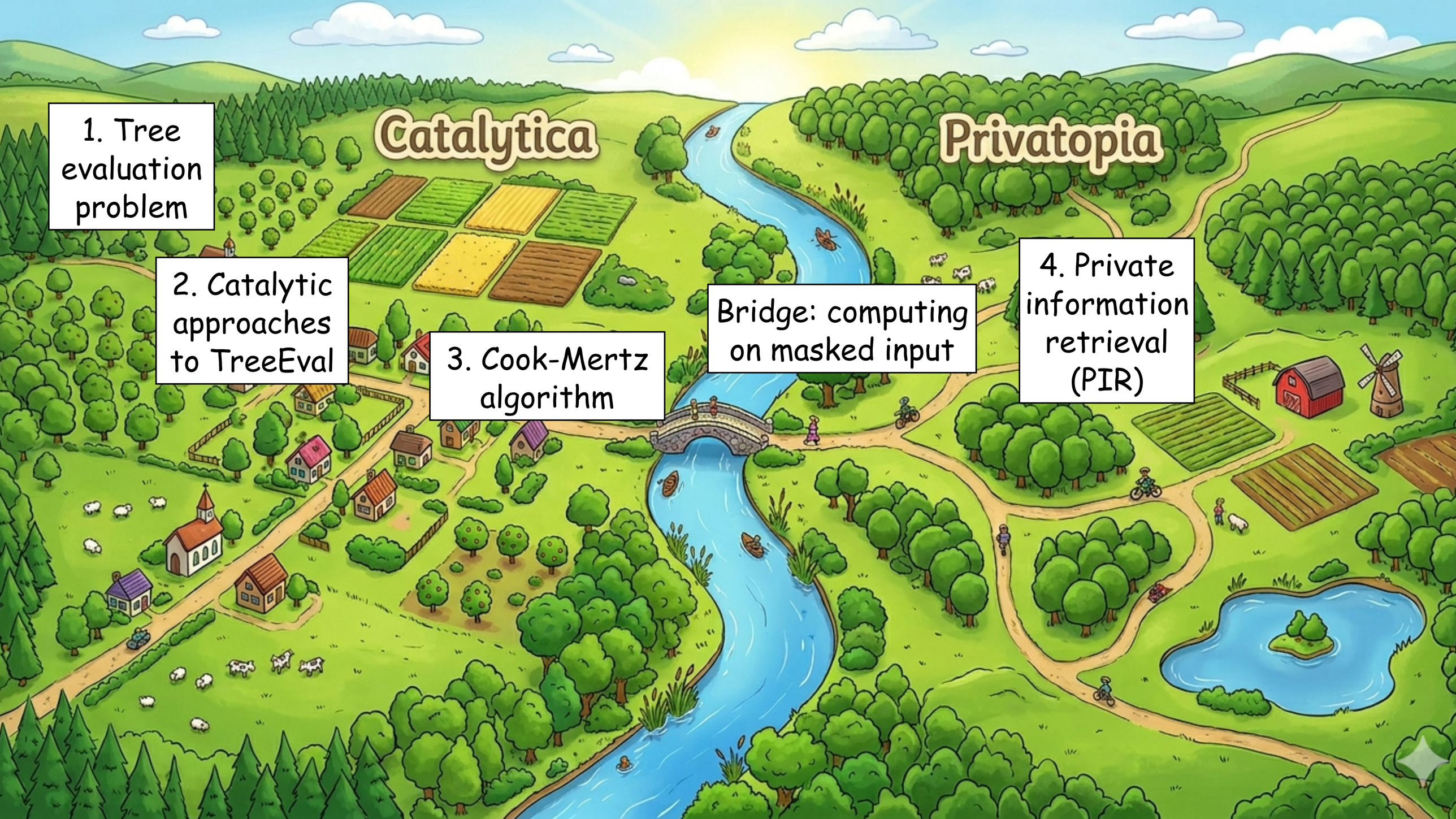
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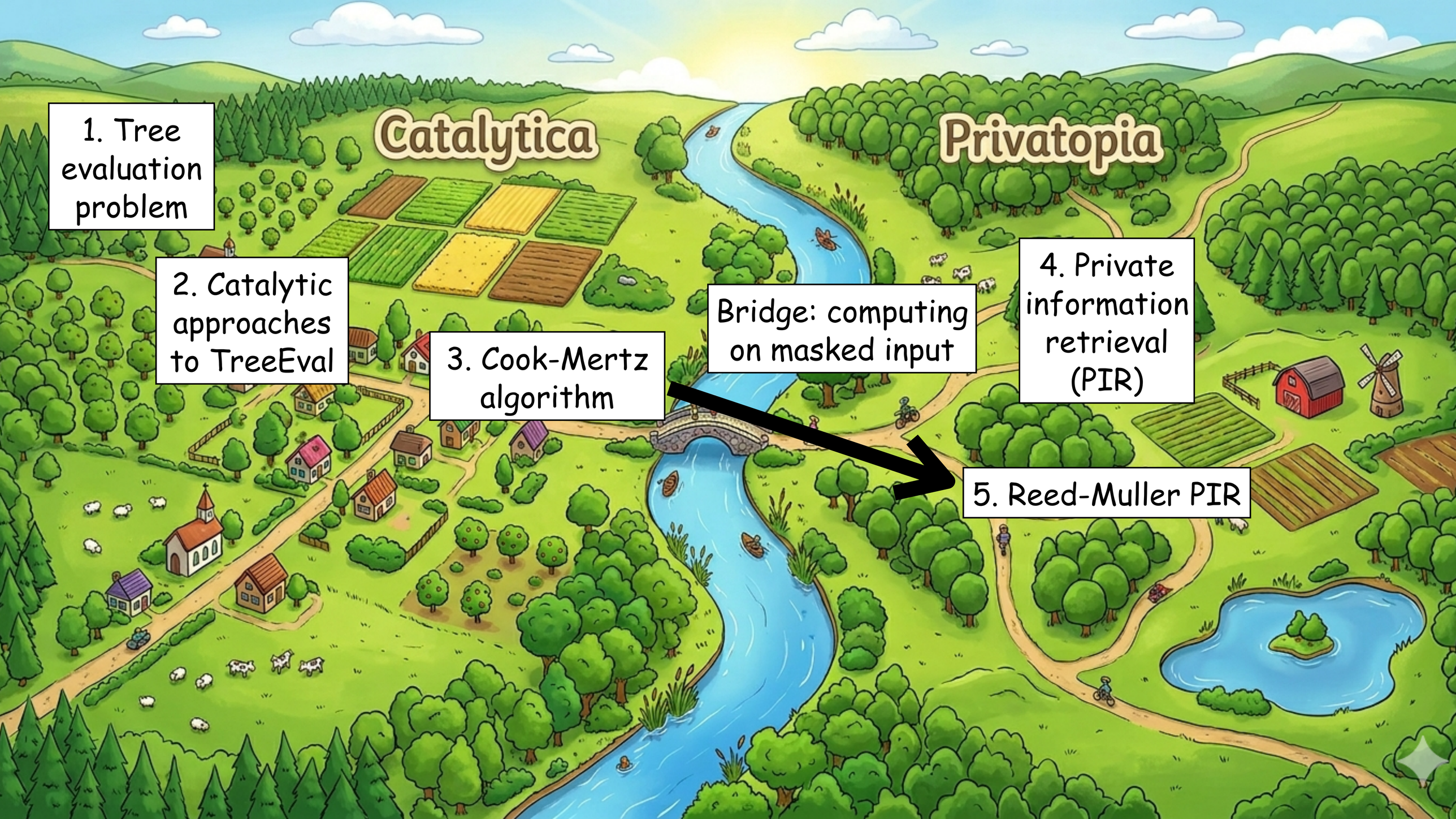
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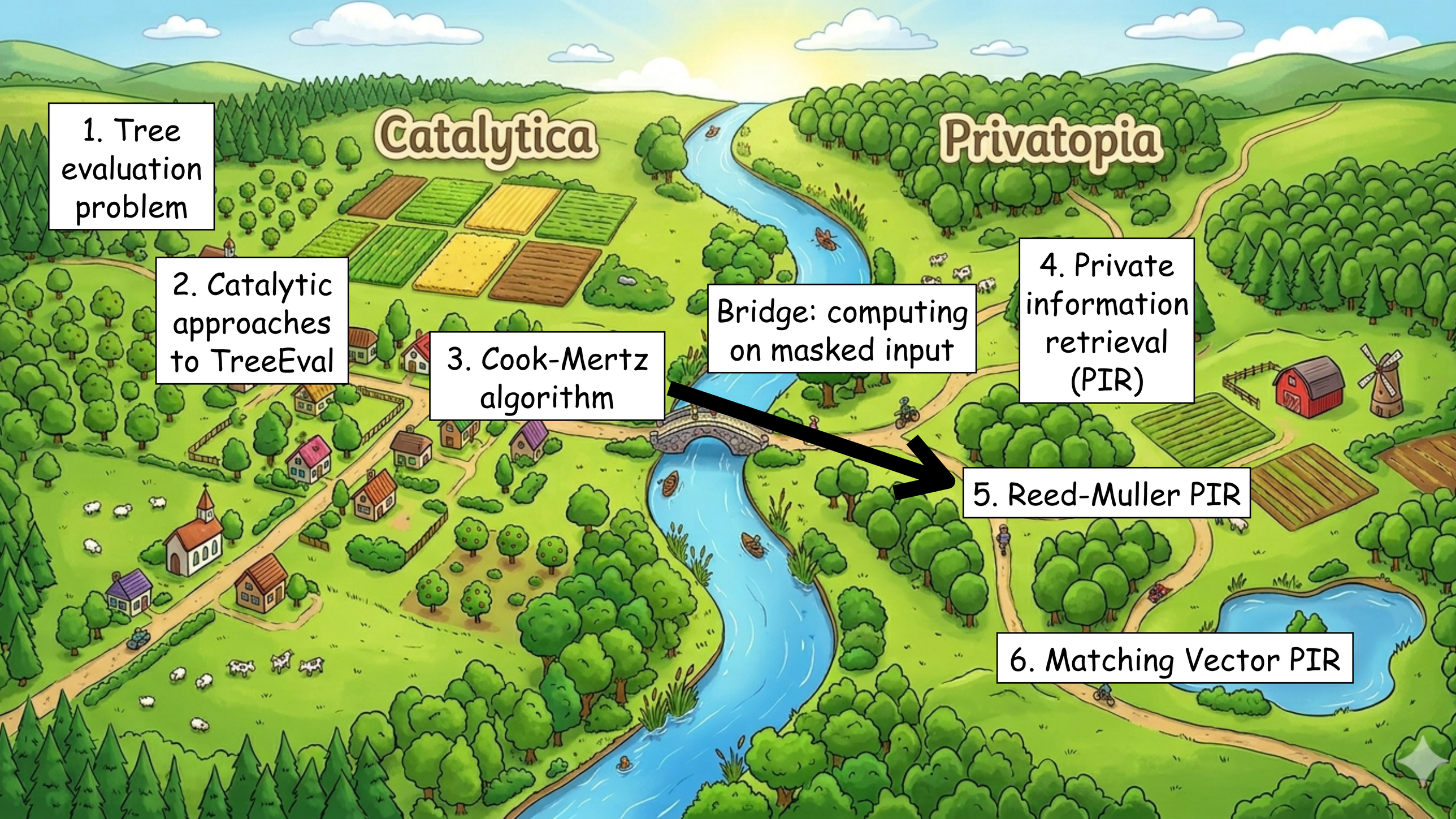
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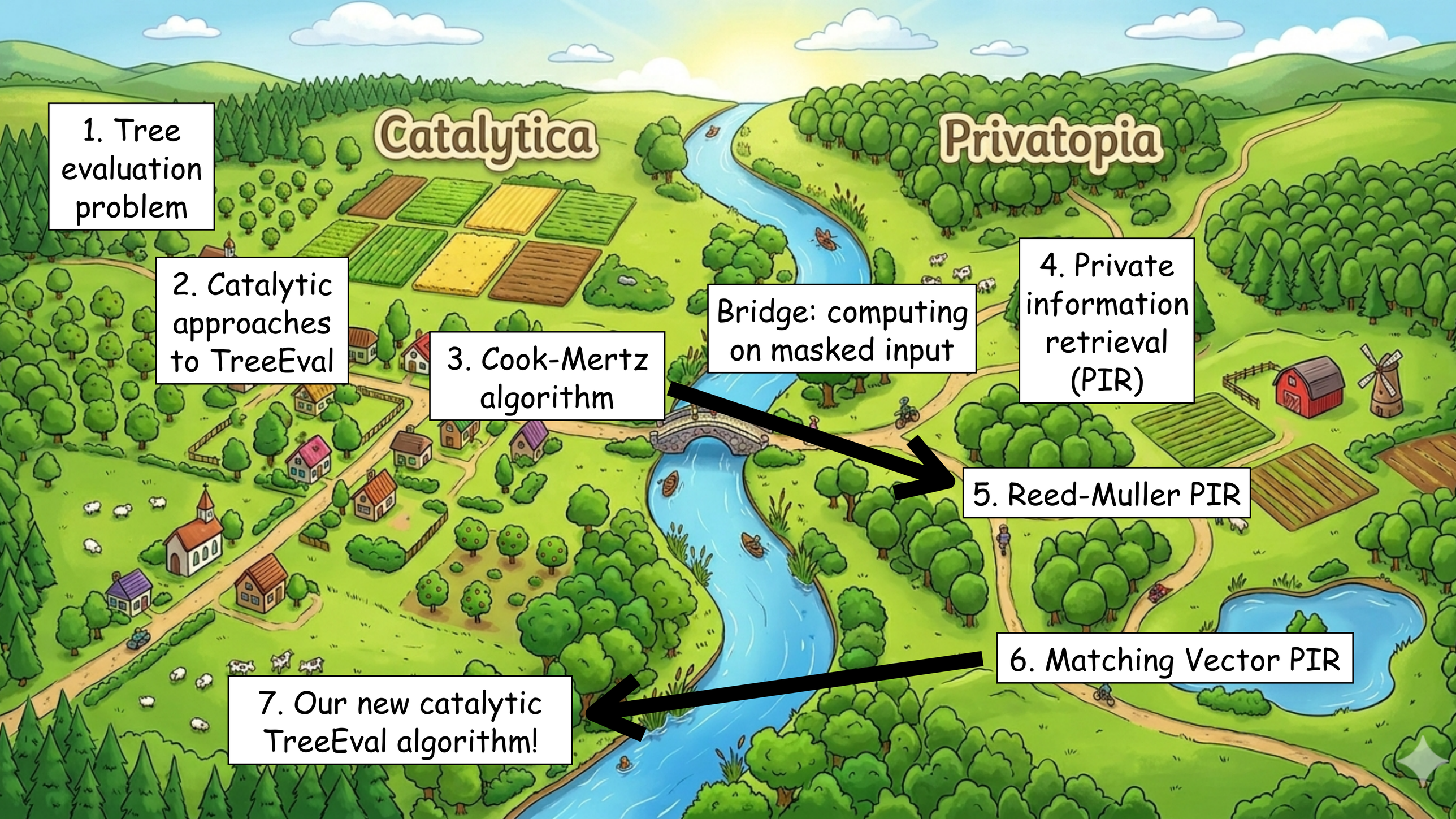
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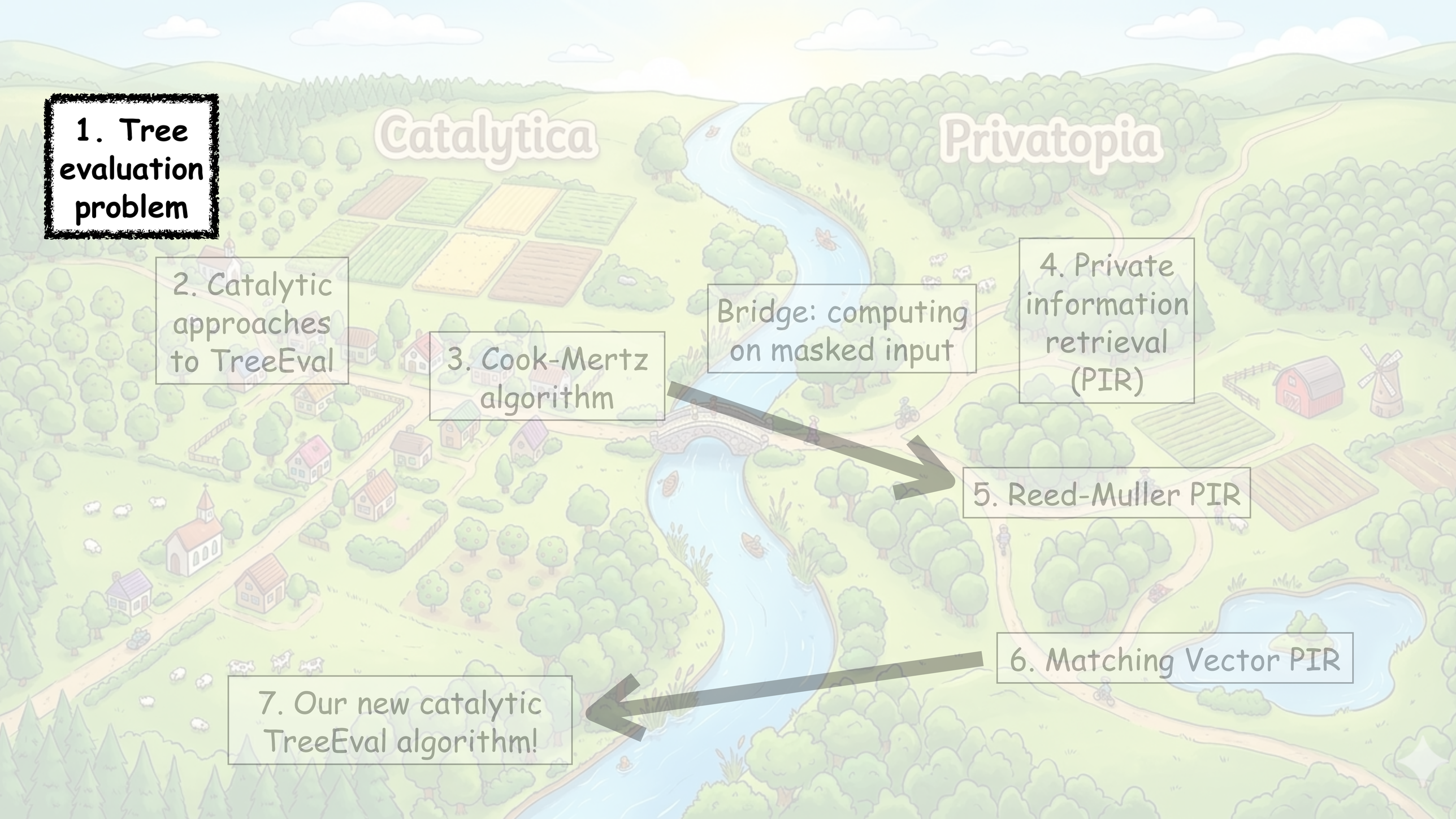
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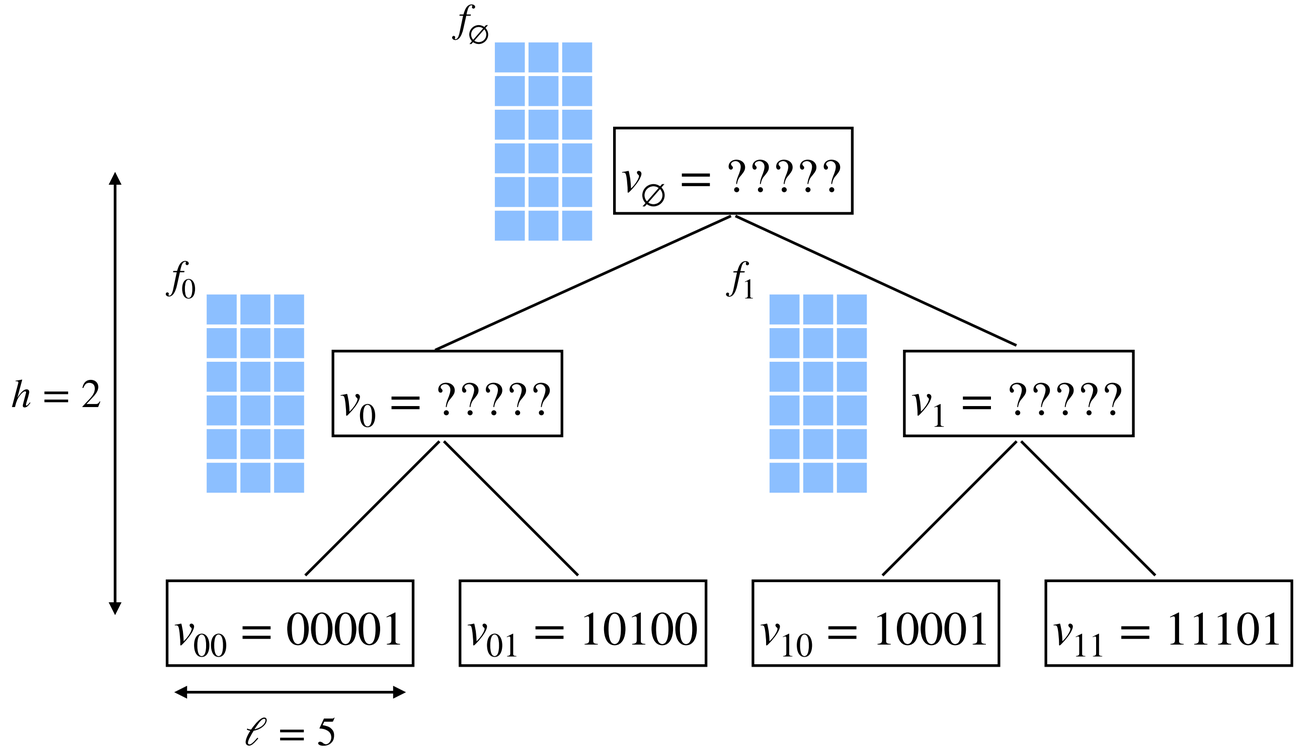

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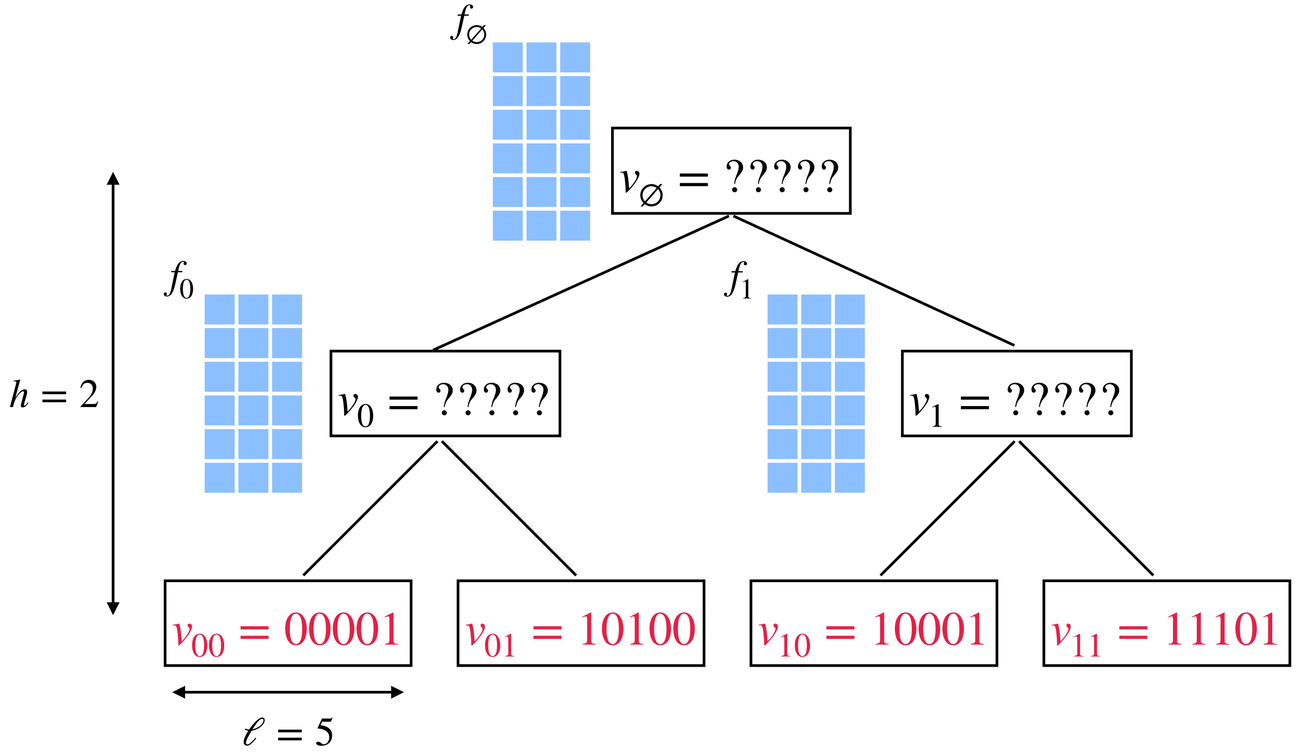

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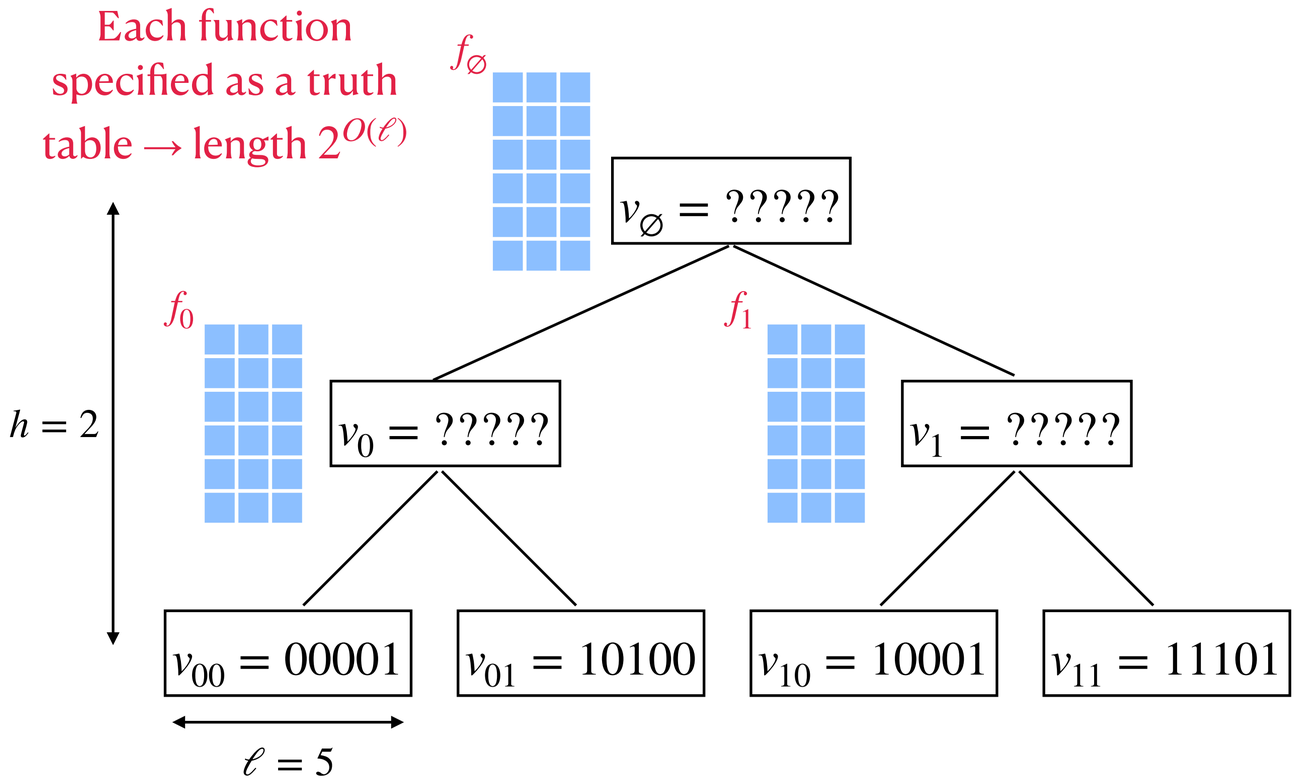

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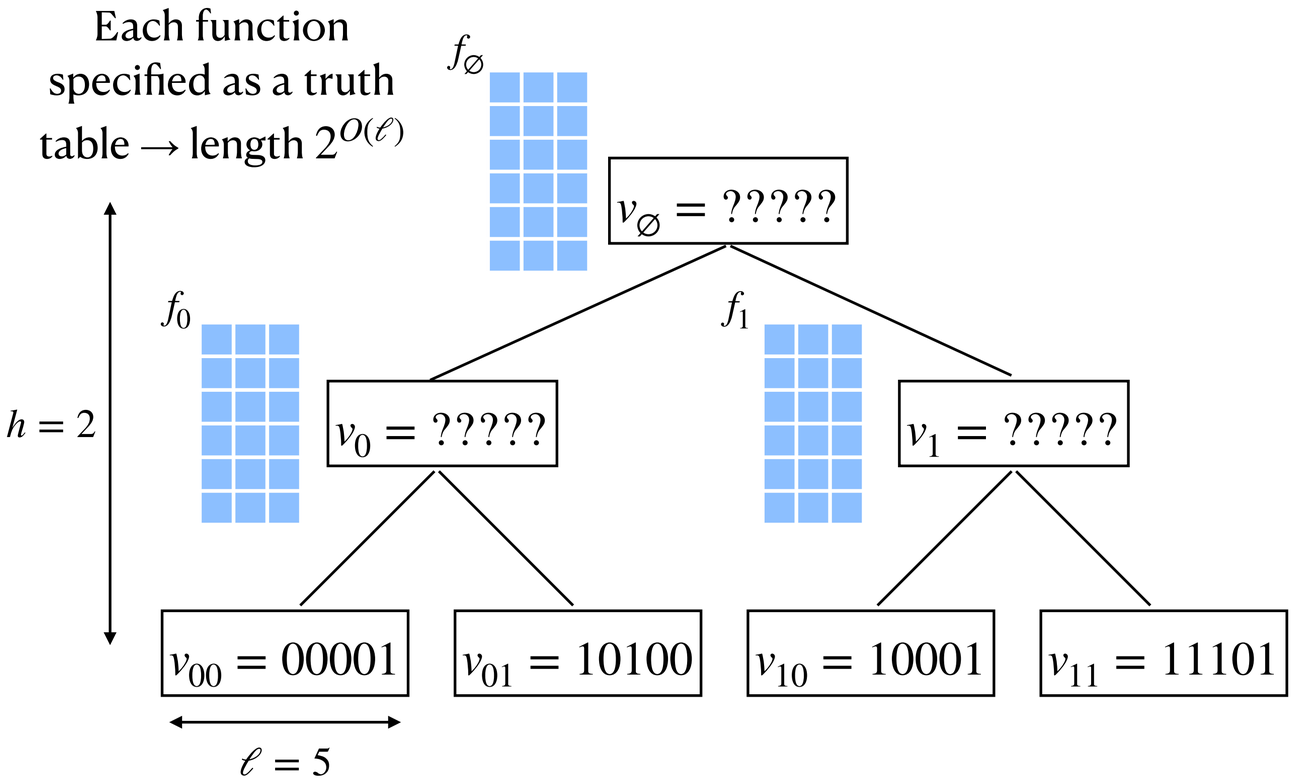

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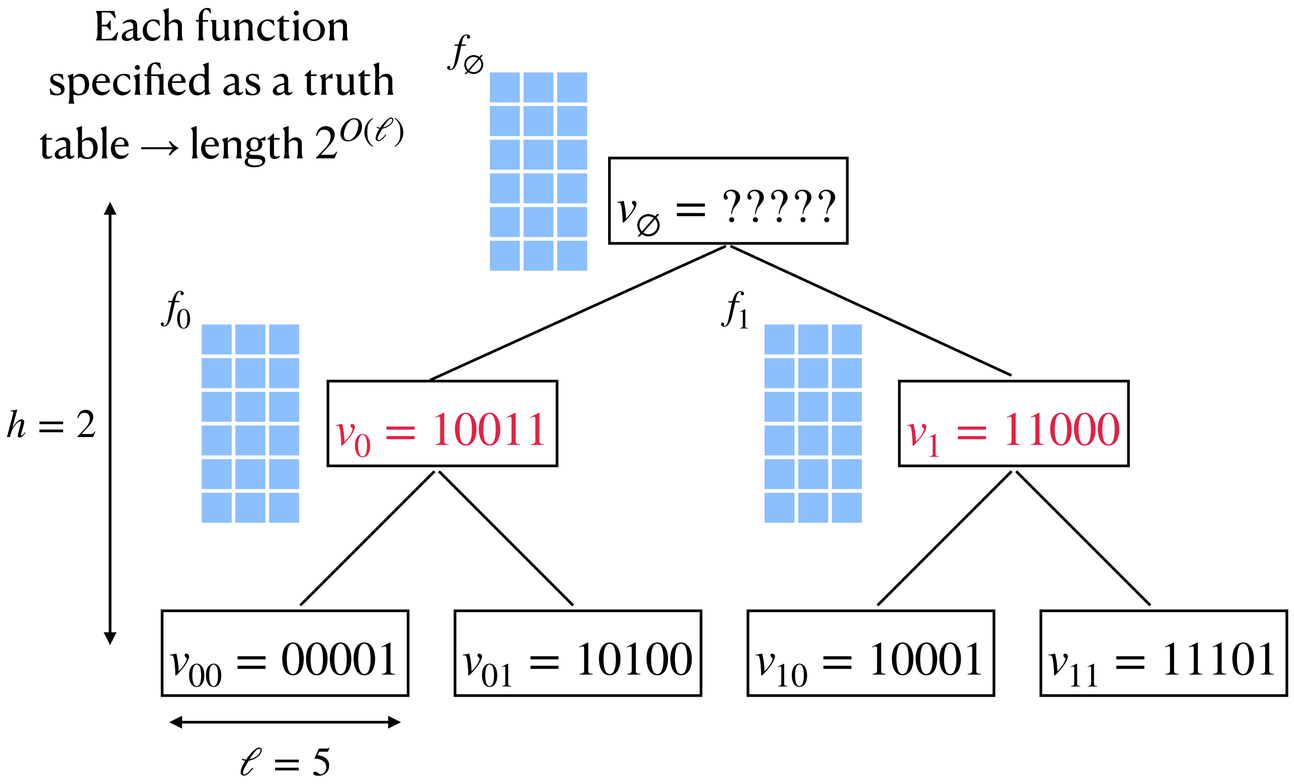

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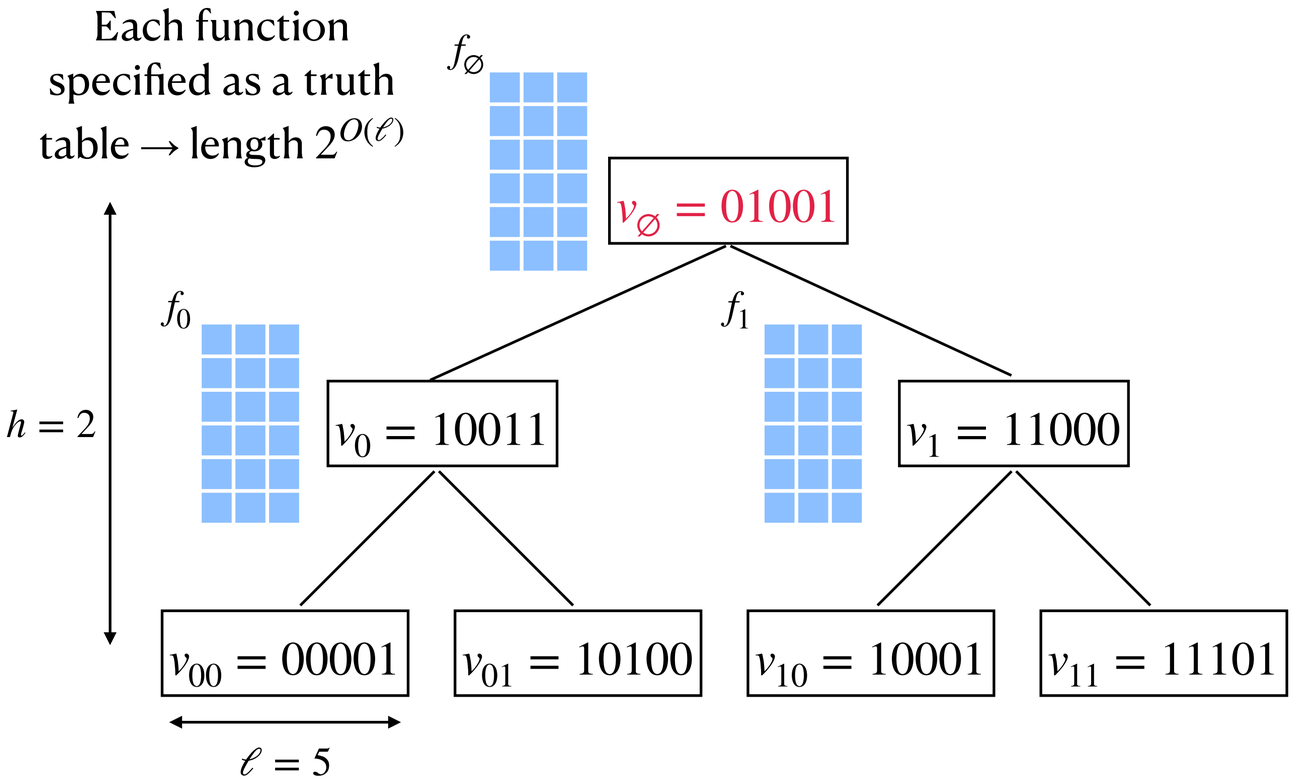

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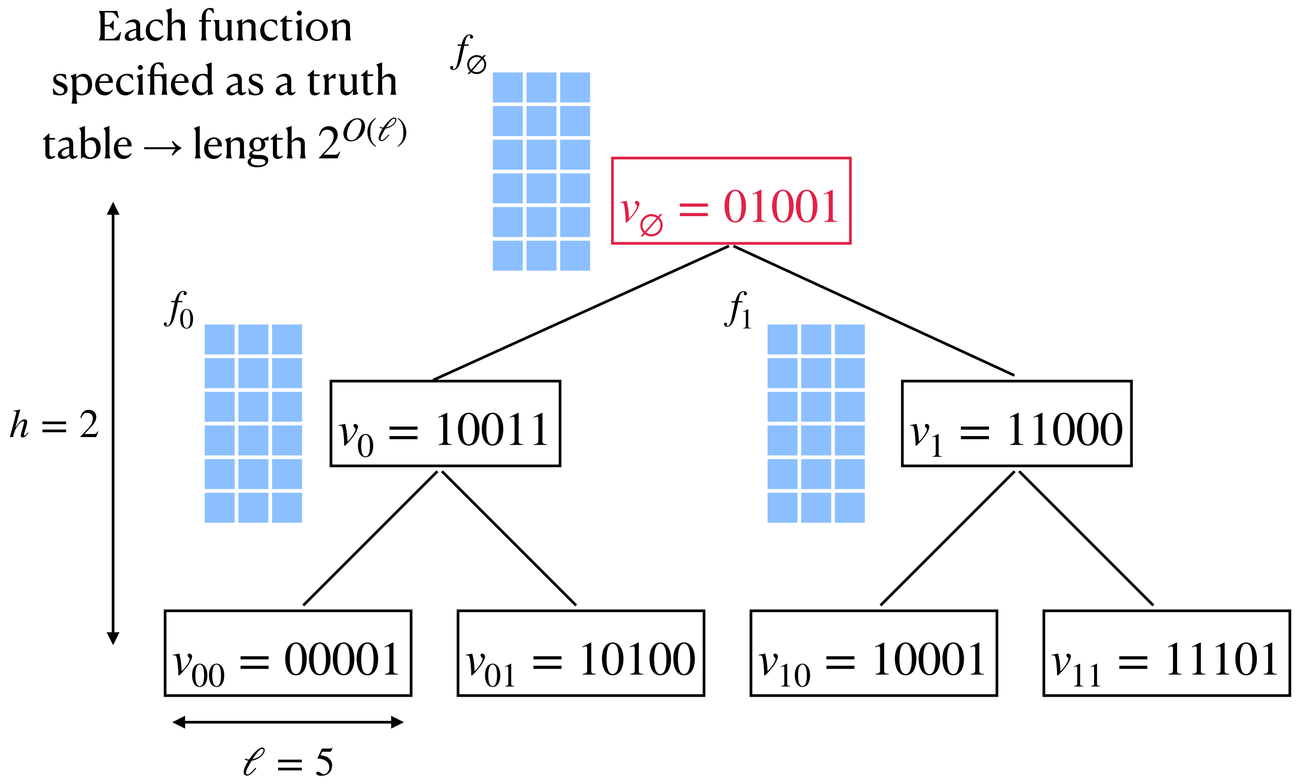

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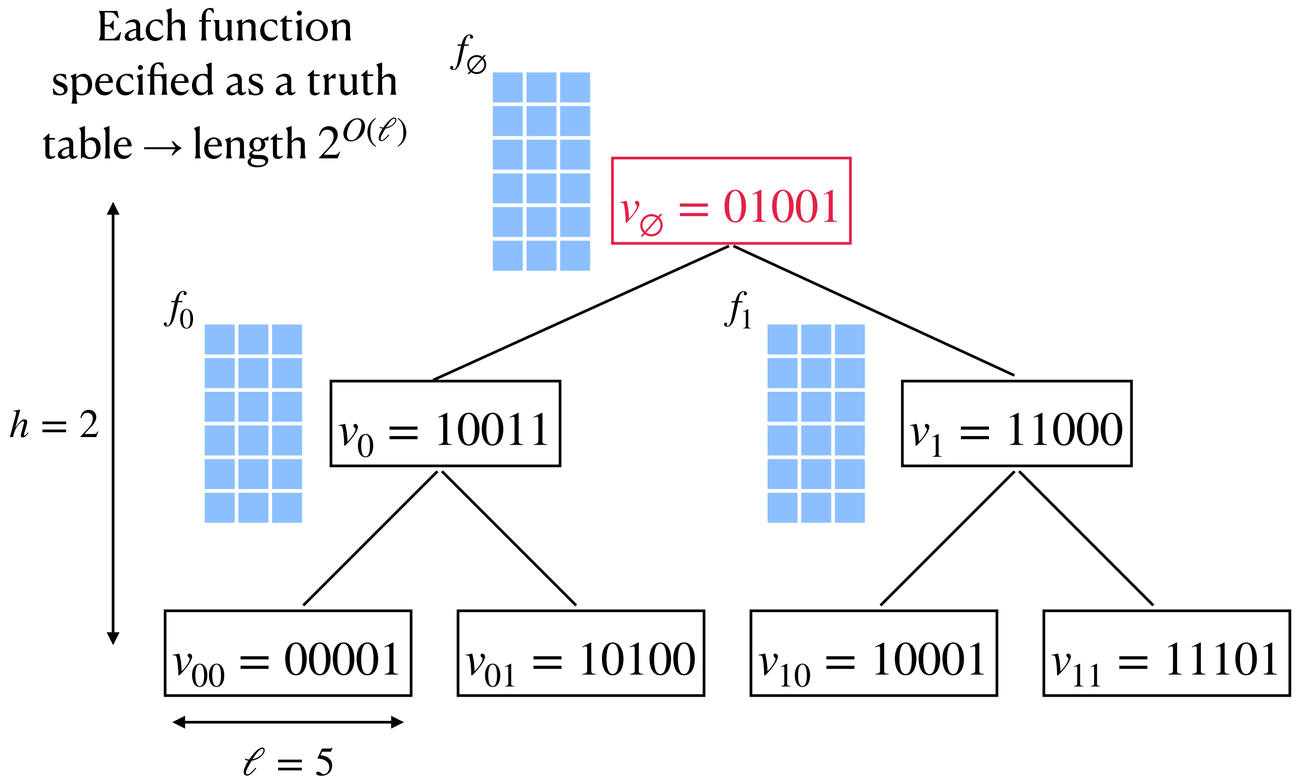

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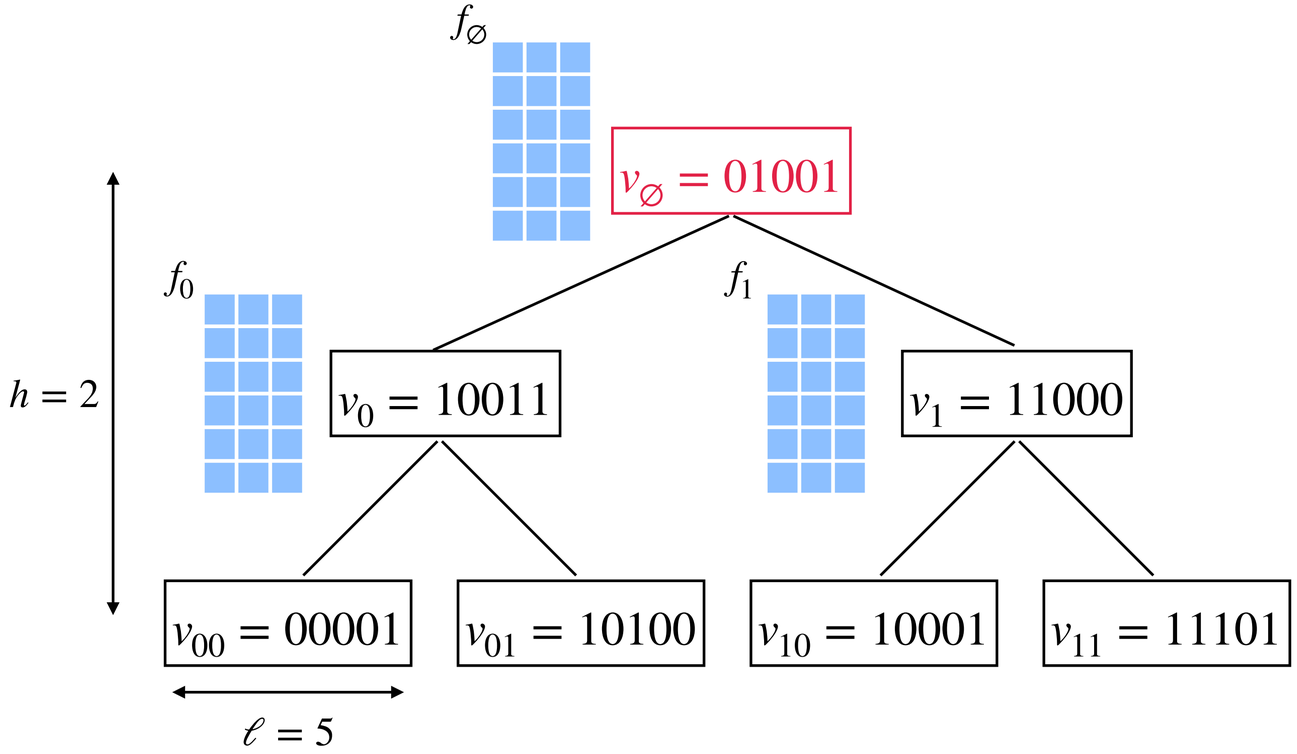
Today's talk: low-space algorithms for Tree Evaluation



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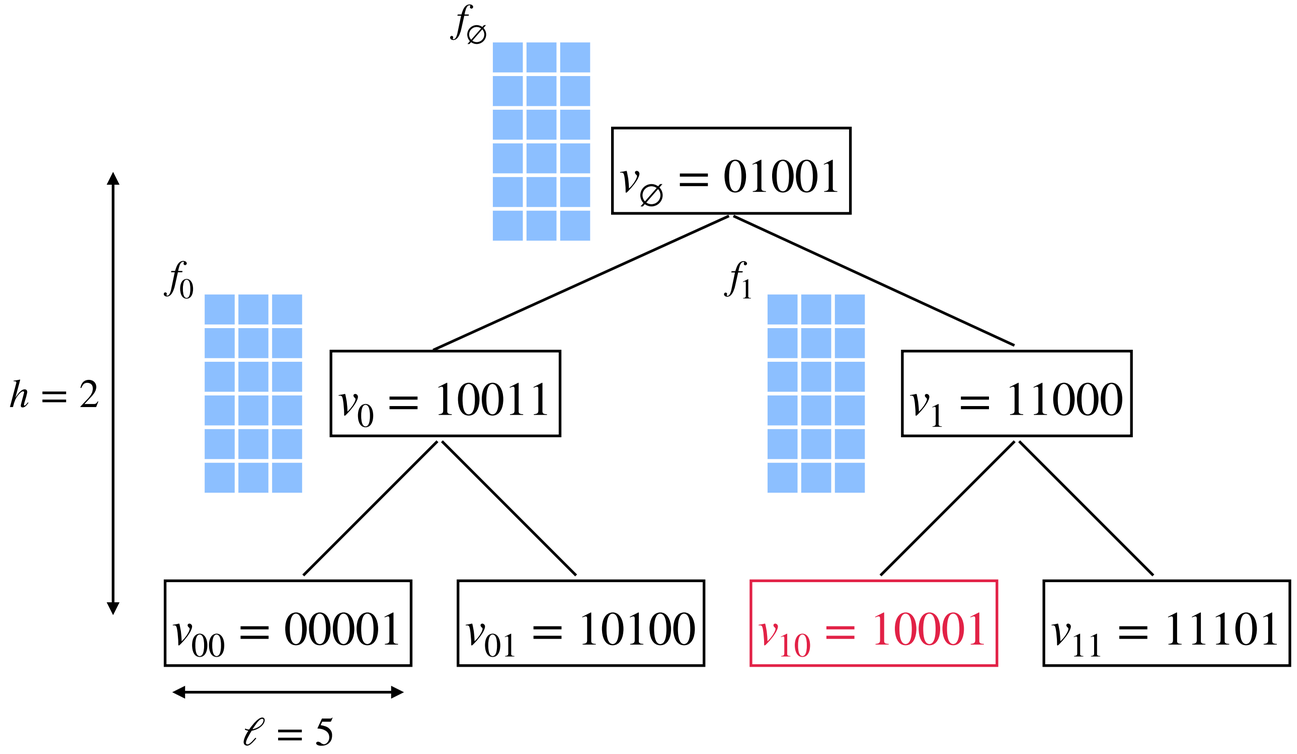

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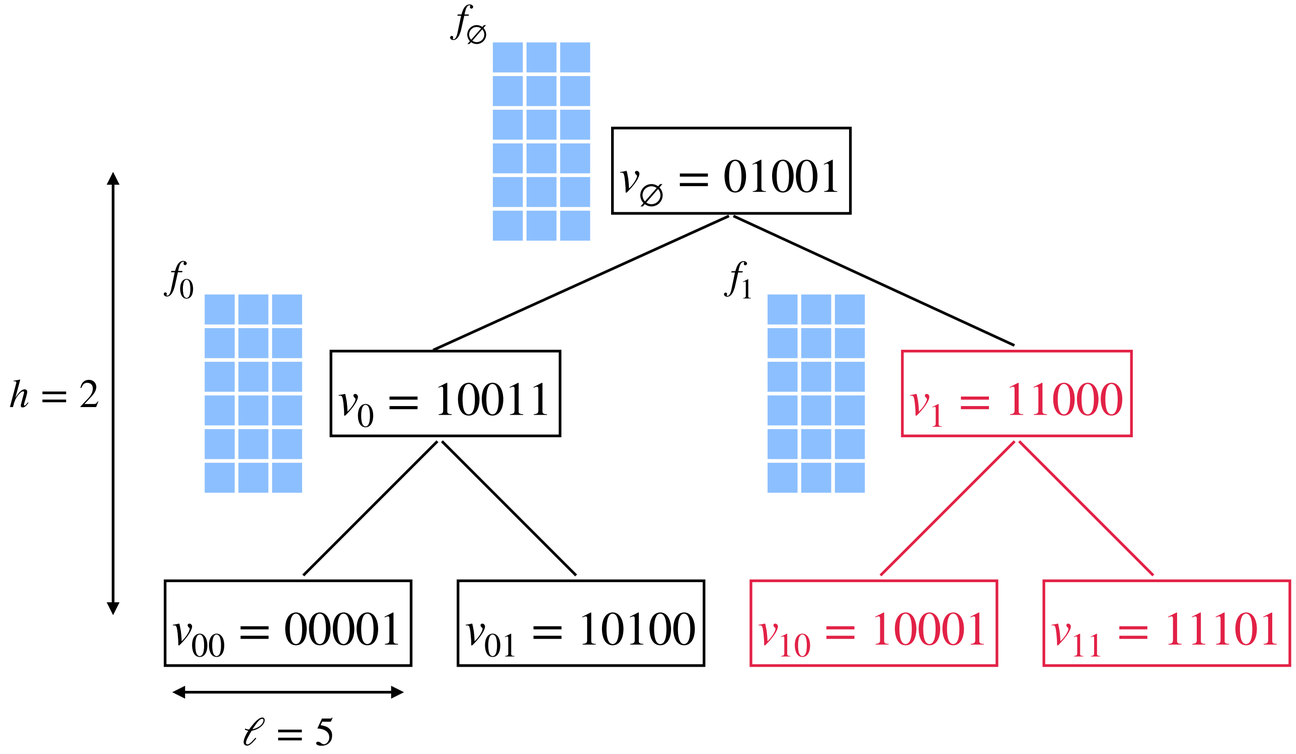

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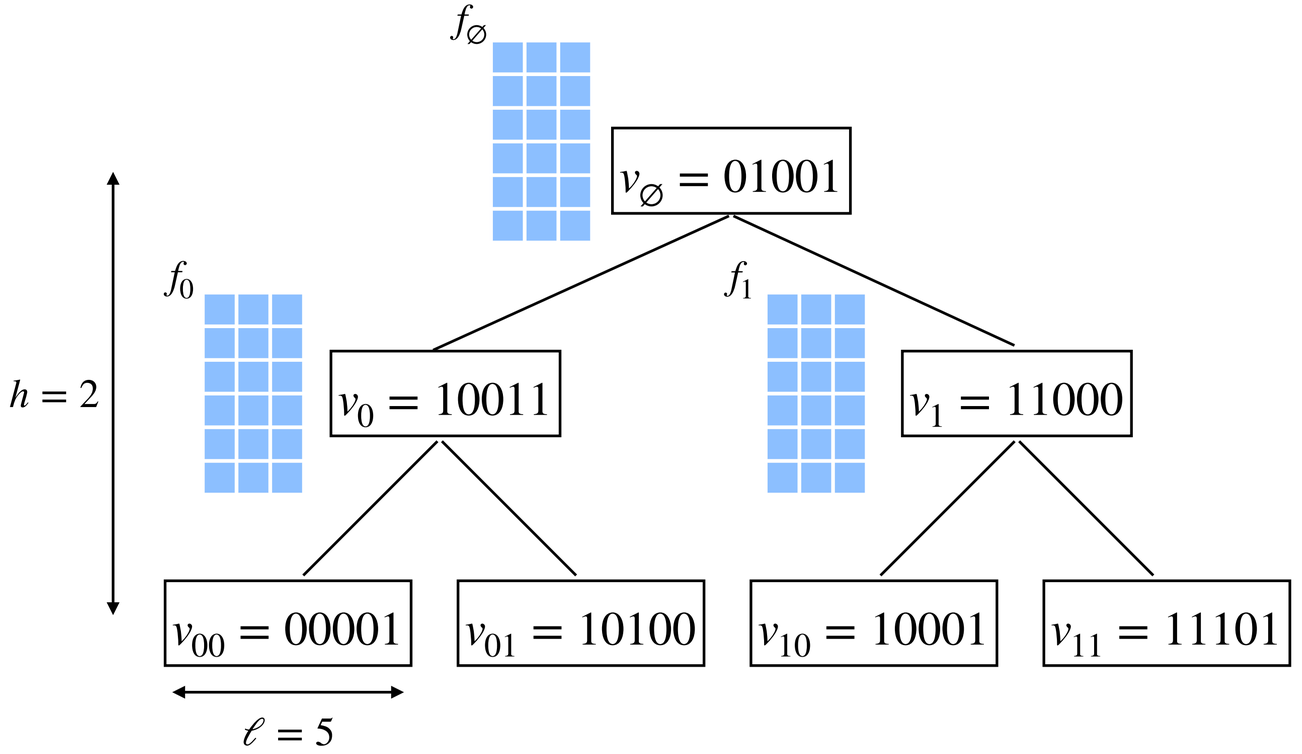

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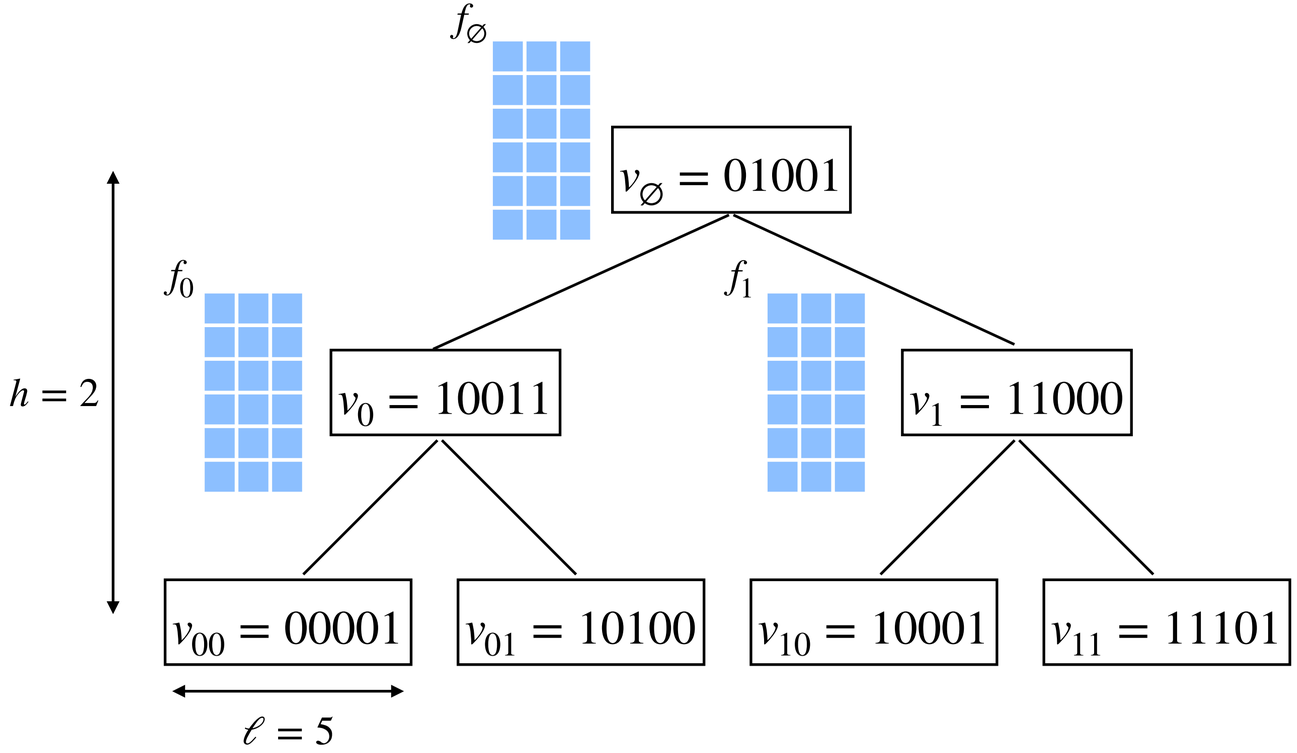

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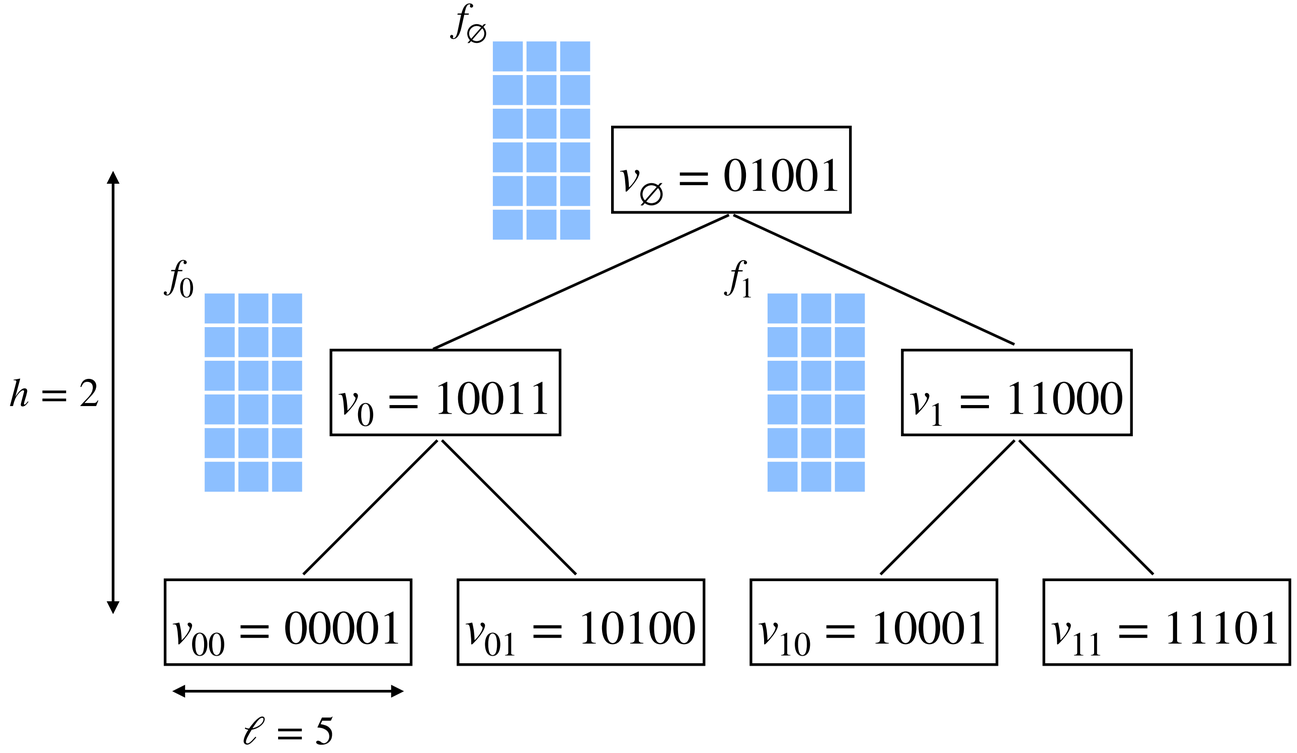

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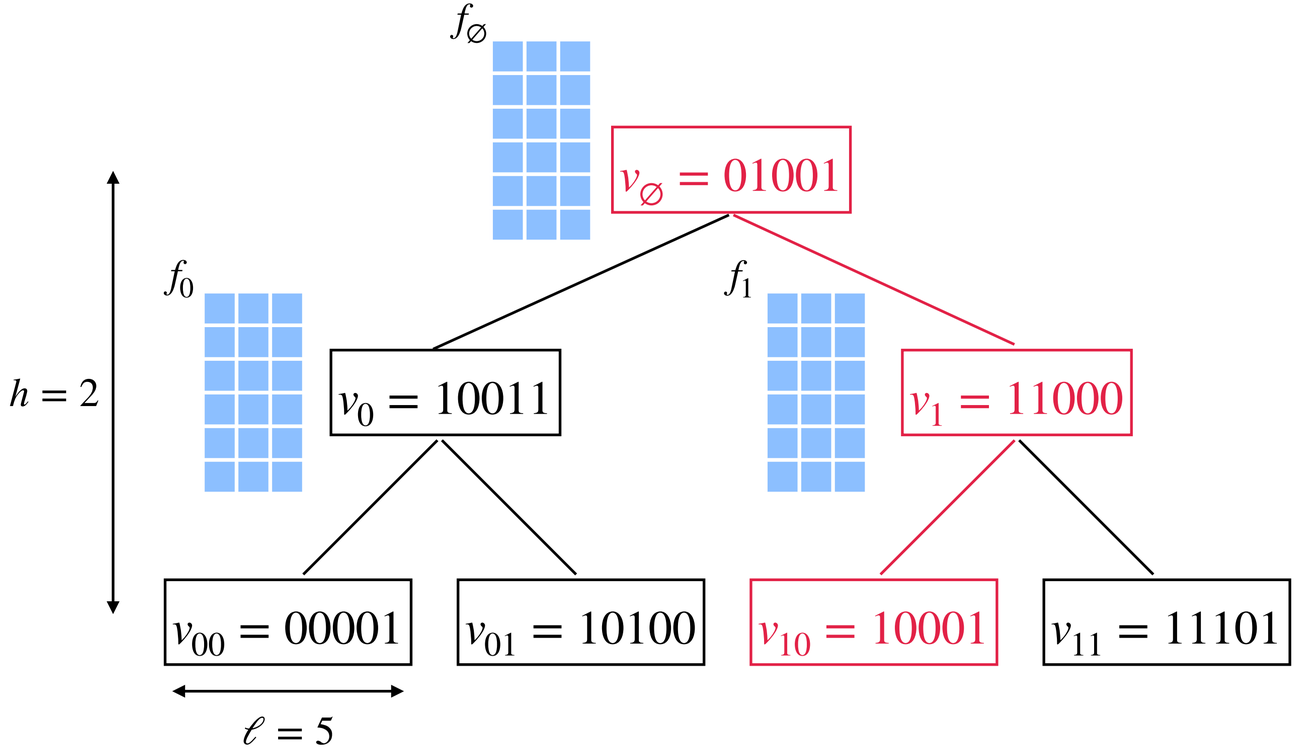

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And applications to time-space tradeoffs

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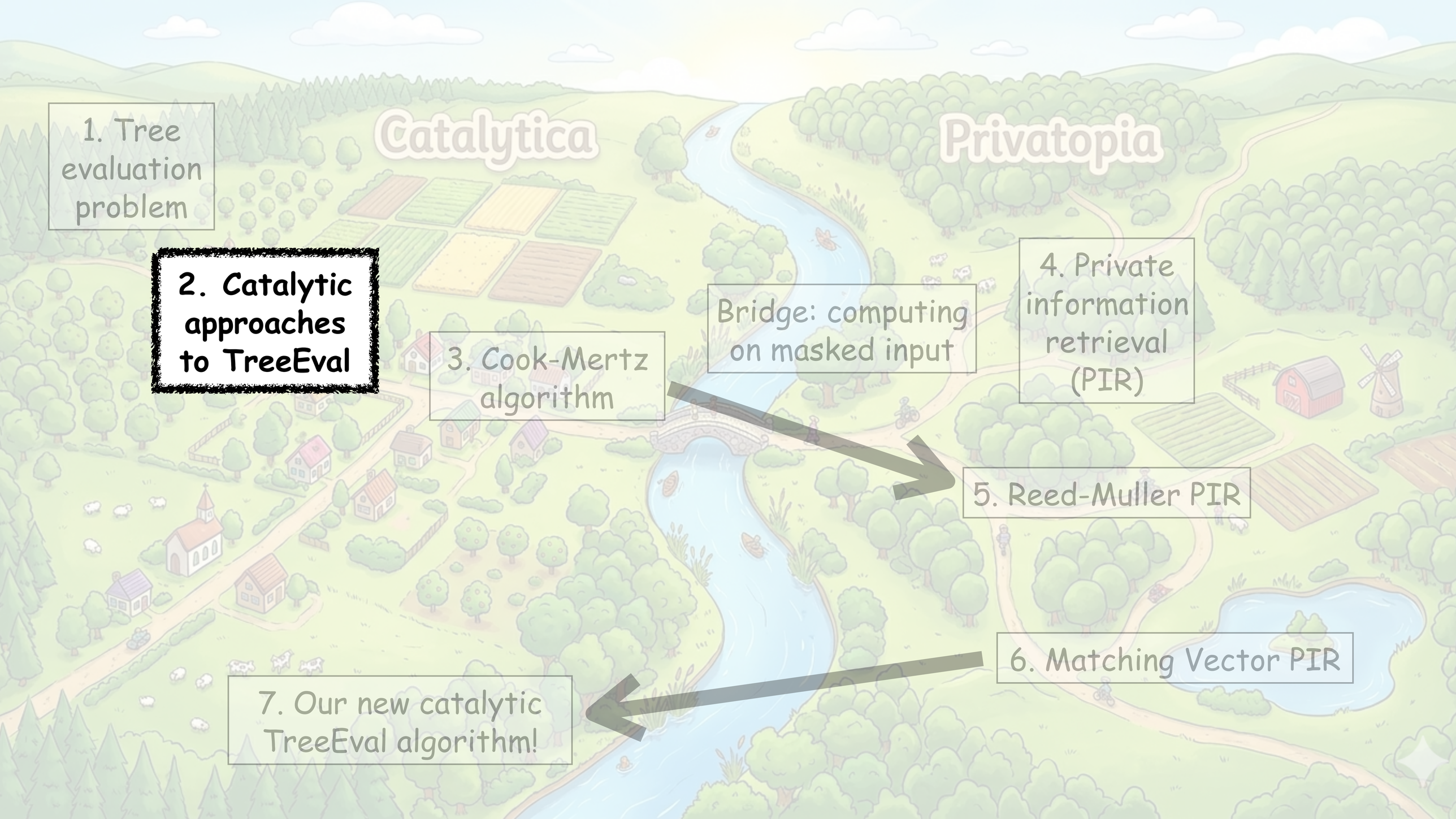
Q: What is the key tool enabling these algorithms by Cook and Mertz?

A: Catalytic computation!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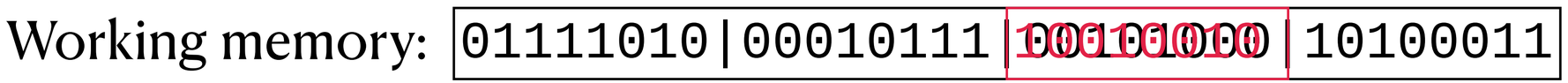
- Catalytic computing ("bad programming"): write this information on top of occupied space!



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  - At the end: the catalytic tape must be restored to the state τ_{init}

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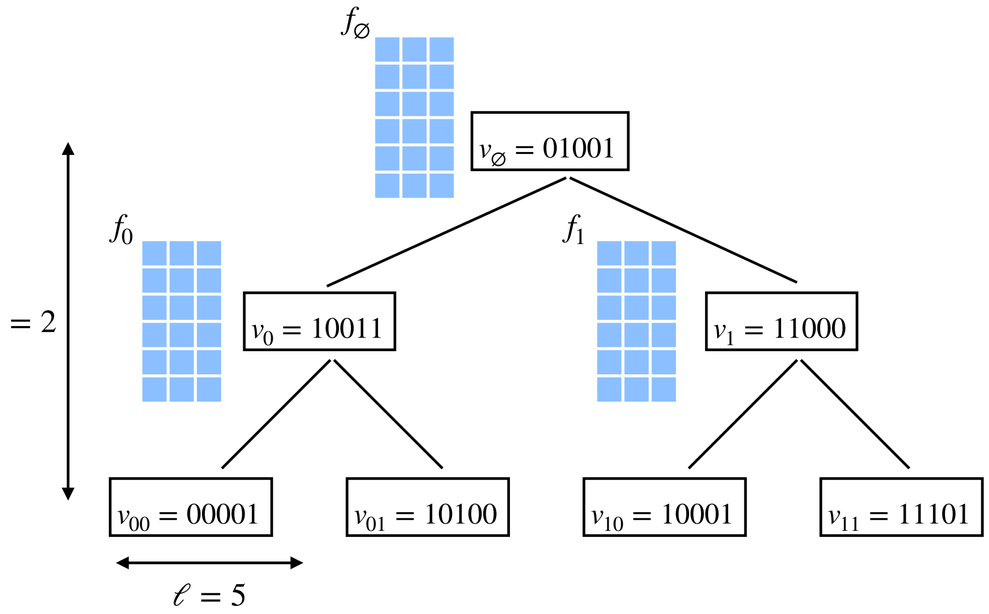
  - Space limit: $O(S(n))$ clean bits + the catalytic tape containing τ
  - At the end: the catalytic tape must be restored to the state τ_{init}
- CatSpaceTime($C(n), S(n), T(n)$) $\subseteq$ CatSpaceTime($0, C(n) + S(n), T(n)$)
  - “Free space is at least as good as catalytic space”

Catalytic Tree Evaluation Through the Years

Algorithm	Catalytic space	Free space	Time
Depth-first		$h\ell = O(\log^2 n)$	poly(n)



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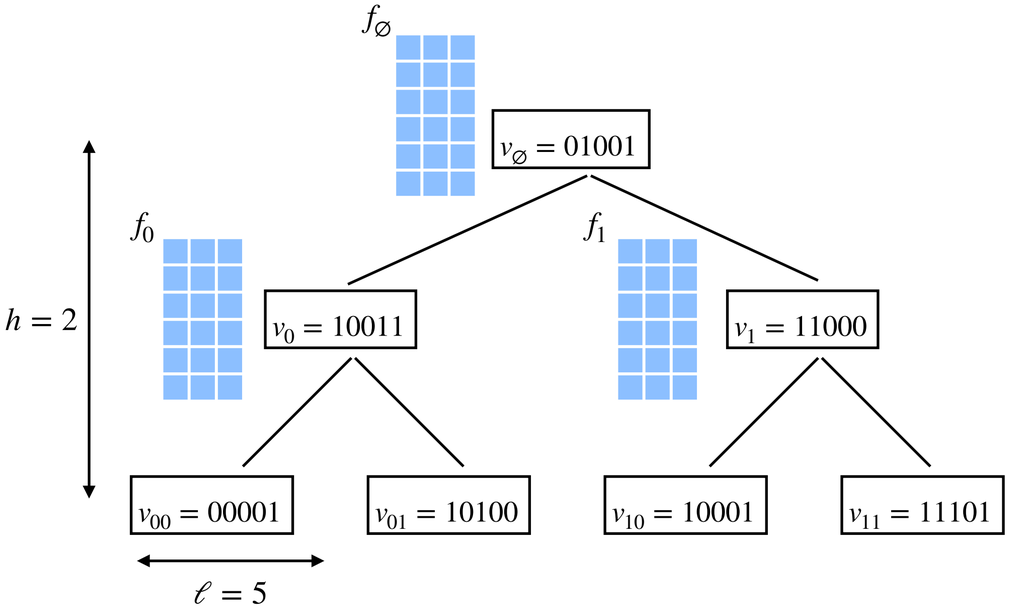

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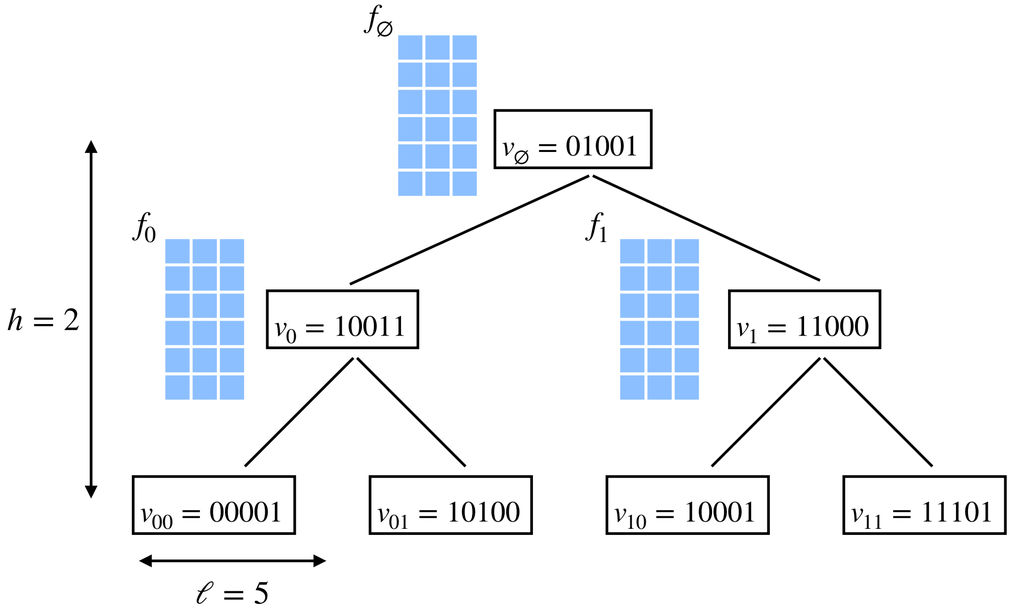

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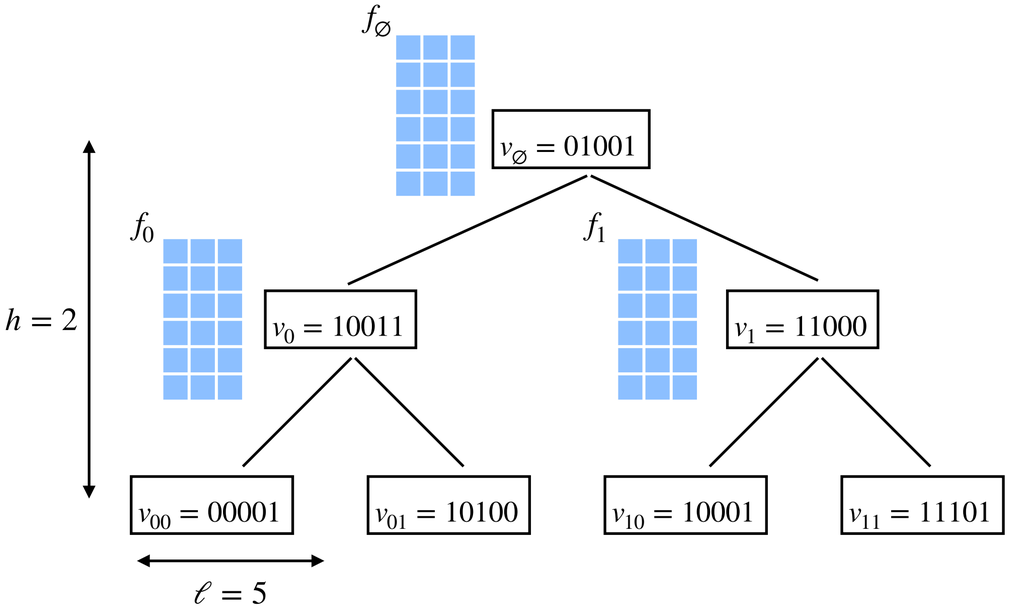

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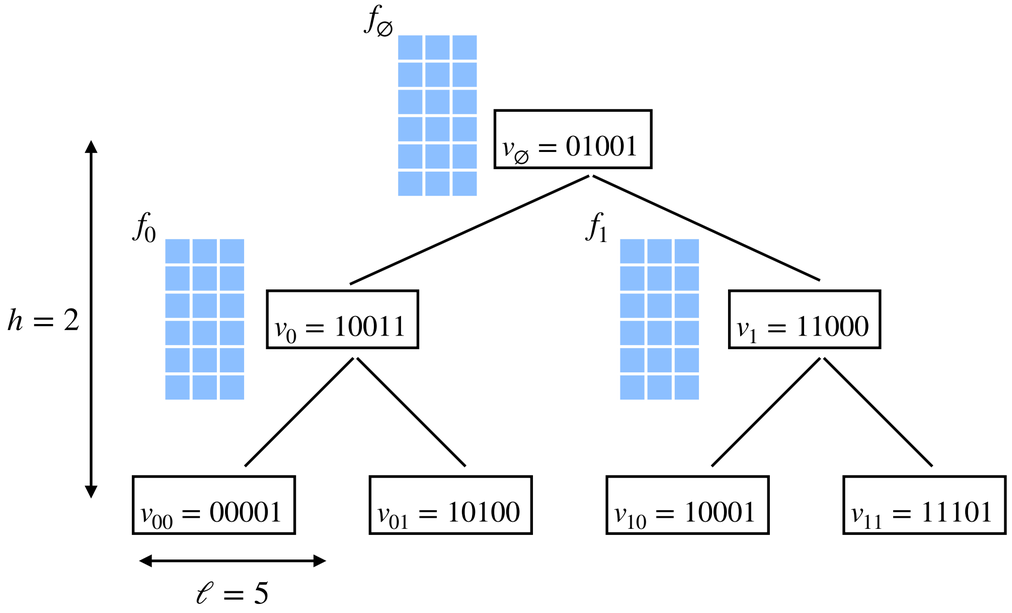

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Strictly better than Buhrman et al.

Incomparable with Cook-Mertz

Cook-Mertz works by evaluating functions on masked inputs.

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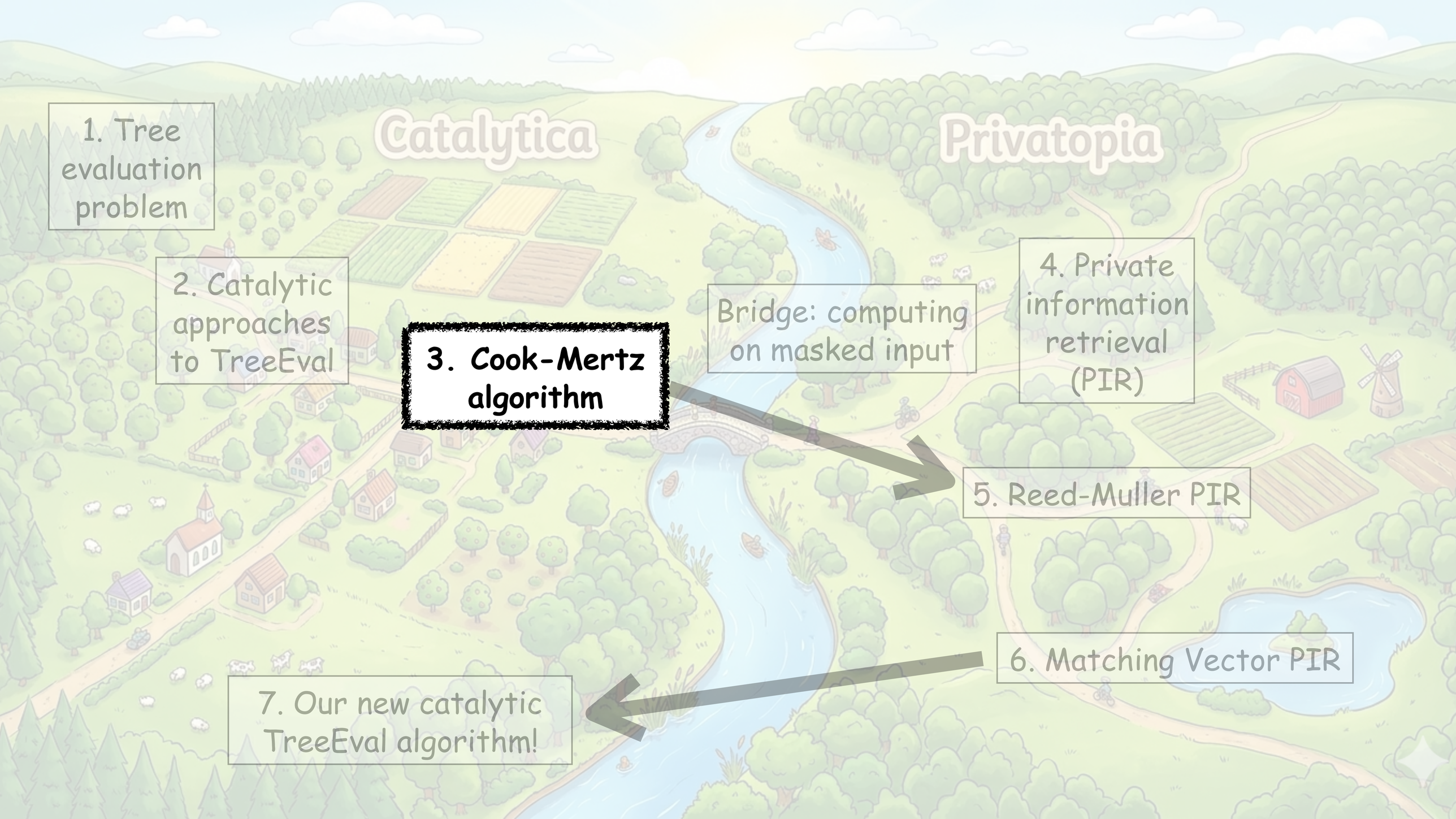
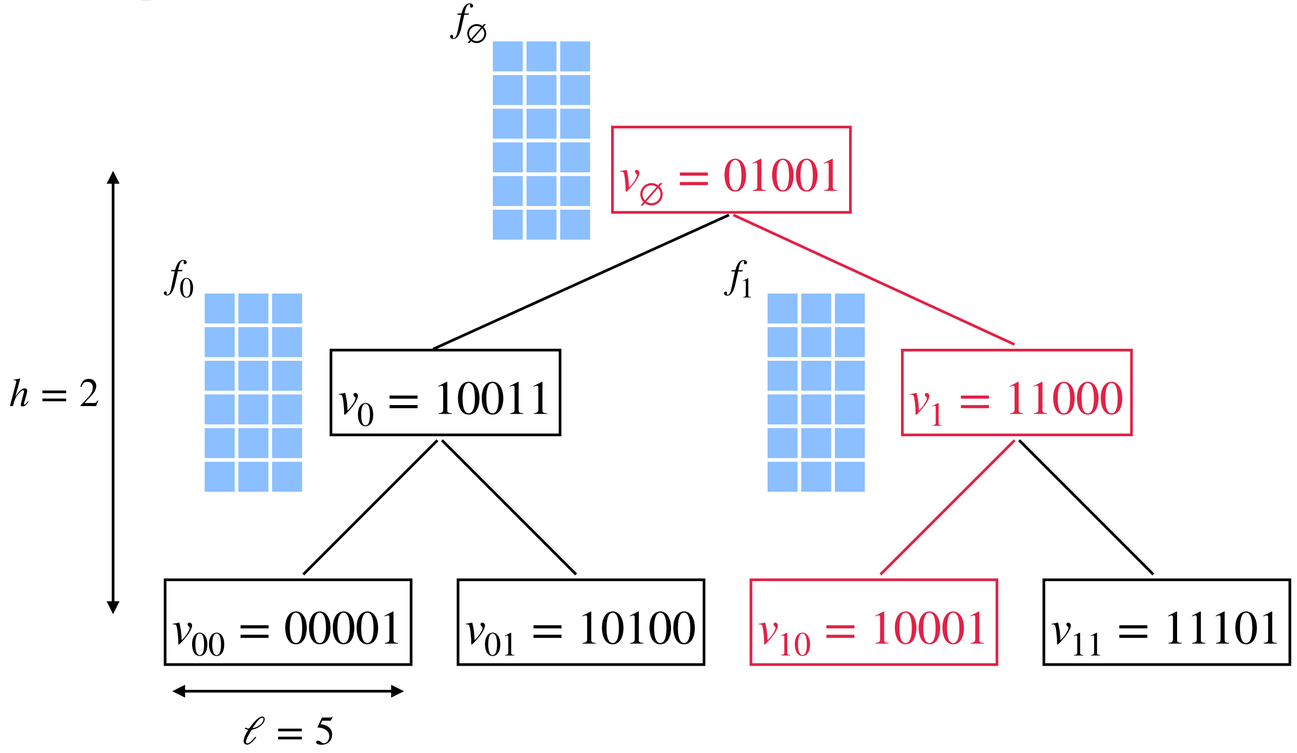
We leverage this connection and use matching-vector PIR to get new algorithms for TreeEval.


Catalytica
Privatopia
1. Tree evaluation problem
2. Catalytic approaches to TreeEval
3. Cook-Mertz algorithm
Bridge: computing on masked input
4. Private information retrieval (PIR)
5. Reed-Muller PIR
6. Matching Vector PIR
7. Our new catalytic TreeEval algorithm!

Bird's Eye View

Recursion + catalytic ideas

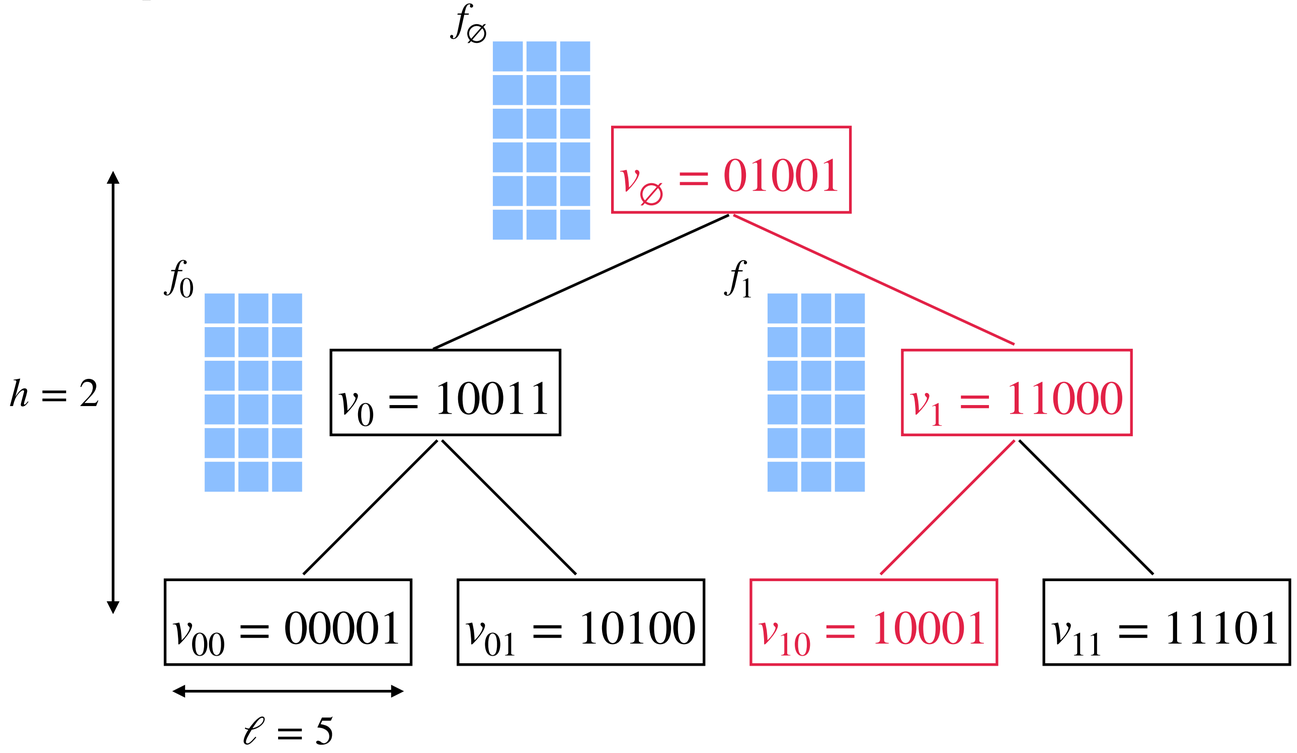


$$n = \text{total input length} = \text{poly}(2^{h+\ell})$$

Bird's Eye View

Recursion + catalytic ideas

- Memory bottleneck of naive recursive algorithm: storing all values (ℓ bits each) along a path (length h) in the tree



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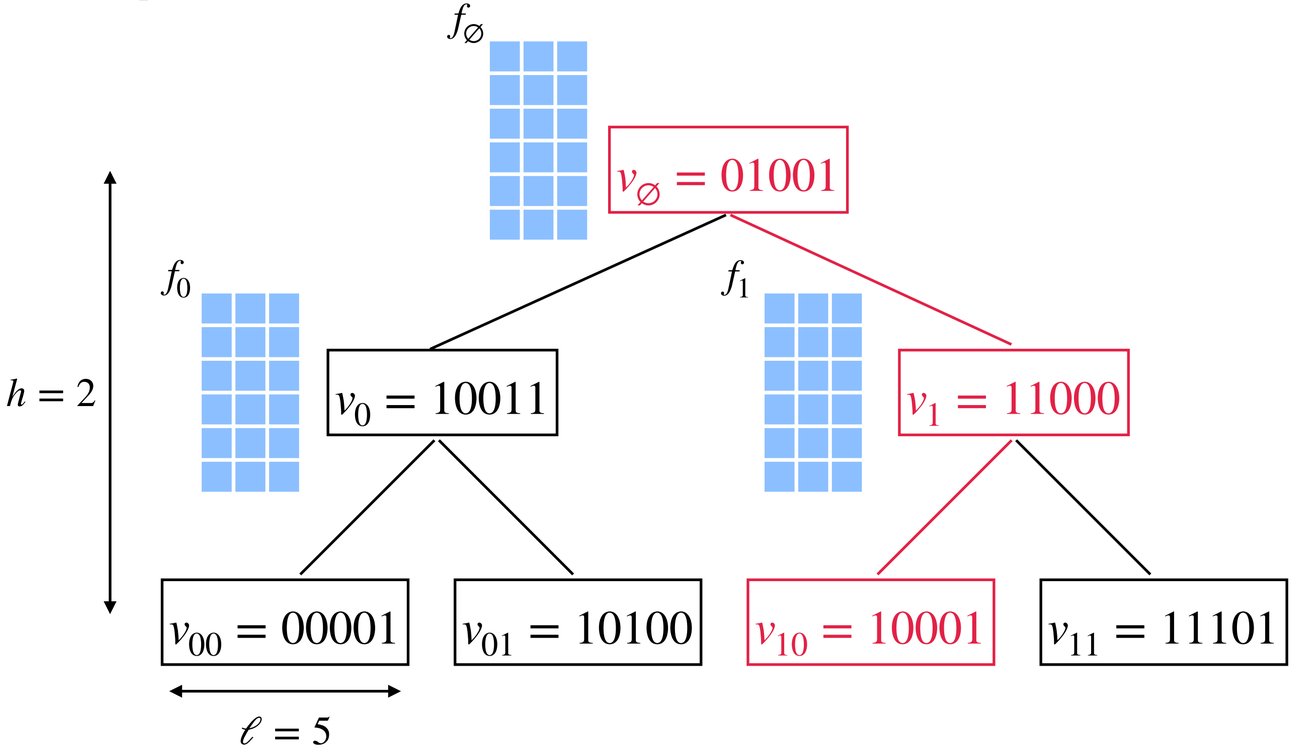

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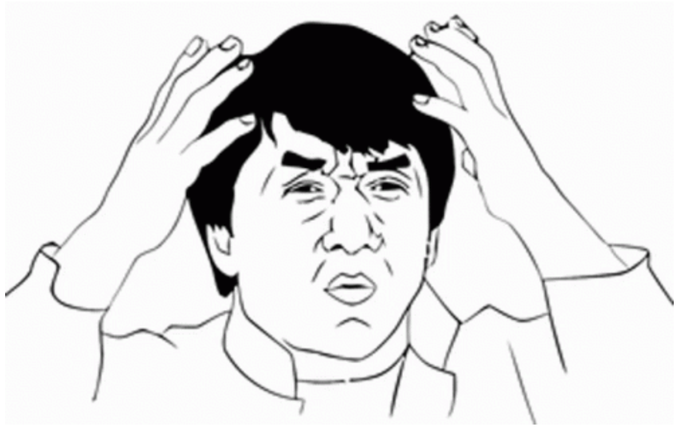

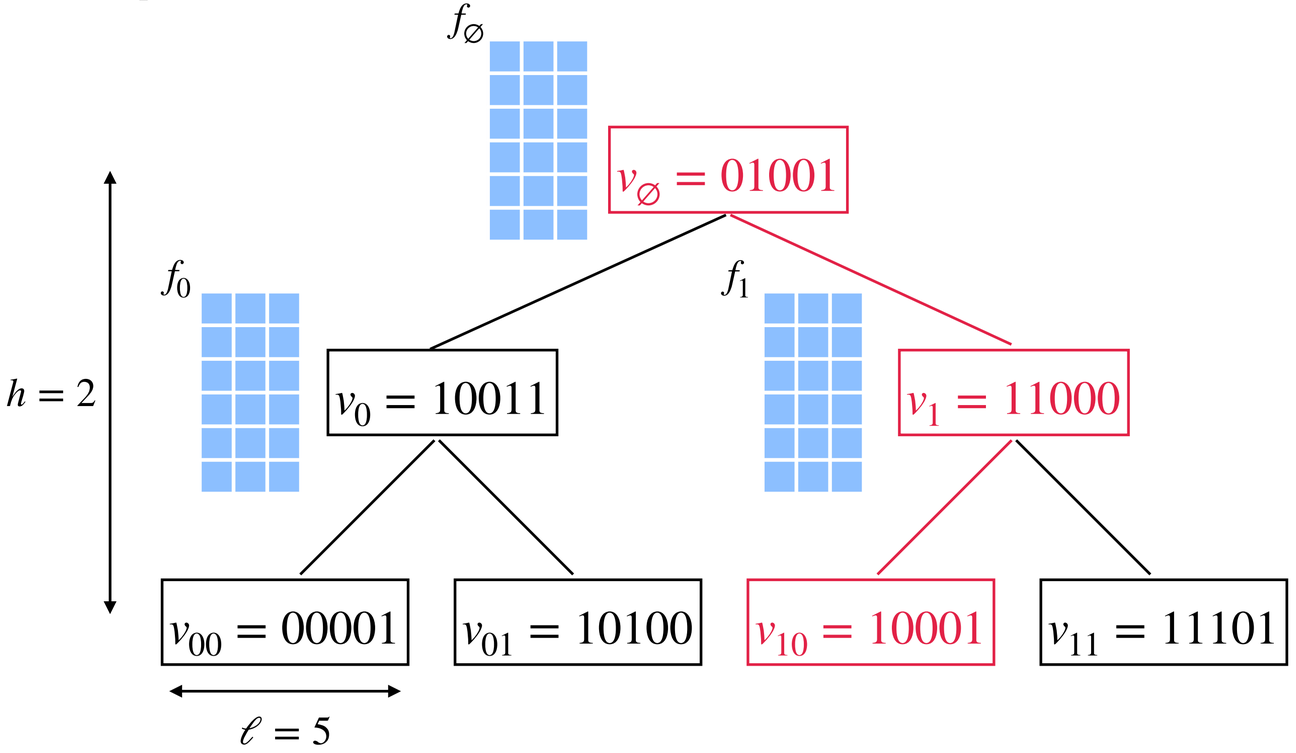

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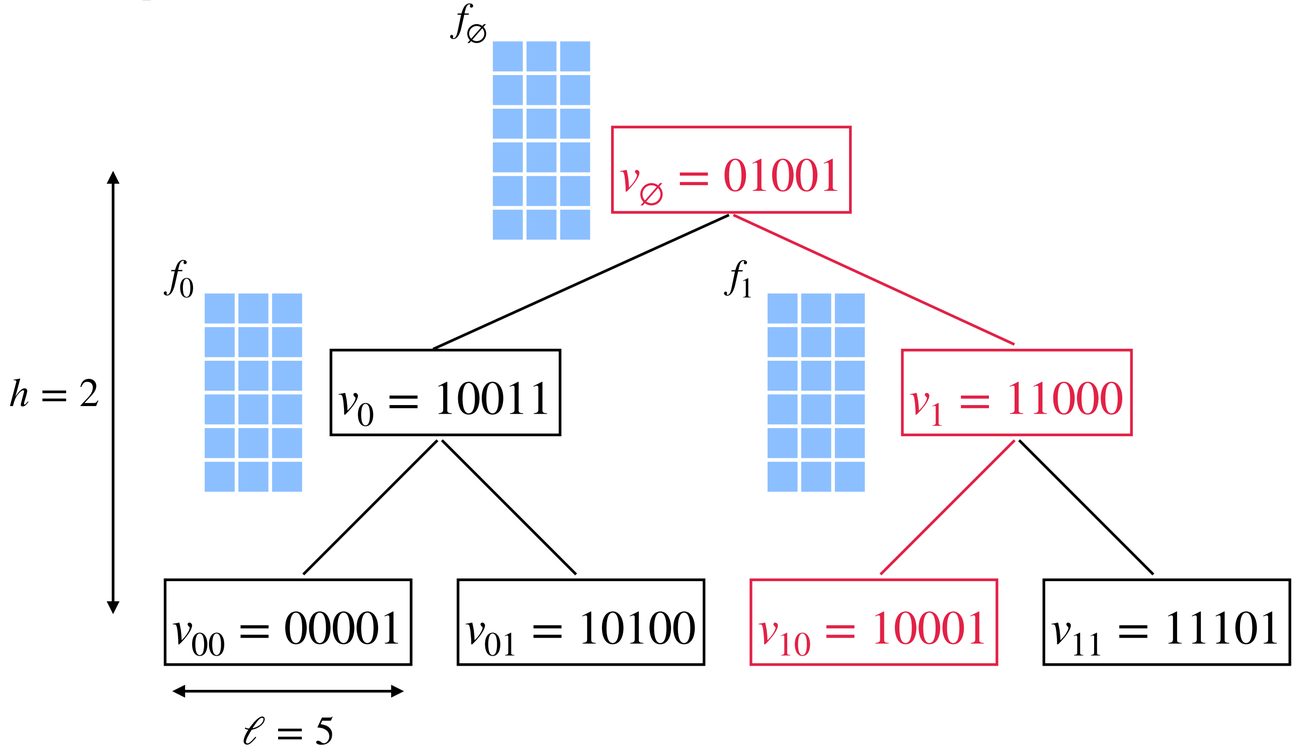
- **Our notion of "computing value v_u": implement an oracle $\mathcal{O}_u$ that on input $c \in \{0,1\}$ updates**

$$\tau \leftarrow \tau + (-1)^c v_u$$



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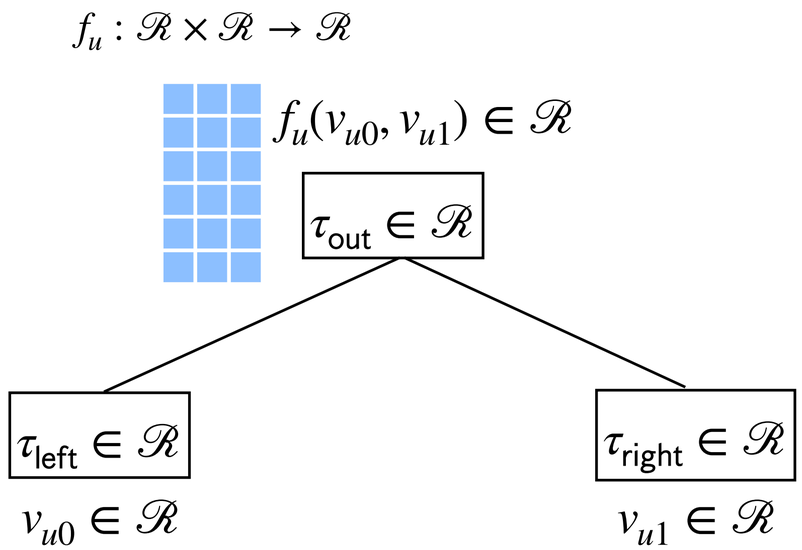
Reduction to a Simpler Problem: The One-Level Gadget

The One-Level Gadget

Or: boosting an oracle one level



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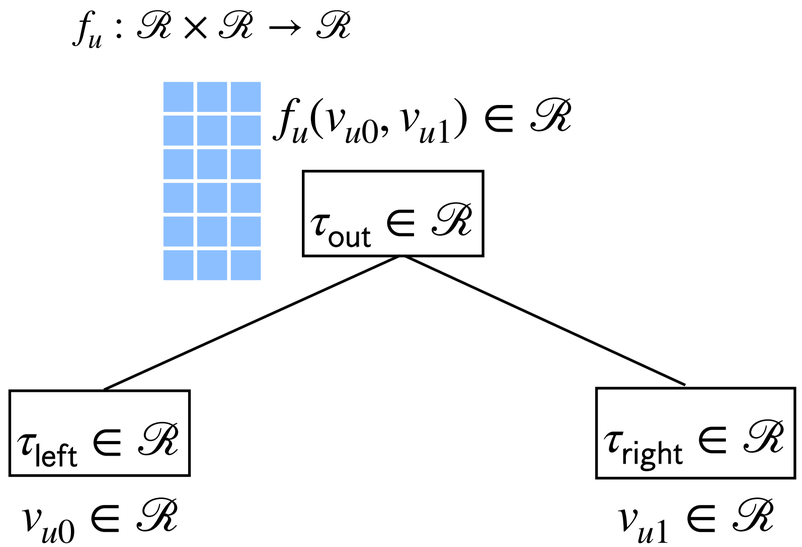

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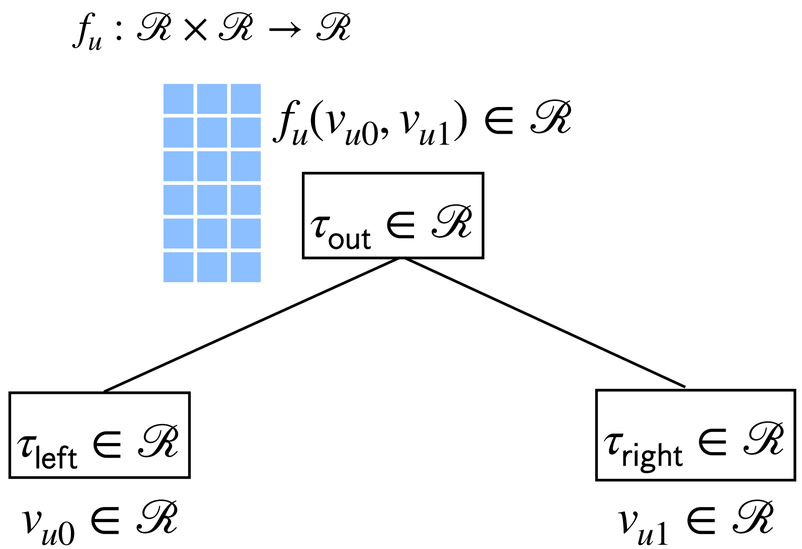

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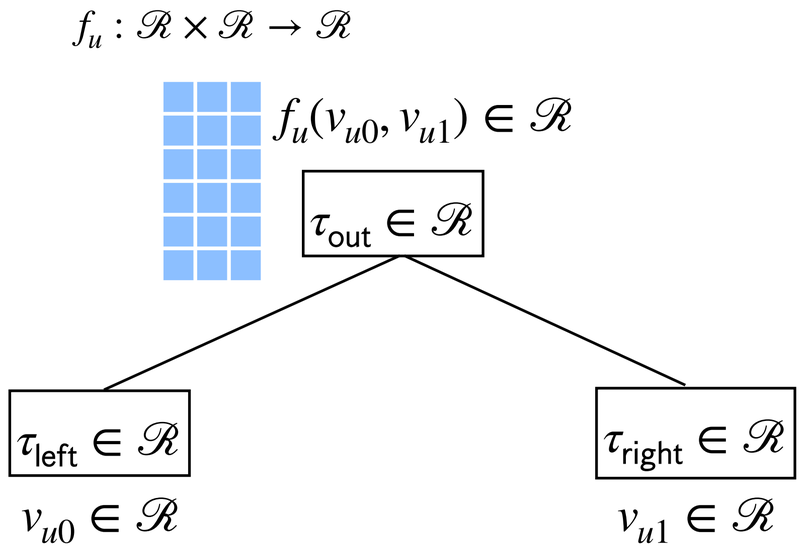

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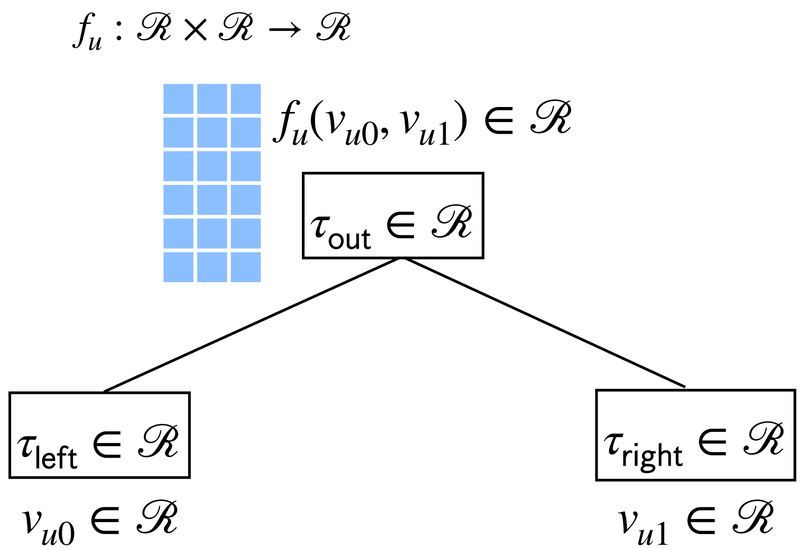

Cheatsheet

$\mathcal{O}_u(c \in \{0,1\}, \tau \in \mathscr{R})$
updates $\tau \leftarrow \tau + (-1)^c v_u$

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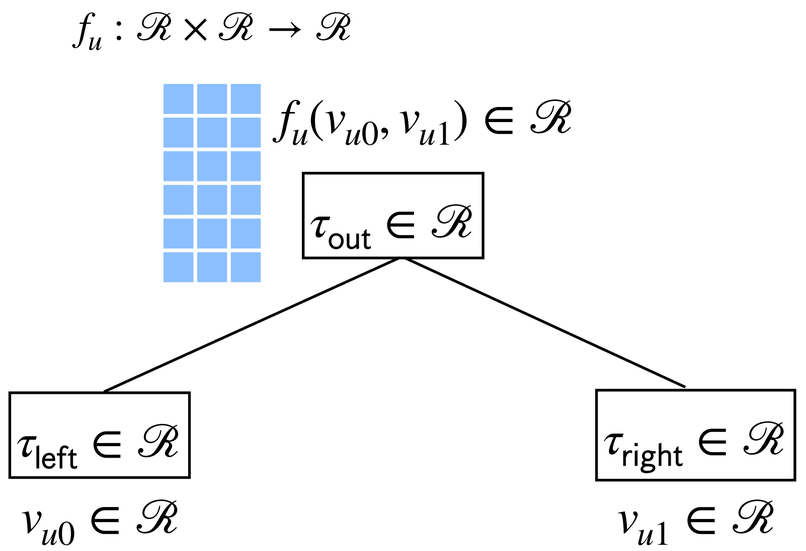
- Goal: implement the oracle $\mathcal{O}_u$ on τ_{out} i.e., want the registers to end up in the state

$$(\tau_{\text{left}}, \tau_{\text{right}}, \tau_{\text{out}} + (-1)^c f_u(v_{u0}, v_{u1}))$$



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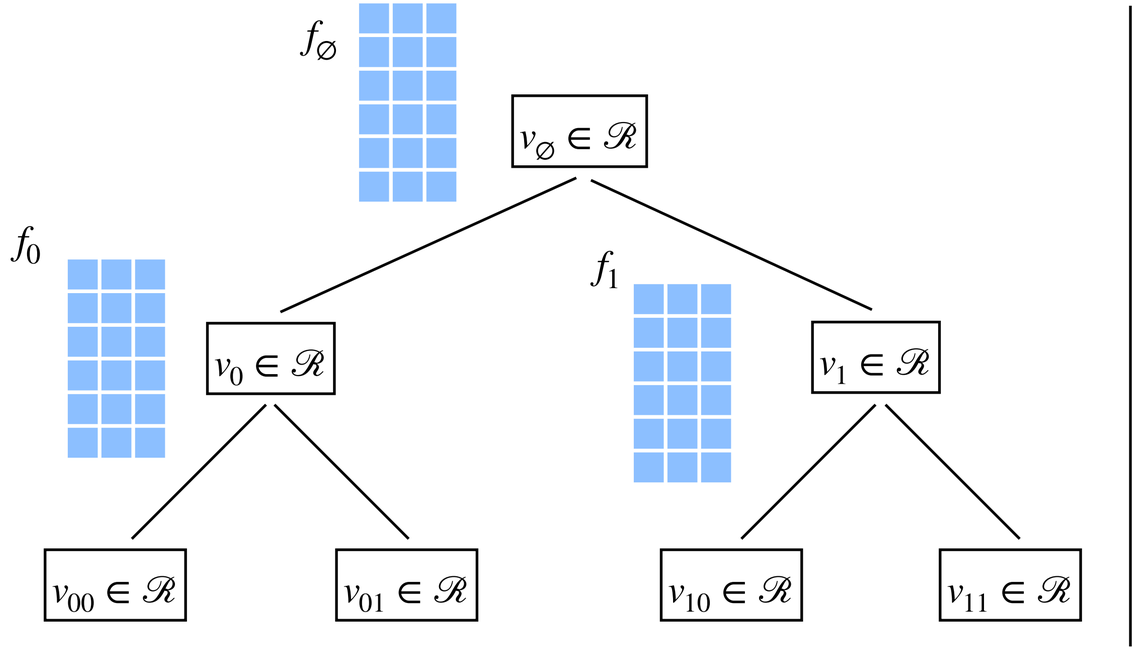
From One Level to Many

Theorem (informal): assume there is a $\mathsf{poly}(2^{\ell})$ time algorithm for the one-level gadget that makes q oracle queries and uses free space s. Then:

$$\mathsf{TreeEval}(h, \ell) \in \mathsf{CatSpaceTime}(\log|\mathscr{R}|, hs+\log\log|\mathscr{R}|+\ell, \mathsf{poly}(q^h, 2^{\ell}))$$



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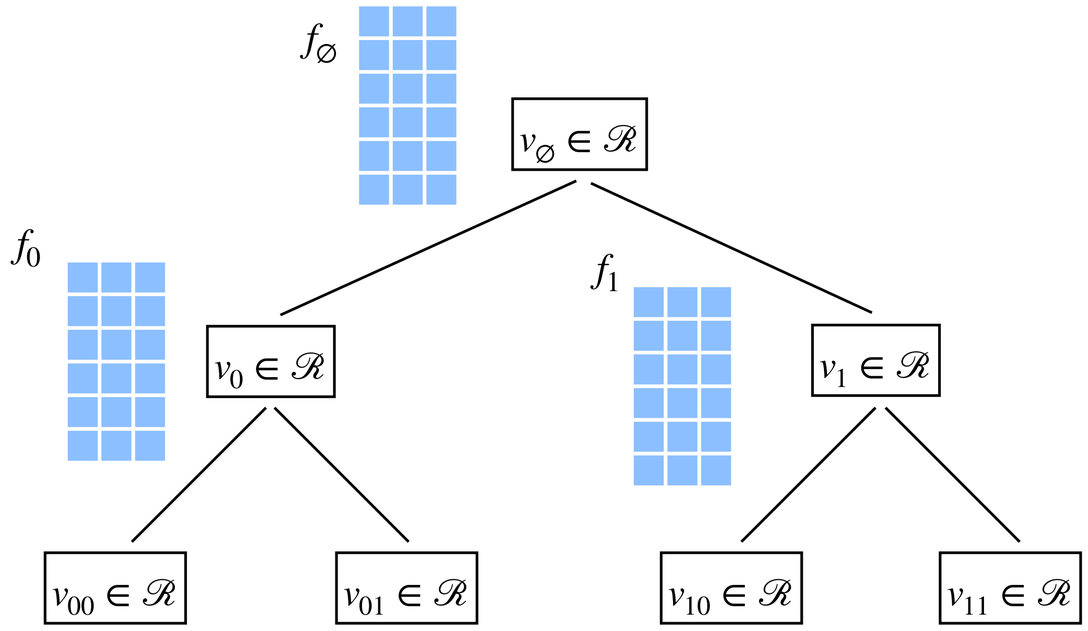

Catalytic tape

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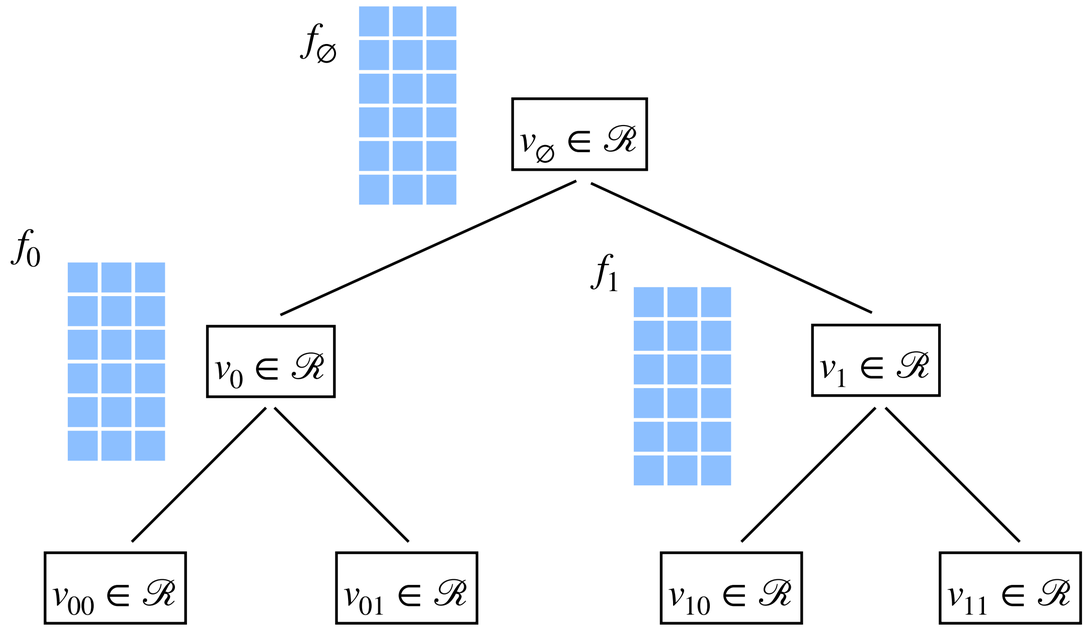

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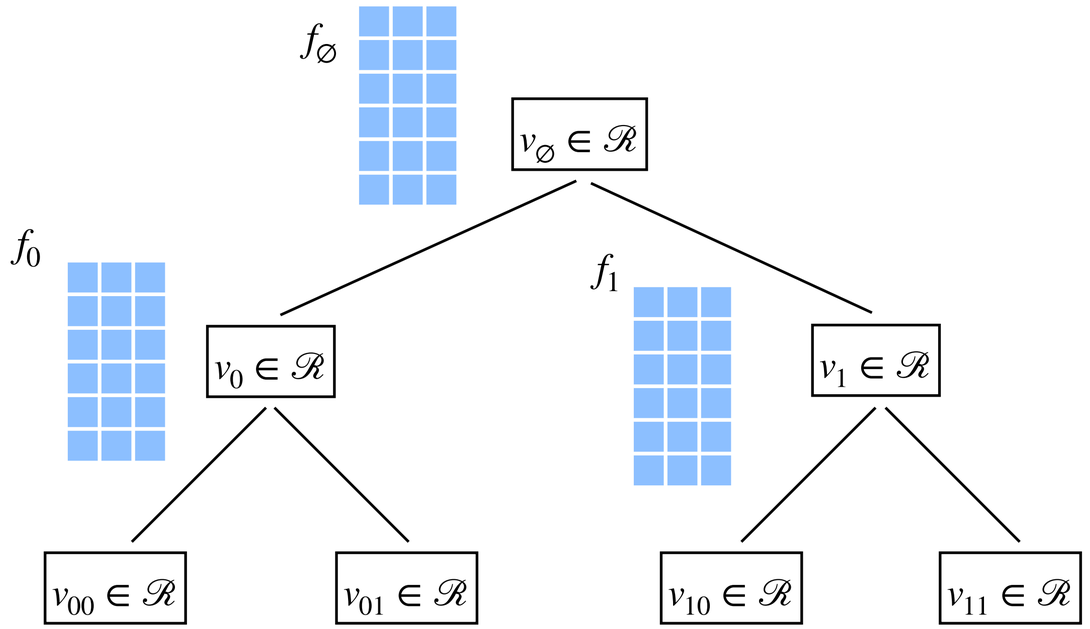

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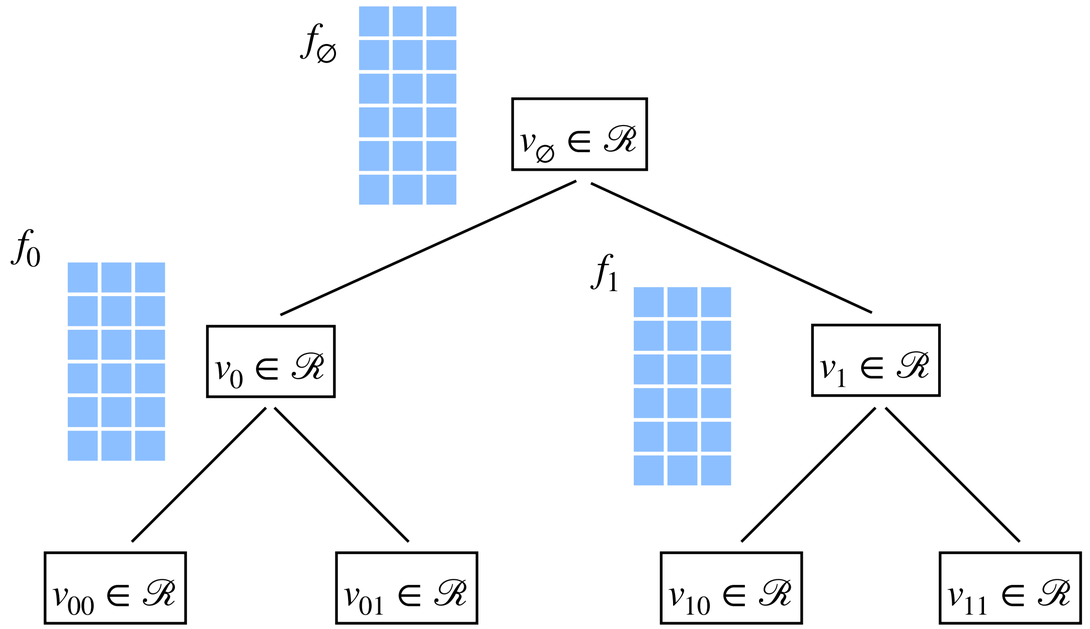

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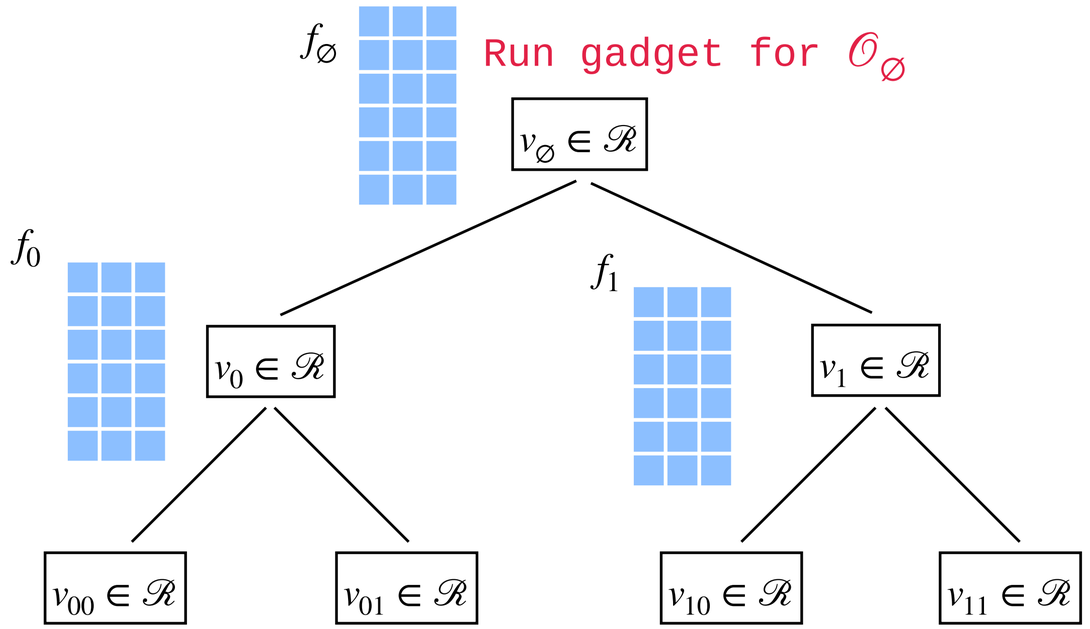

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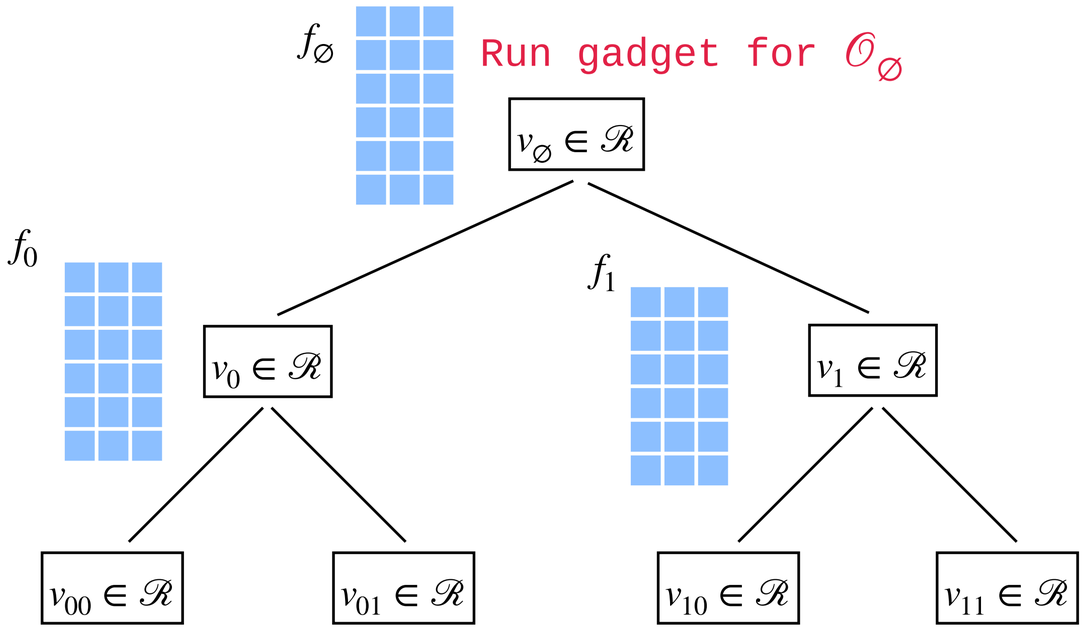

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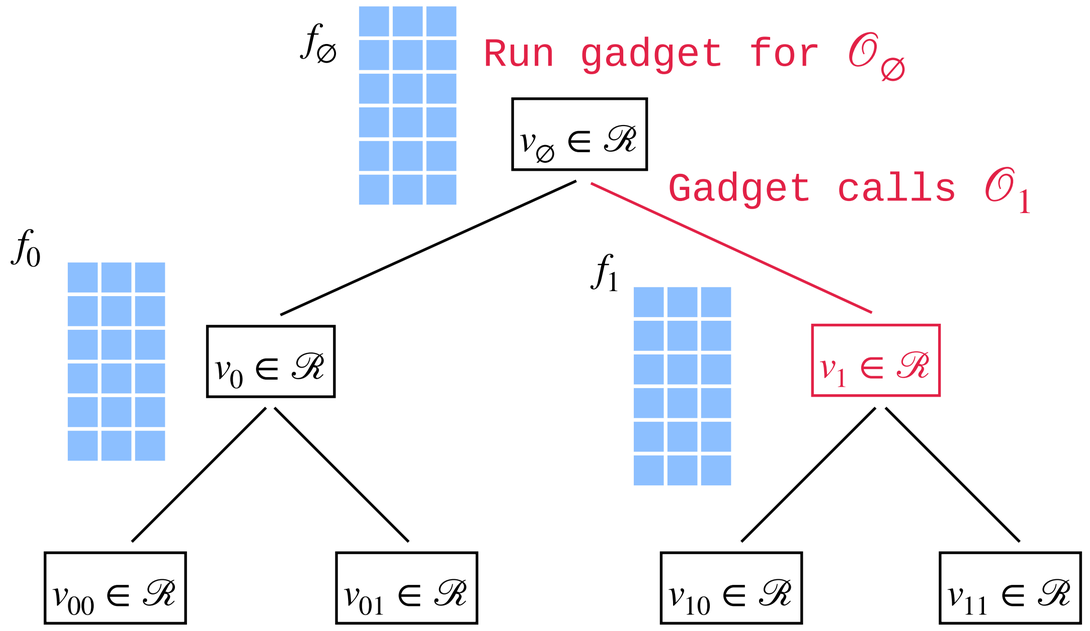

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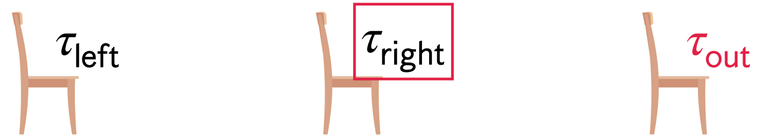

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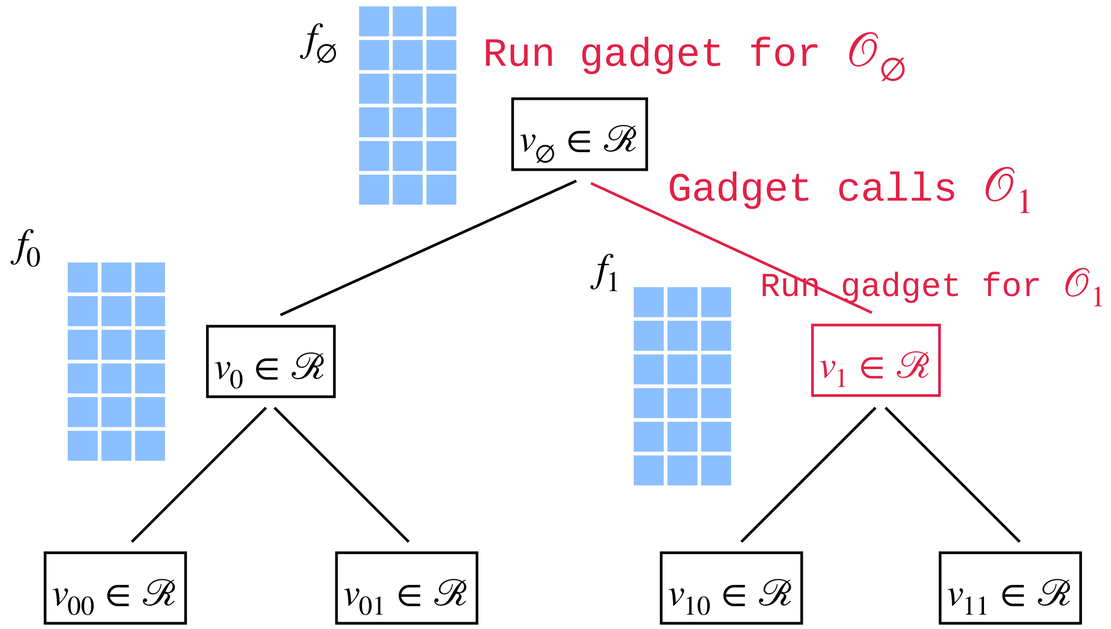

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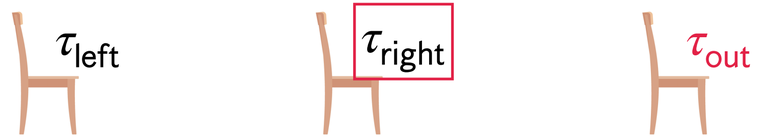

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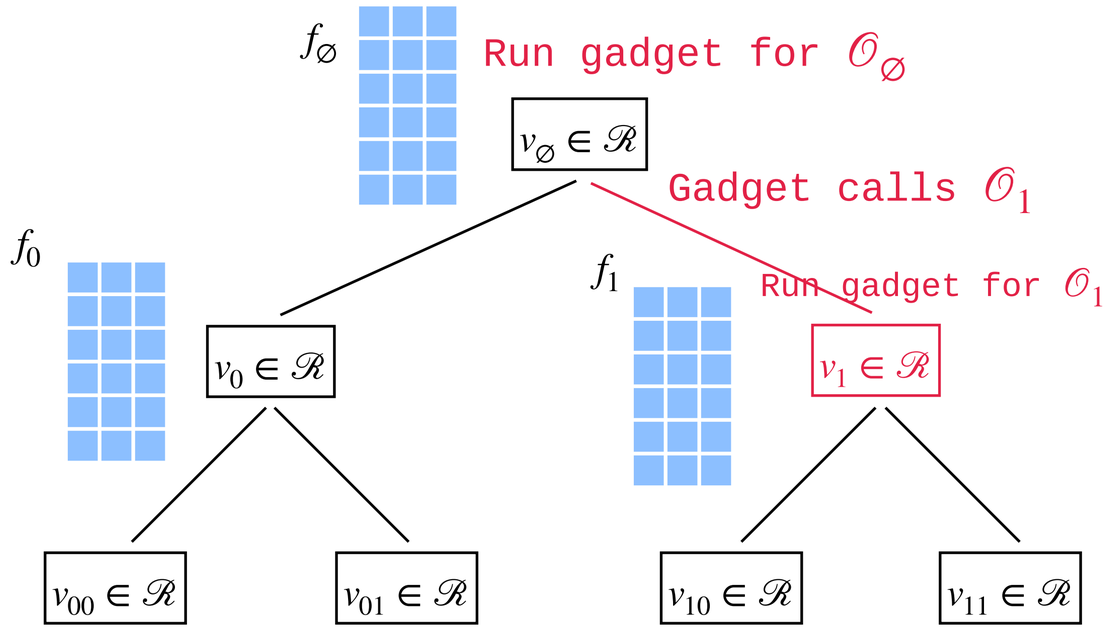

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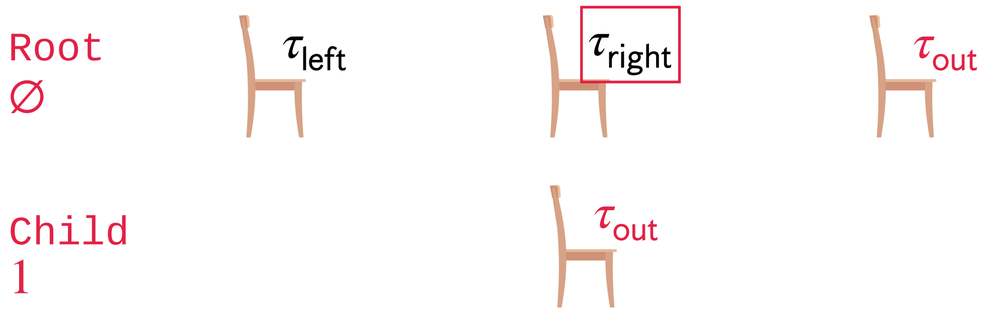

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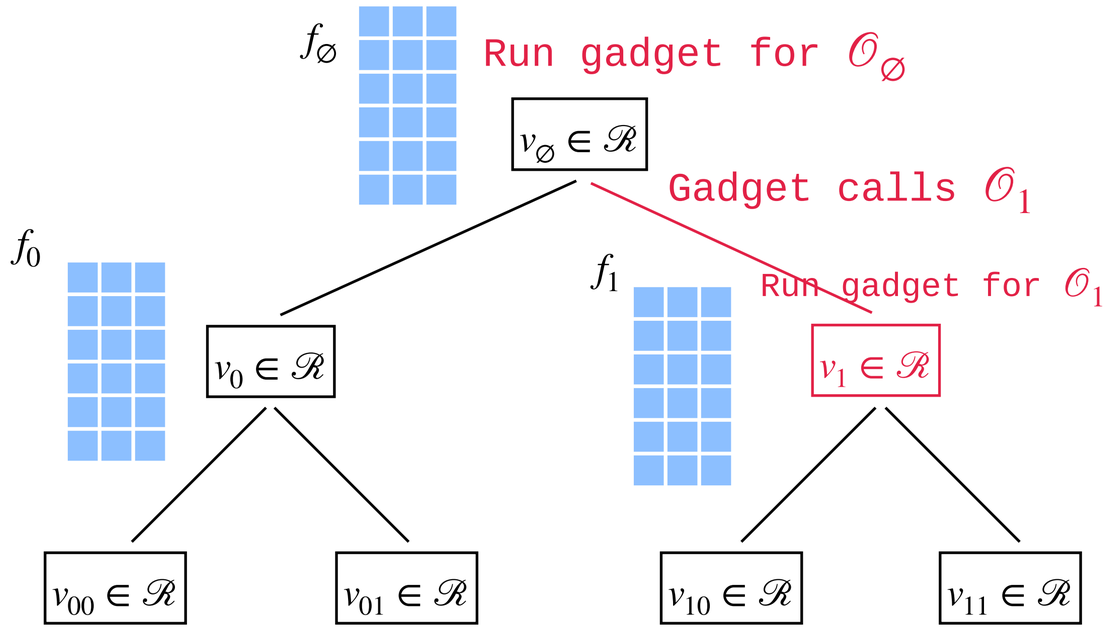

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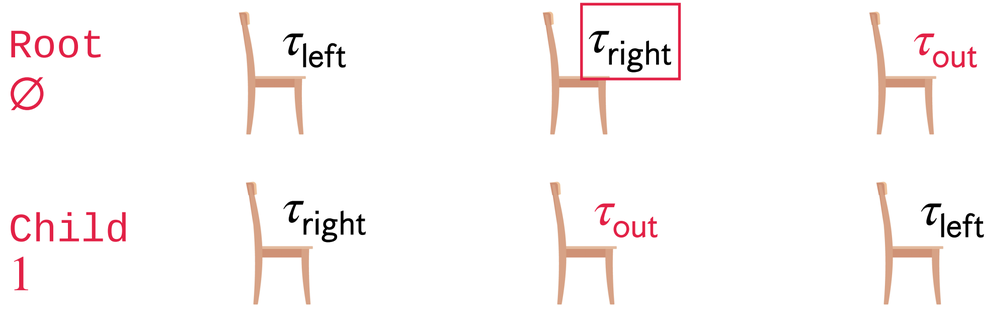

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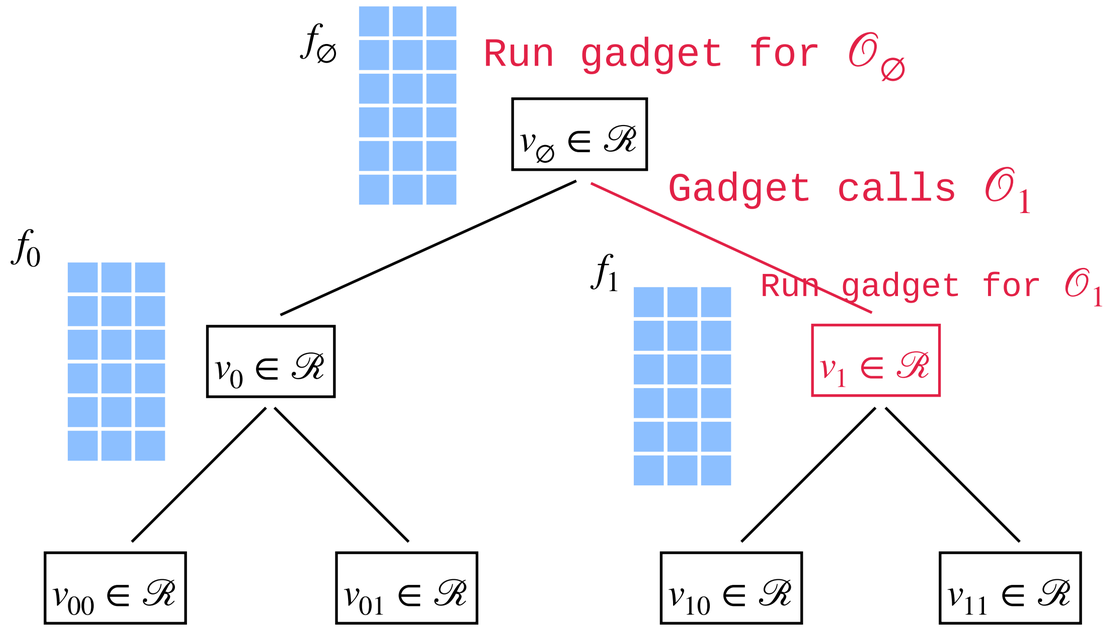
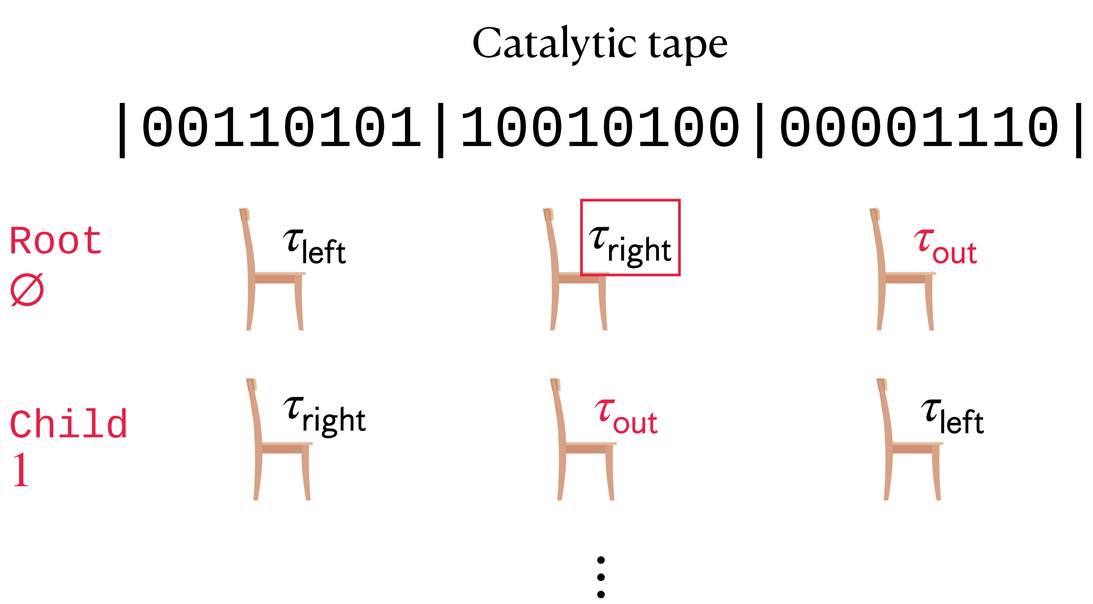


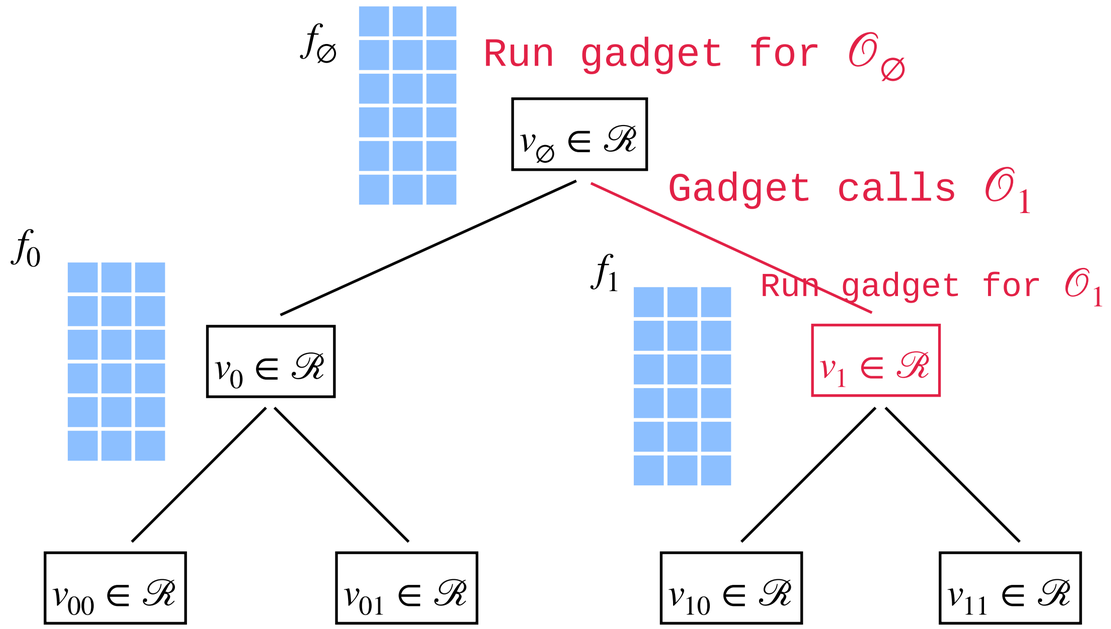


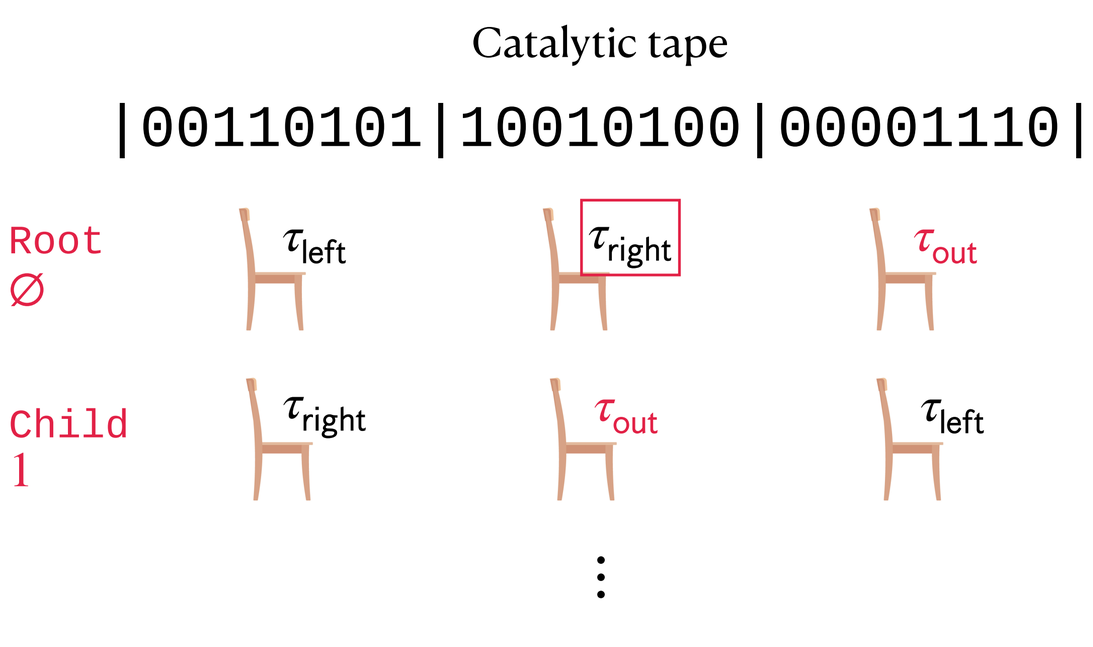
From One Level to Many

Cheatsheet

$\mathcal{O}_u(c \in \{0,1\}, \tau \in \mathcal{R})$
updates $\tau \leftarrow \tau + (-1)^c v_u$

- Catalytic space: $O(\log |\mathcal{R}|)$
- Free space (for low-level information): s per node in a call stack of height $h \to hs$
- Time: q recursive calls at each node in a tree of height $h \to q^h$





The One-Level Gadget: Summary

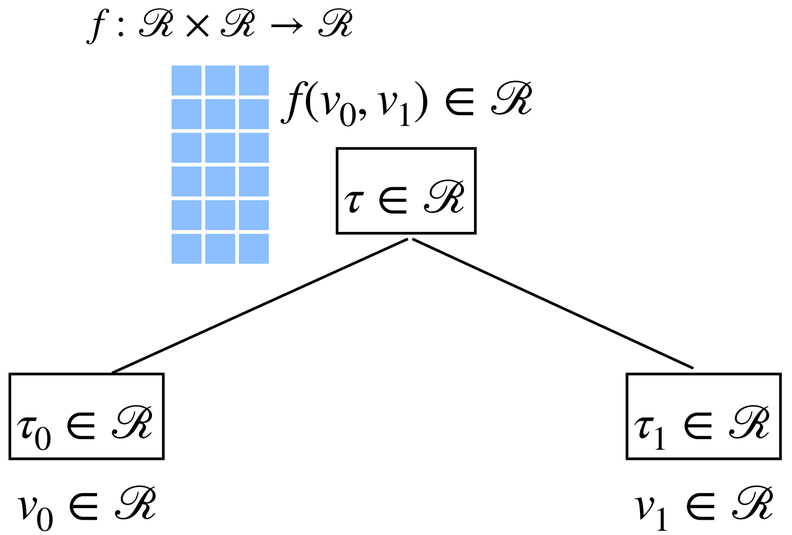
Centrepiece of Cook-Mertz and our work: all I will focus on for the rest of the talk

- Embed $\{0,1\}^{\ell}$ in some ring $\mathscr{R}$
- Inputs:
  - Control bit $c' \in \{0,1\}$
  - Arbitrary catalytic registers $\tau_0, \tau_1, \tau \in \mathscr{R}$
  - Oracle $\mathcal{O}_{v_0,v_1}(b, c \in \{0,1\})$: updates

$$\tau_b \leftarrow \tau_b + (-1)^c v_b$$

- Output: want the registers to end up in the state

$$(\tau_0, \tau_1, \tau) = (\tau_{0,\text{init}}, \tau_{1,\text{init}}, \tau_{\text{init}} + (-1)^{c'} f(v_0, v_1))$$

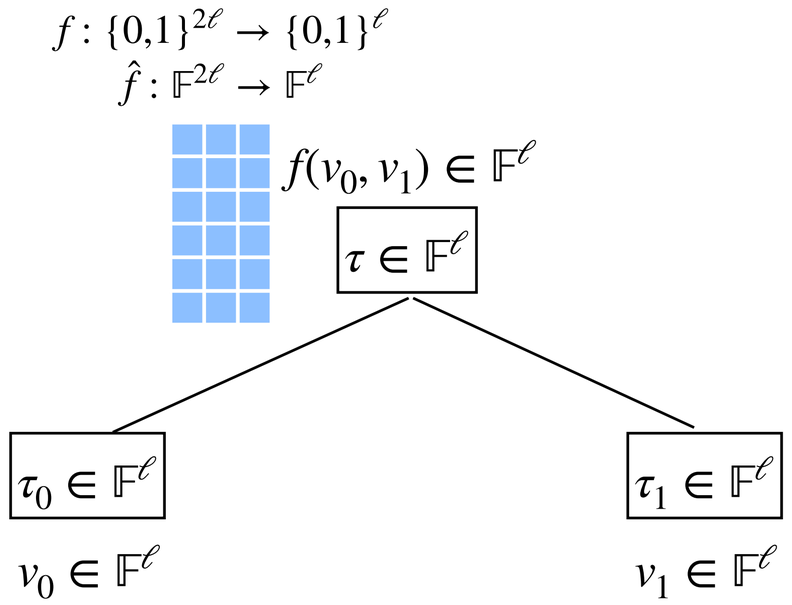


Metrics

- $\log |\mathscr{R}|$: final catalytic space
- $h \cdot$ [free space]: final free space
- [oracle queries]^h: final runtime

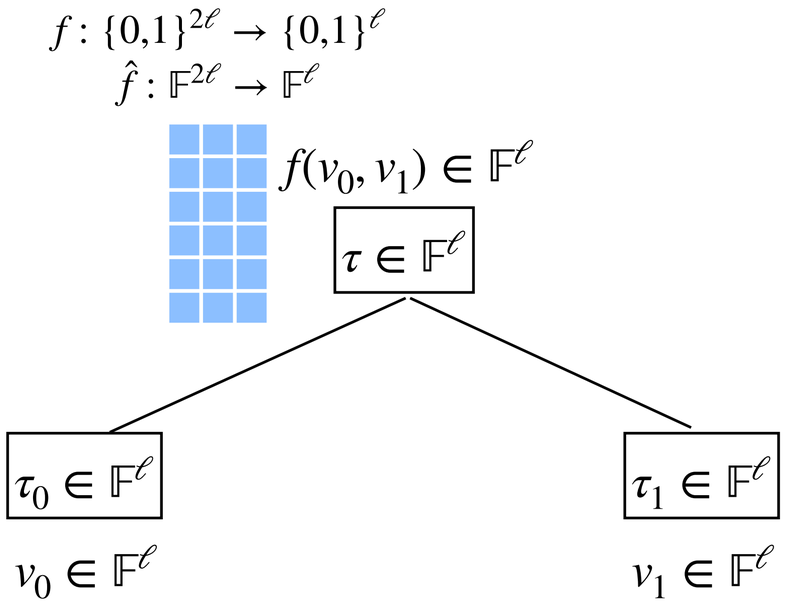
Cook-Mertz's One-Level Gadget

Setup


$f : \{0,1\}^{2\ell} \to \{0,1\}^{\ell}$
$\hat{f} : \mathbb{F}^{2\ell} \to \mathbb{F}^{\ell}$
$f(v_0, v_1) \in \mathbb{F}^{\ell}$
$\tau \in \mathbb{F}^{\ell}$
$\tau_0 \in \mathbb{F}^{\ell}$
$\tau_1 \in \mathbb{F}^{\ell}$
$v_0 \in \mathbb{F}^{\ell}$
$v_1 \in \mathbb{F}^{\ell}$

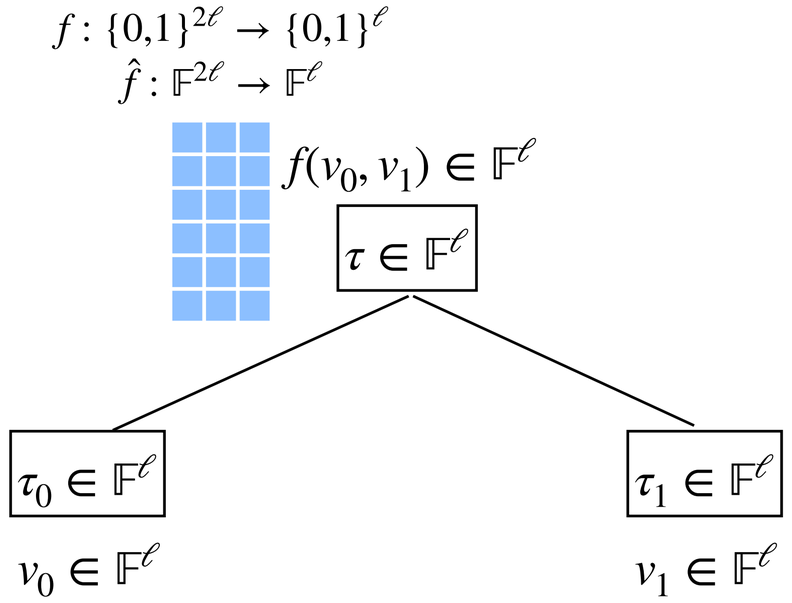
Setup

- Embed $\{0,1\}$ into a field $\mathbb{F}$ of size $\geq 2\ell + 2$



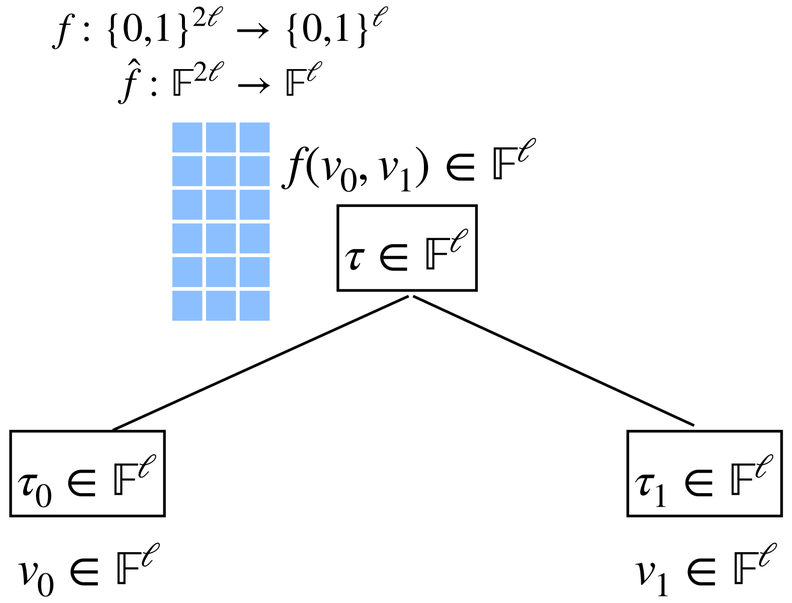
Setup

- Embed $\{0,1\}$ into a field $\mathbb{F}$ of size $\geq 2\ell + 2$
- Ring $\mathscr{R} = \mathbb{F}^{\ell}$



Setup

- Embed $\{0,1\}$ into a field $\mathbb{F}$ of size $\geq 2\ell + 2$
- Ring $\mathscr{R} = \mathbb{F}^{\ell}$
- Multilinear extension $\hat{f} : \mathbb{F}^{2\ell} \to \mathbb{F}^{\ell}$



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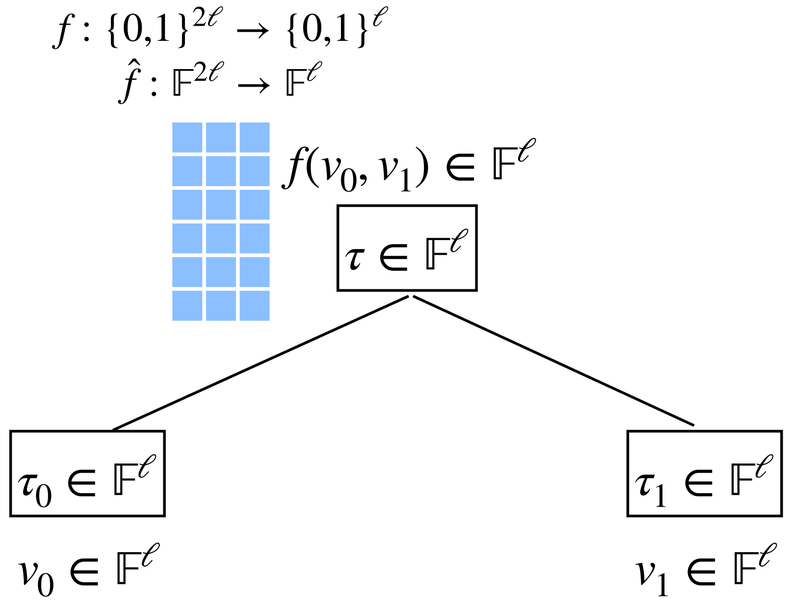

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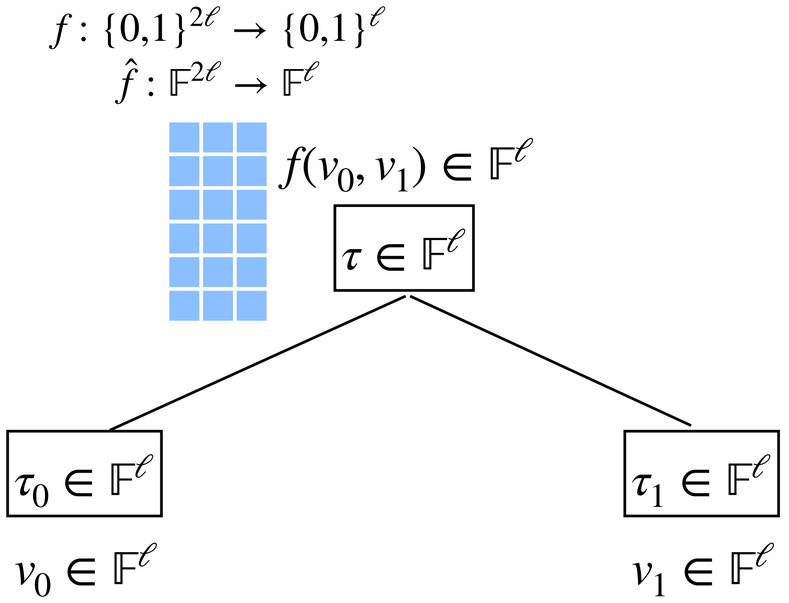

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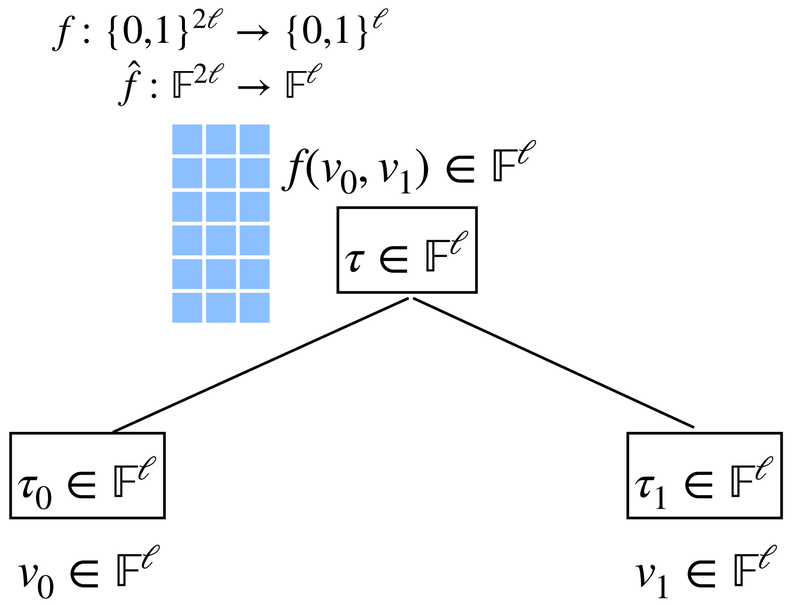
- Idea: recover $g = \hat{f}\Big|_L$ via Lagrange interpolation



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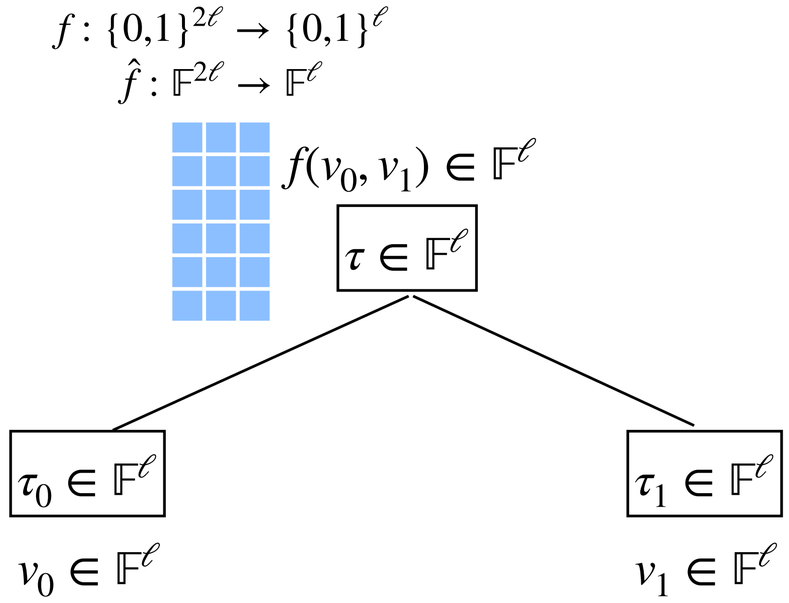
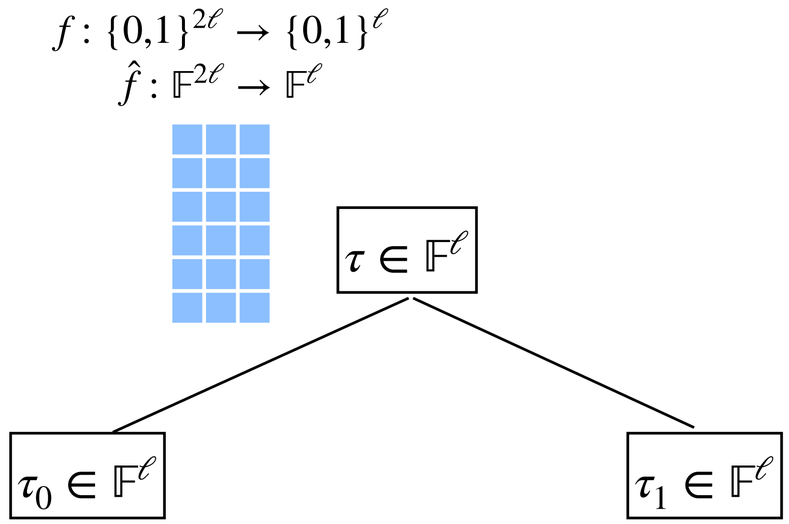
  - Then can just add $g(0)$ into τ



Details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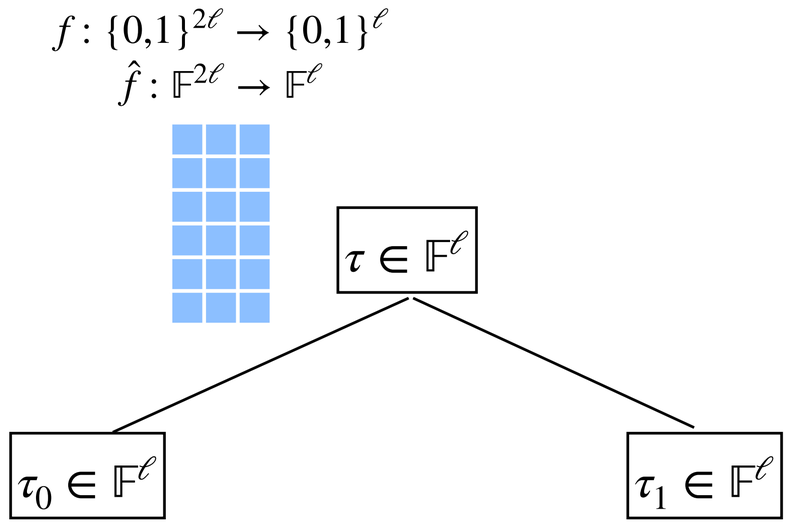

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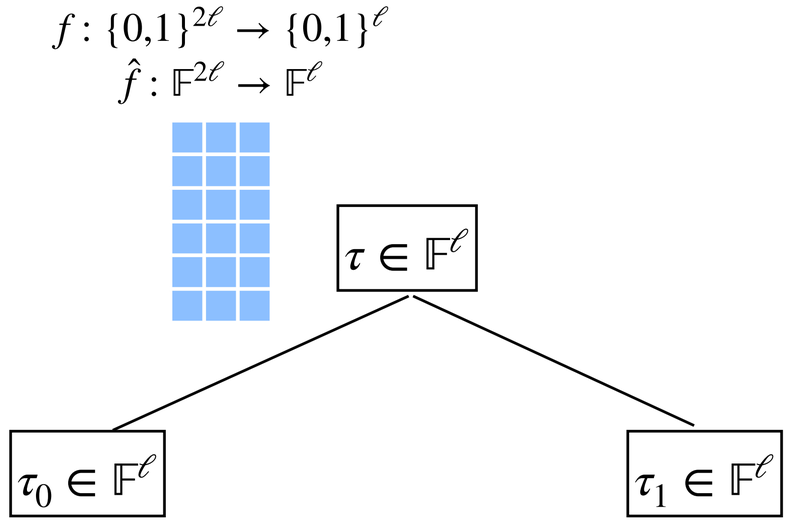

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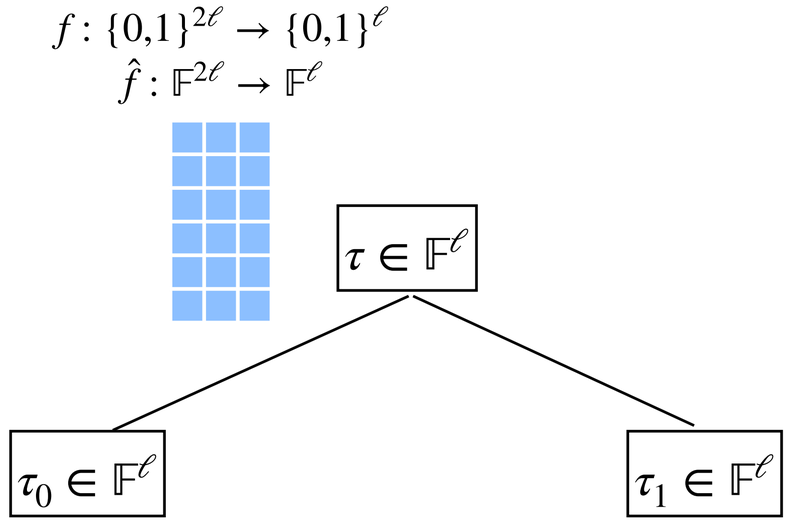

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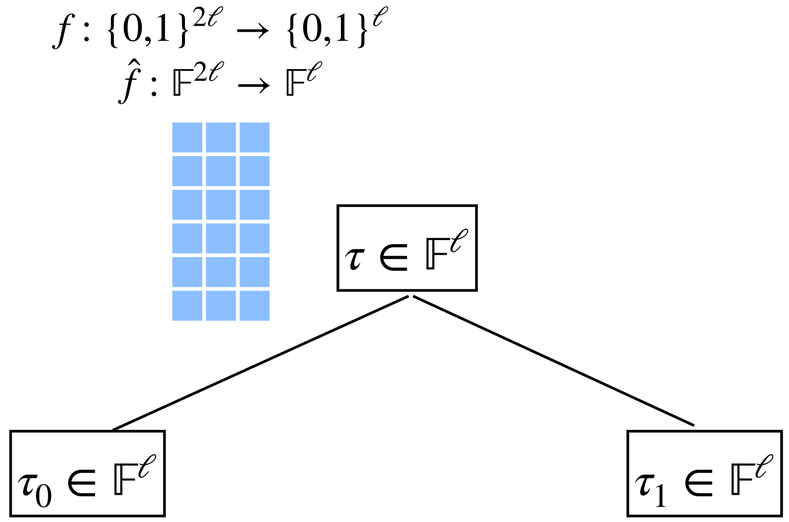

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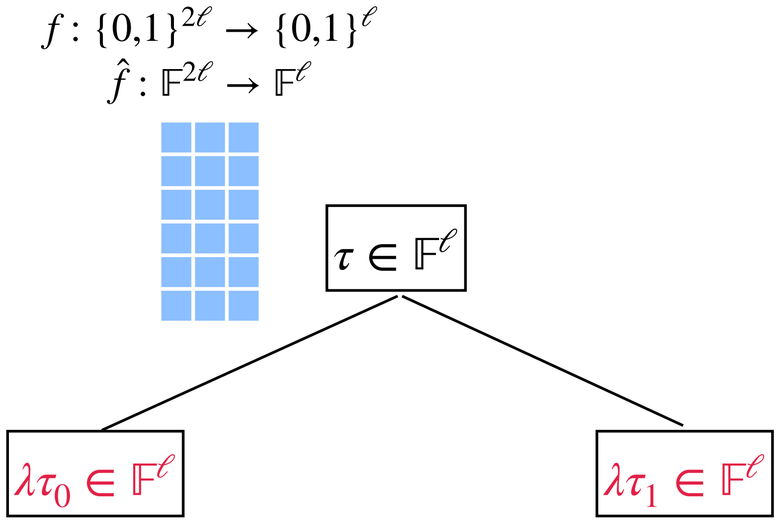

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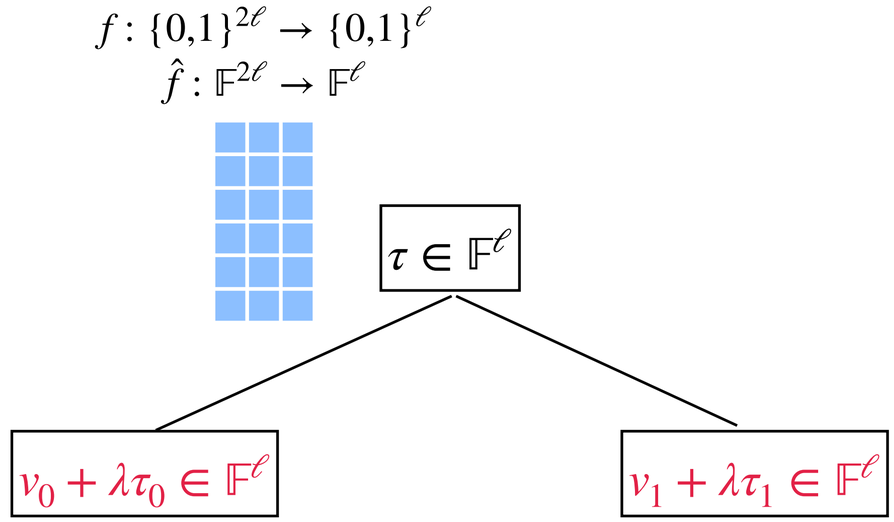

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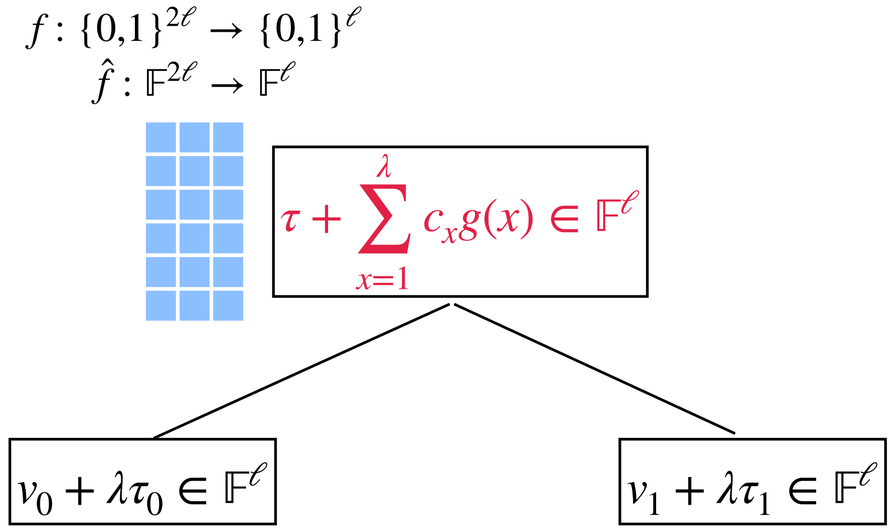

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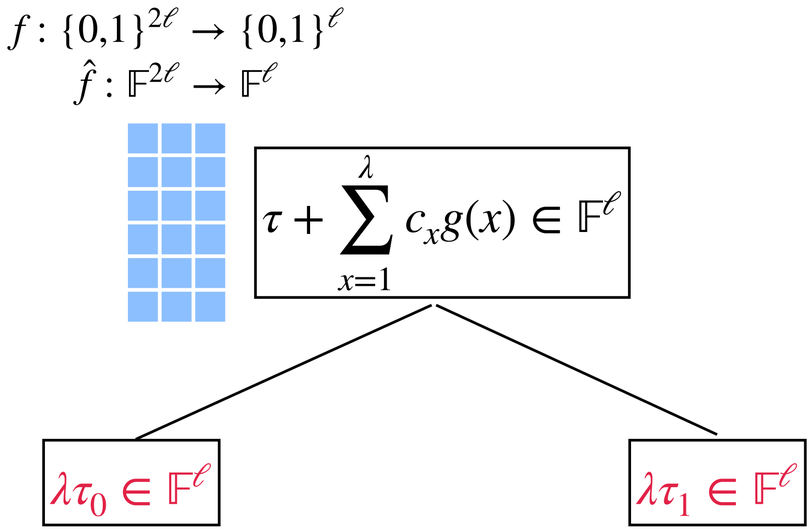

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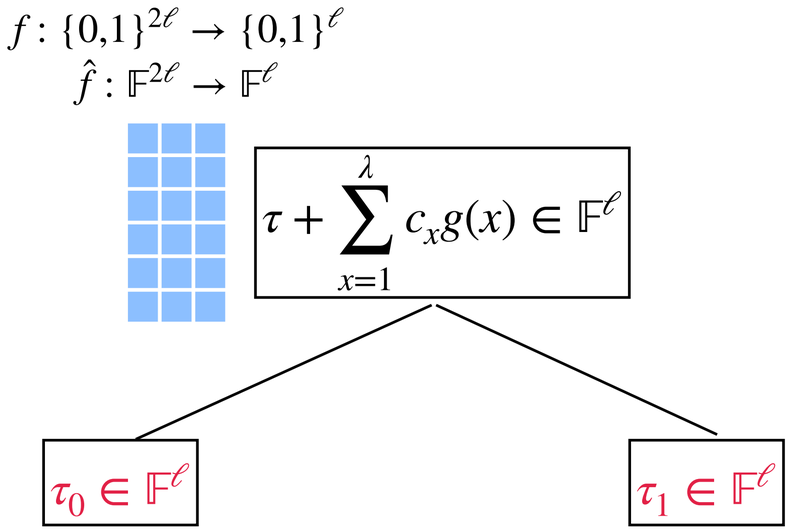
  - Reset (τ_0, τ_1) by dividing by λ



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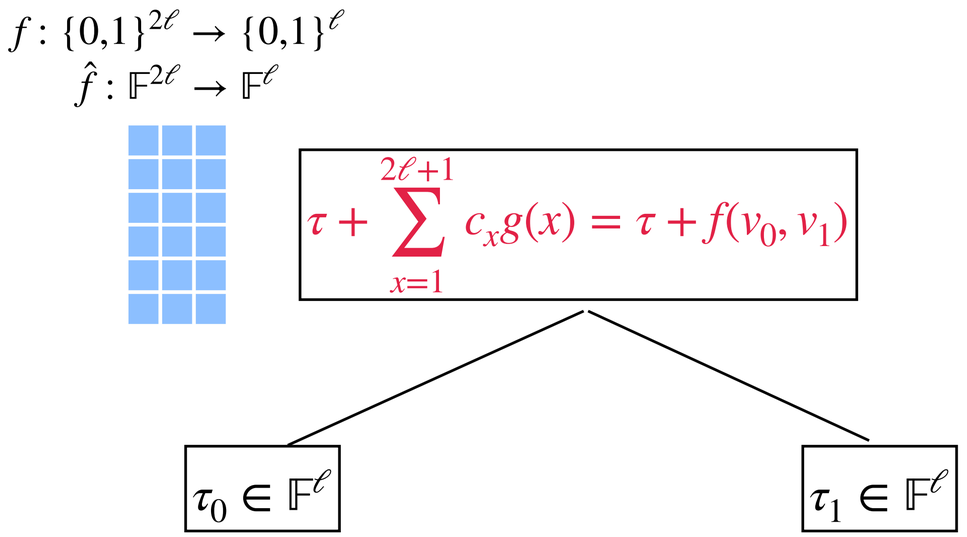

Efficiency Analysis

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Metrics

- $\mathscr{R} = \mathbb{F}^{\ell}$

Efficiency Analysis

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Metrics

- $\mathscr{R} = \mathbb{F}^{\ell}$
  - Final catalytic space
    $\log|\mathscr{R}| = \ell \log \ell$
    $= O(\log n \log\log n)$

Efficiency Analysis

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  - Multilinear $\hat{f} : \mathbb{F}^{2\ell} \to \mathbb{F}^{\ell}$
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Metrics

- $\mathscr{R} = \mathbb{F}^{\ell}$
  - Final catalytic space
    $\log|\mathscr{R}| = \ell \log \ell$
    $= O(\log n \log\log n)$
- Free space: $\log \ell$

Efficiency Analysis

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Metrics

- $\mathscr{R} = \mathbb{F}^{\ell}$
  - Final catalytic space
    $\log|\mathscr{R}| = \ell \log \ell$
    $= O(\log n \log\log n)$
- Free space: $\log \ell$
  - Final free space:
    $h \log \ell = O(\log n \log\log n)$

Efficiency Analysis

- Setup
  - Multilinear $\hat{f} : \mathbb{F}^{2\ell} \to \mathbb{F}^{\ell}$
  - Line $L = \{(v_0 + \lambda\tau_0, v_1 + \lambda\tau_1) : \lambda \in \mathbb{F}\}$, $g = \hat{f}\Big|_L$
  - Lagrange interpolation: $g(0) = \sum_{\lambda=1}^{2\ell+1} c_\lambda g(\lambda)$
- For each $\lambda = 1,2,\ldots,2\ell + 1$:
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  - Two oracle calls $\rightarrow$ add v_0, v_1 into the input registers
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Metrics

- $\mathscr{R} = \mathbb{F}^{\ell}$
  - Final catalytic space
    $\log|\mathscr{R}| = \ell \log \ell$
    $= O(\log n \log\log n)$
- Free space: $\log \ell$
  - Final free space:
    $h \log \ell = O(\log n \log\log n)$
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Efficiency Analysis

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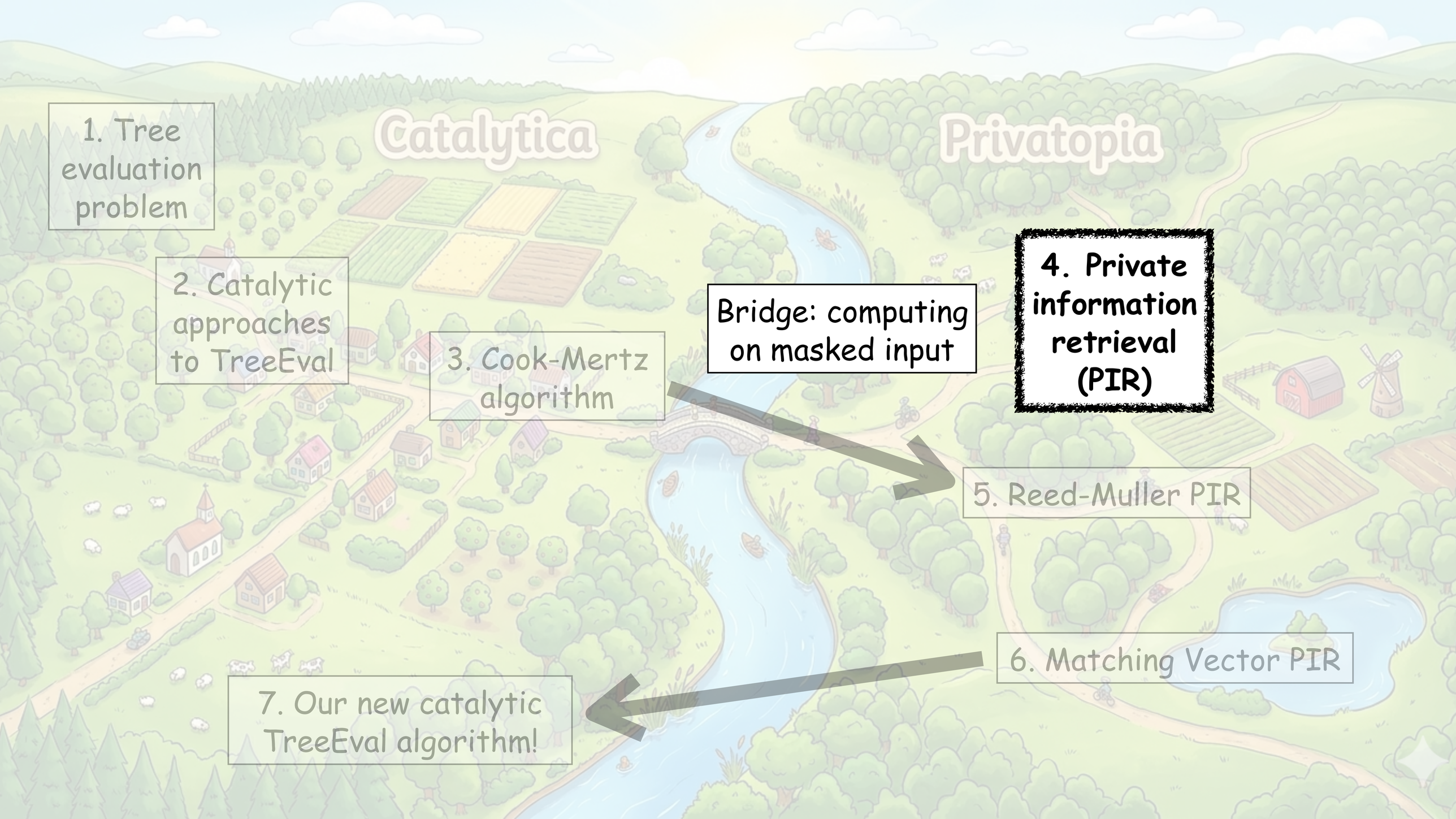
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Another reason why one would use masking is **privacy...**

Private Information Retrieval (PIR)

Closely related to locally-decodable codes (LDCs)

DB $\in \{0,1\}^n$

Server 1 

Server 2 

Server S ... 

Client

$i \in \{1,2,\ldots,n\}$

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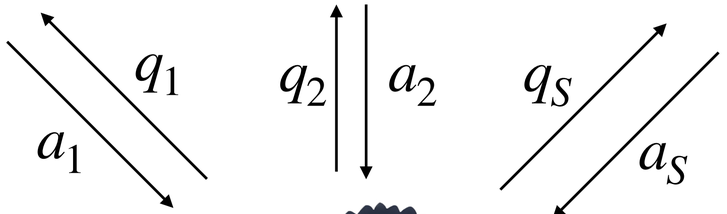

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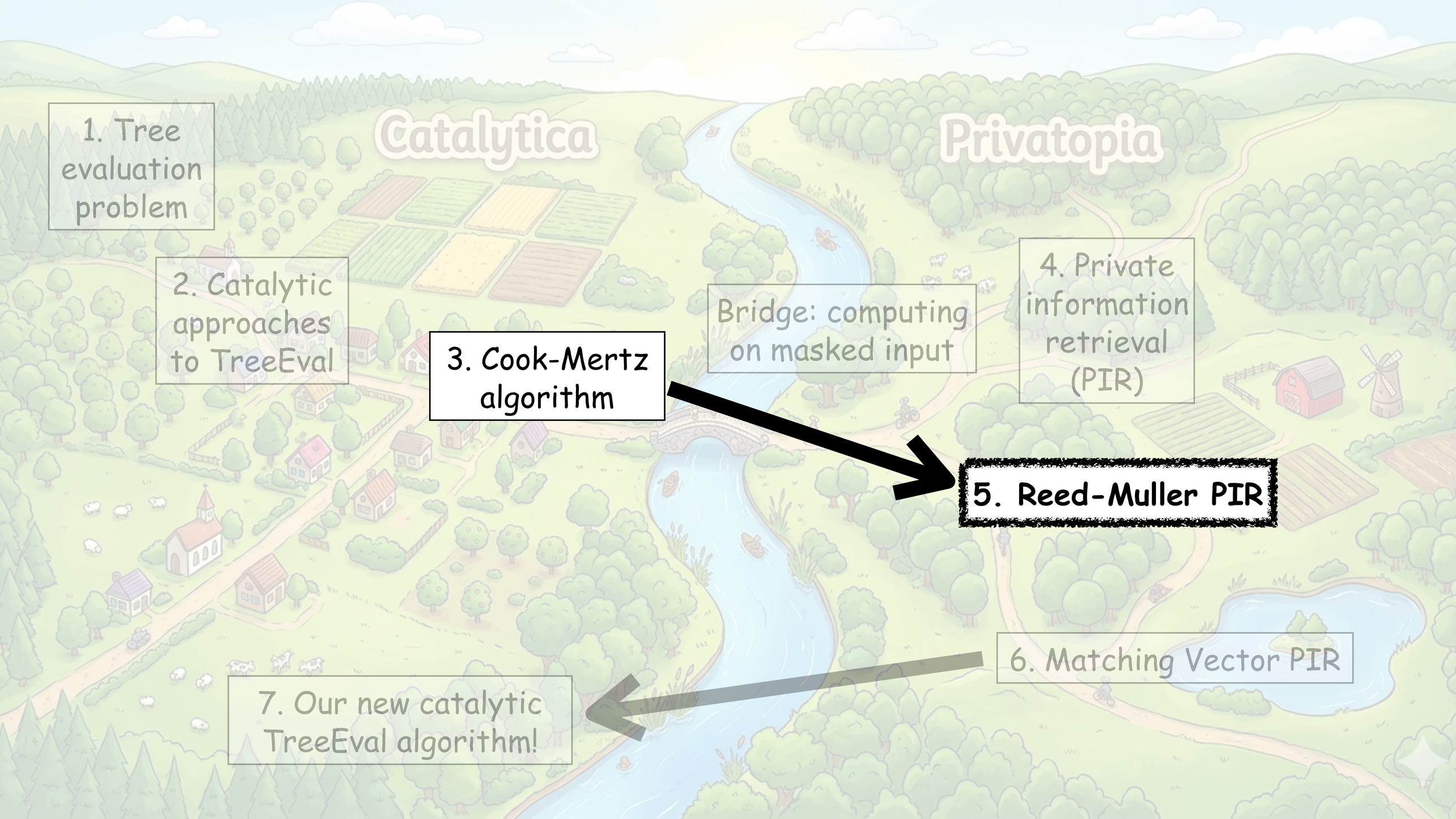
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  - Alice recovers $\hat{f}(v)$ with Lagrange interpolation

Reed-Muller PIR Efficiency Metrics

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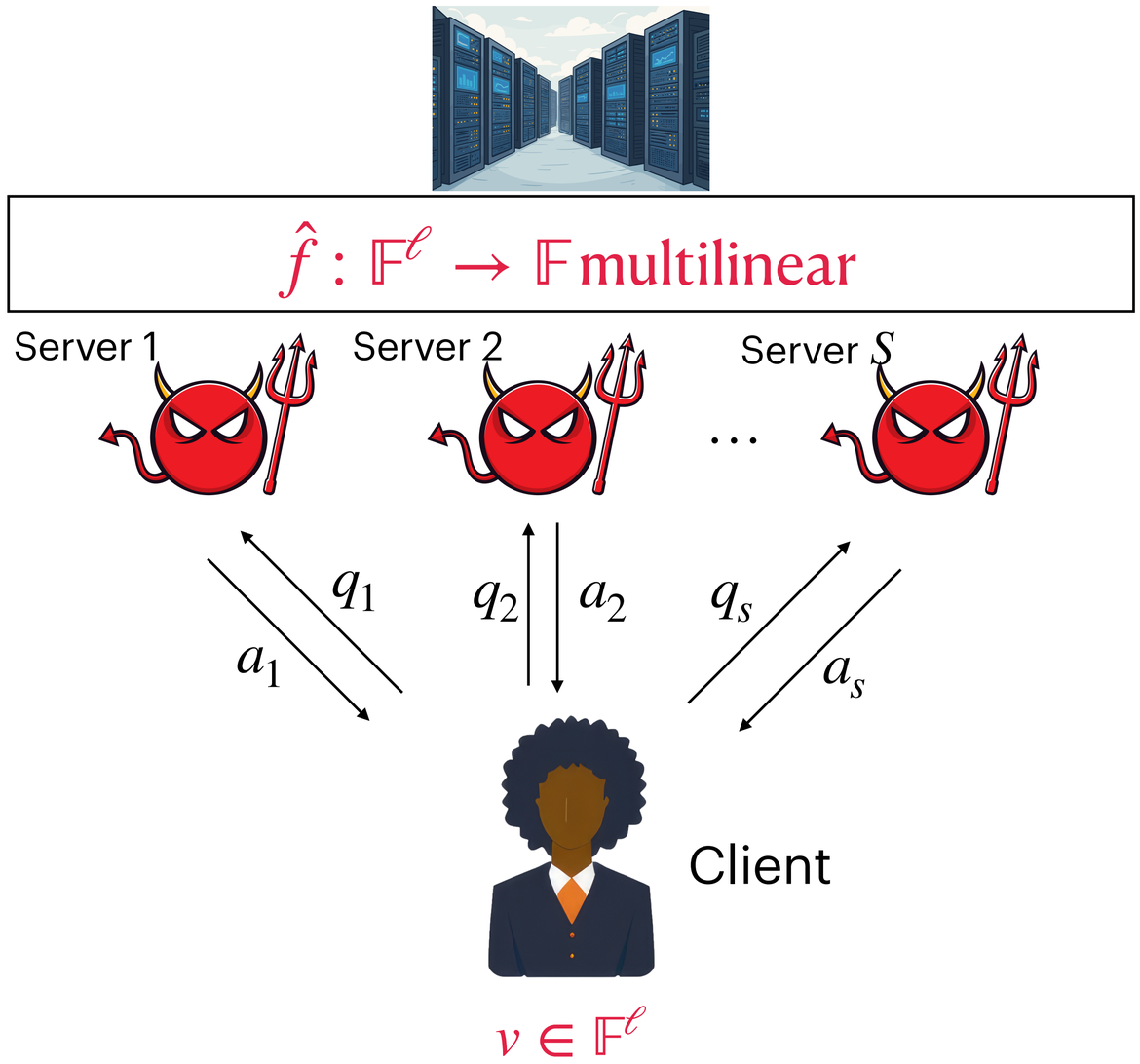
Reed-Muller PIR Efficiency Metrics



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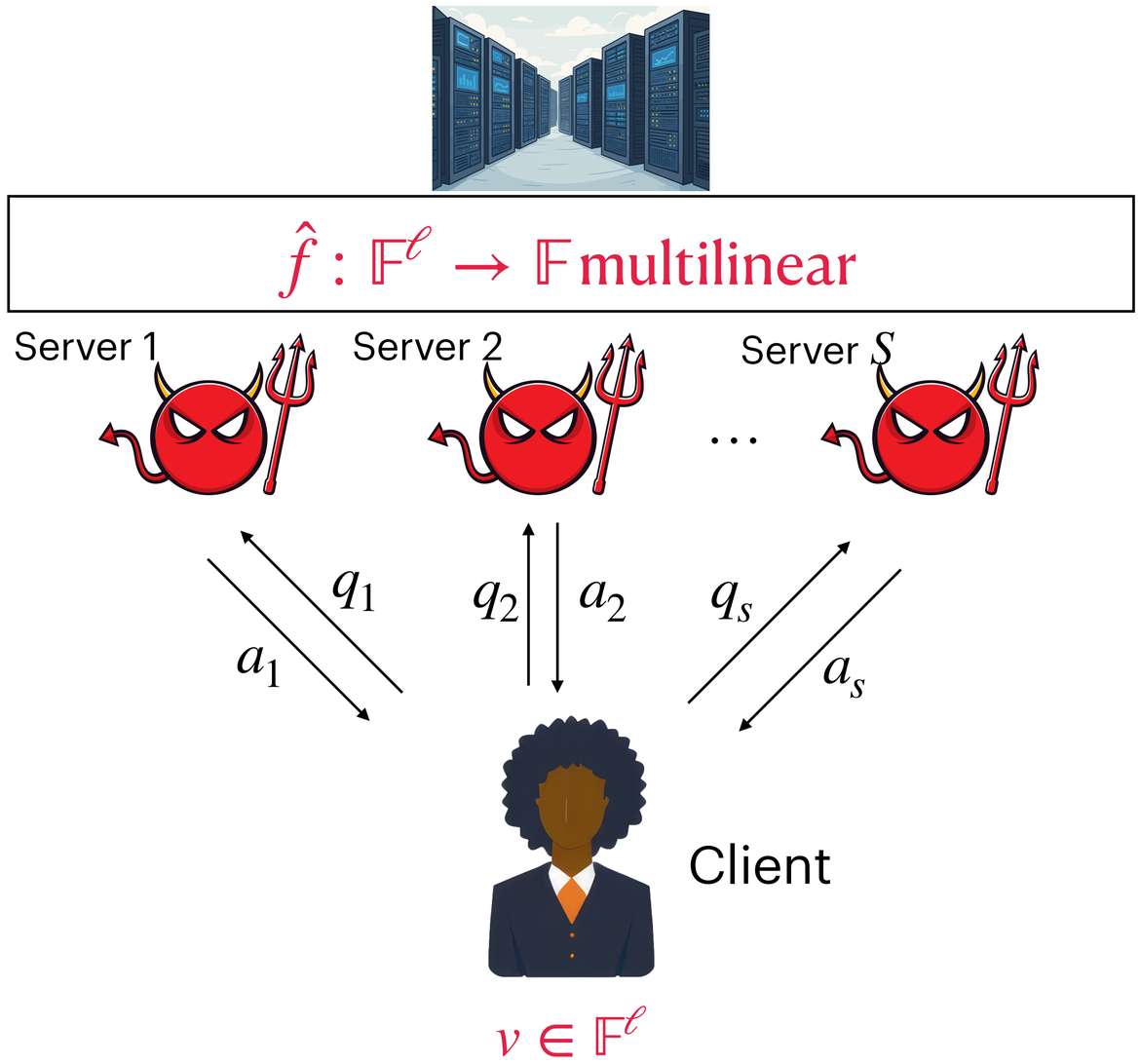
Reed-Muller PIR Efficiency Metrics



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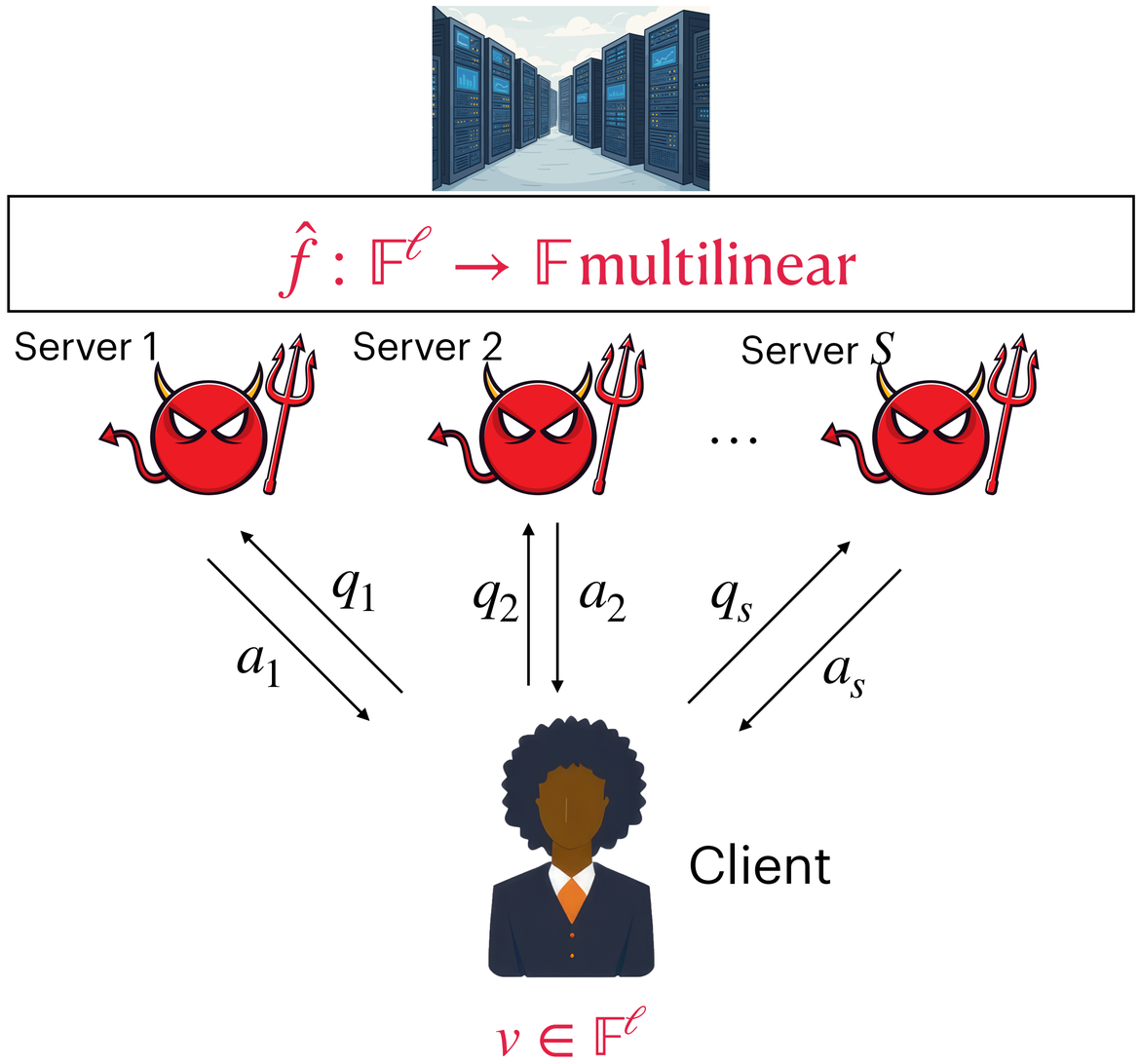

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$$\mathsf{CC} = O(\ell \log \ell)$$

Informal Dictionary: PIR ↔ TreeEval

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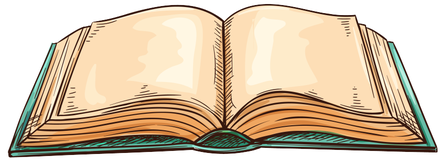

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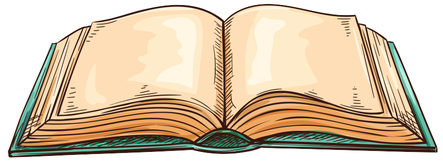

PIR	TreeEval One-Level Gadget

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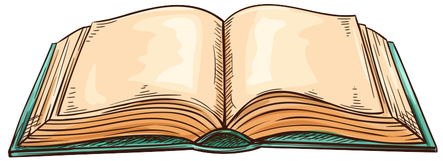

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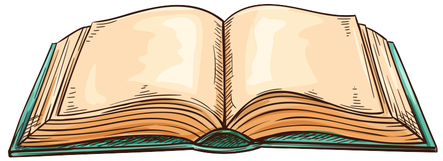

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Client's answer	f(v)

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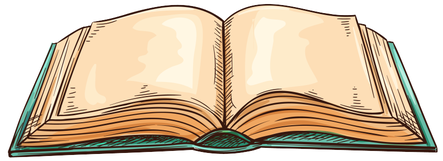

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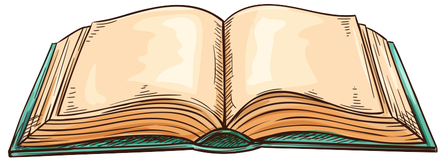

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# of servers	# of oracle calls
Question	Written to catalytic tape by child oracle

$q_j = v + j\tau$ for Reed-Muller

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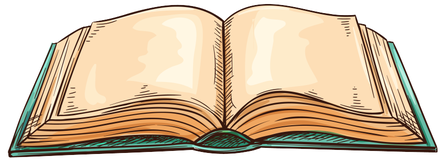

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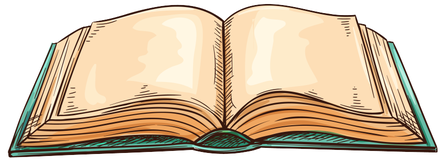

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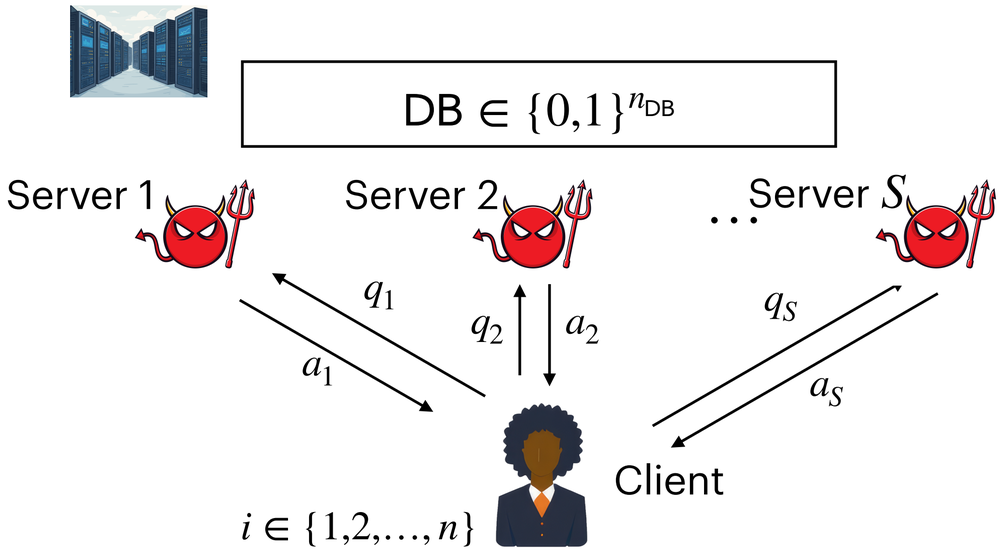
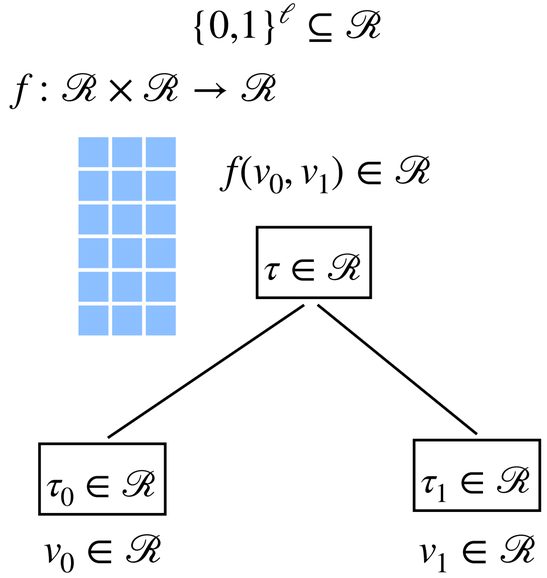
CC	Catalytic tape size	

PIR



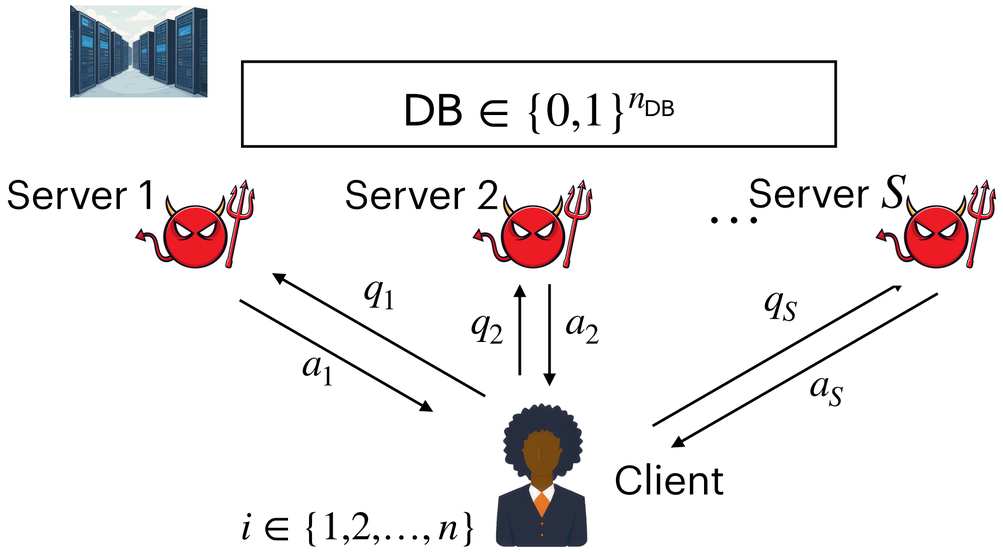
- **Mask:** sampled randomly by client to ensure privacy

TreeEval **One-Level Gadget**



- **Mask:** the existing contents of the catalytic tape

PIR



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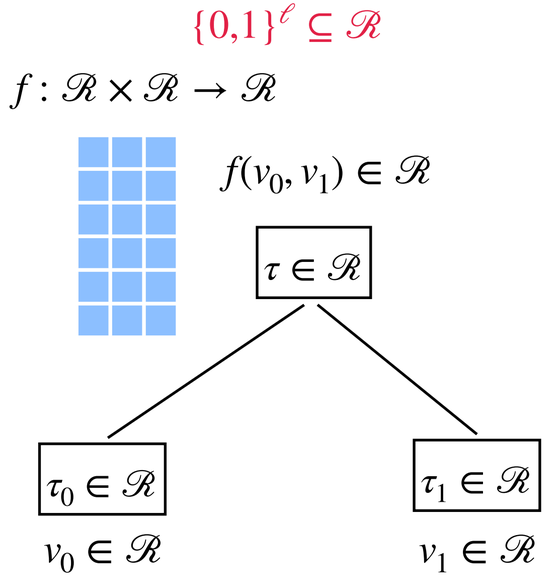

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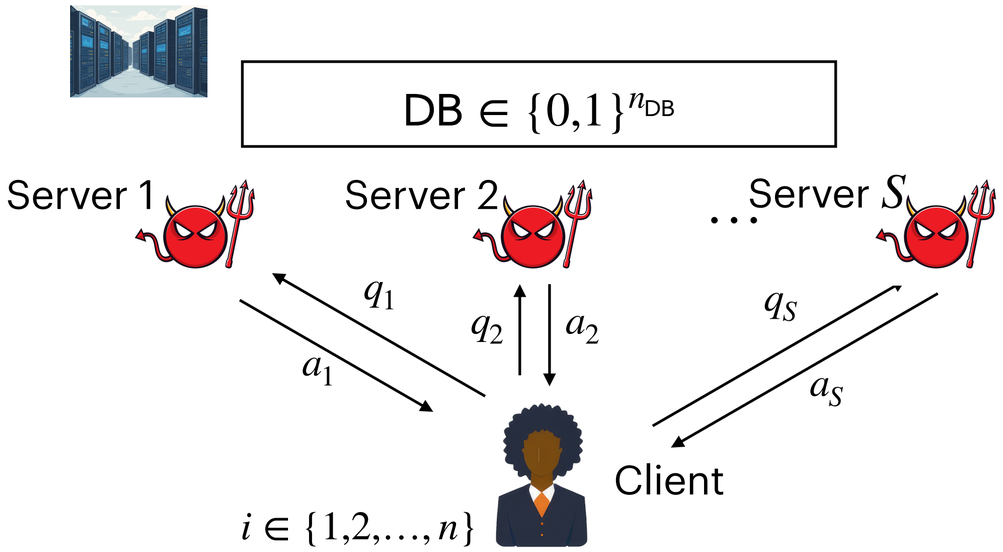

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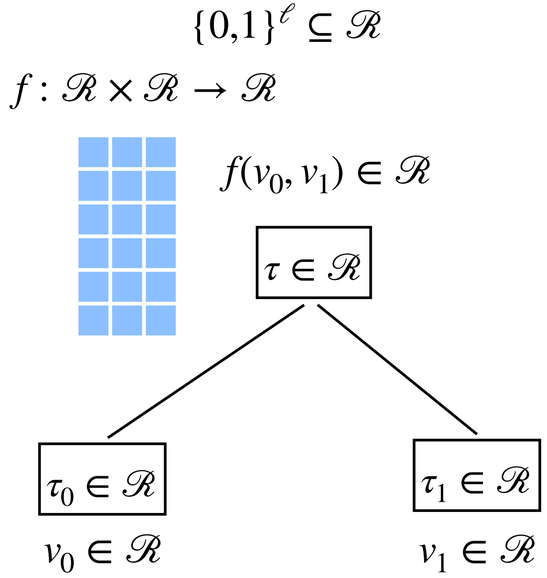
TreeEval **One-Level Gadget**



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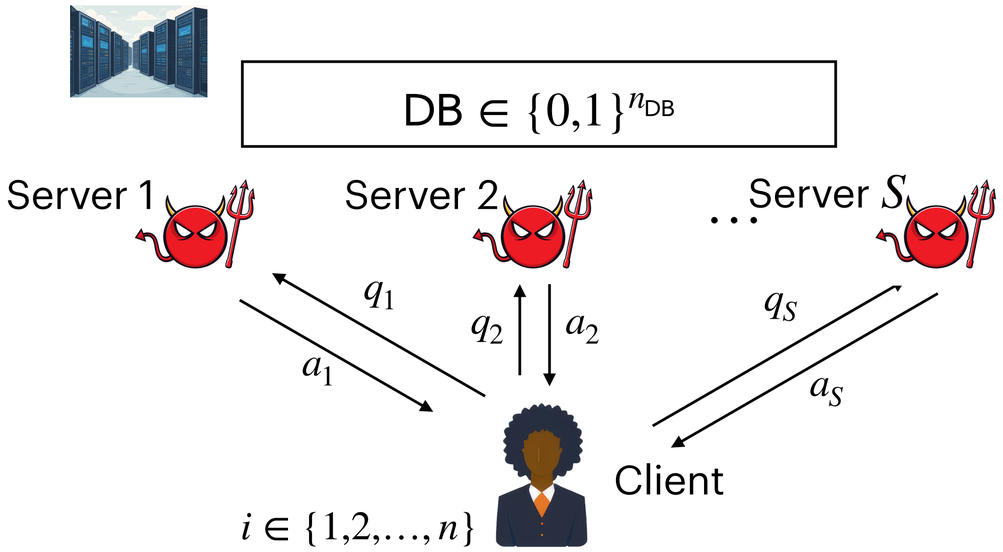

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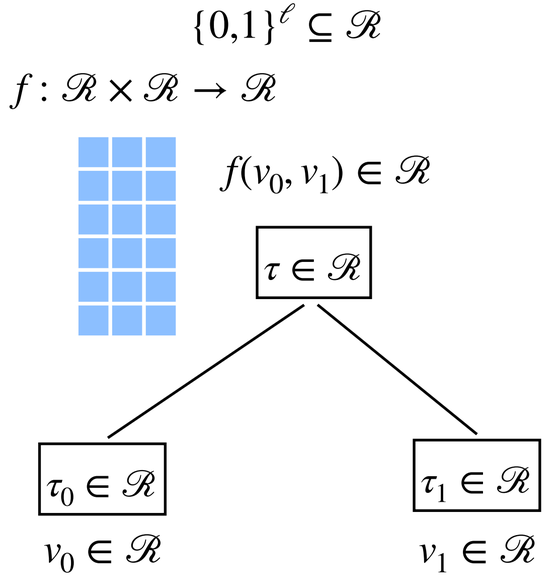

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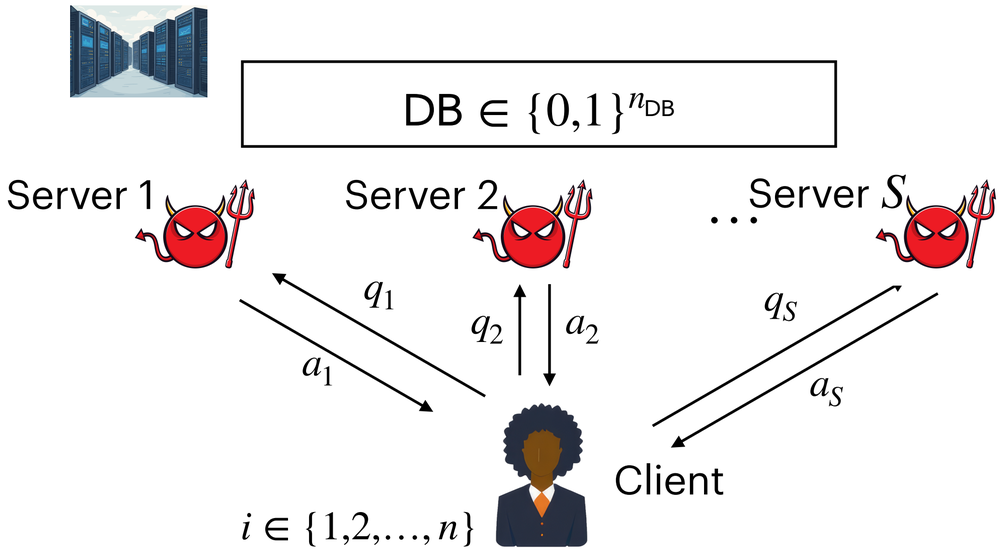
- Free space: $O(\log S)$

PIR



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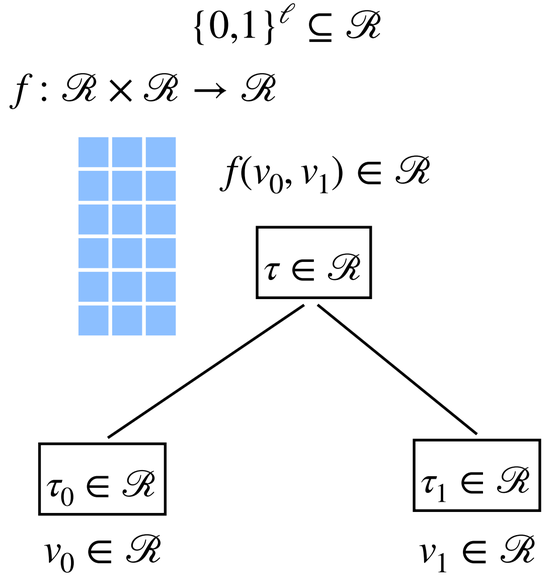
TreeEval **One-Level Gadget**



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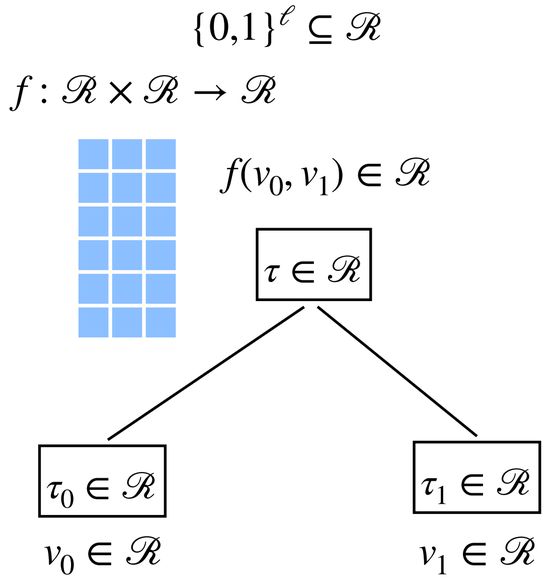

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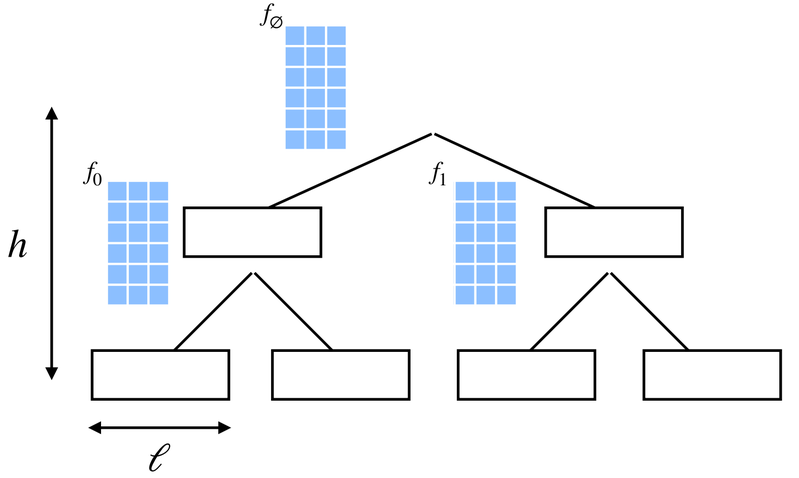

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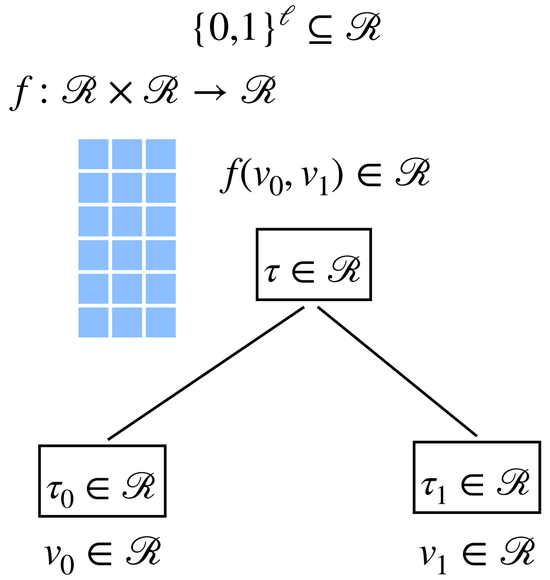

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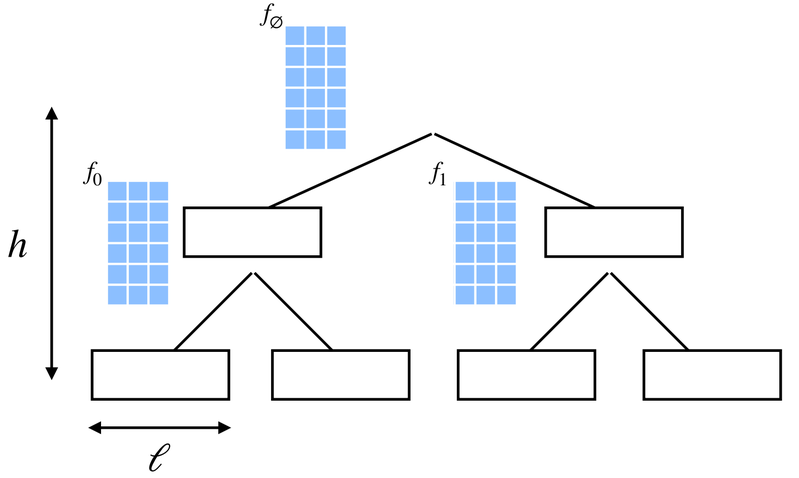

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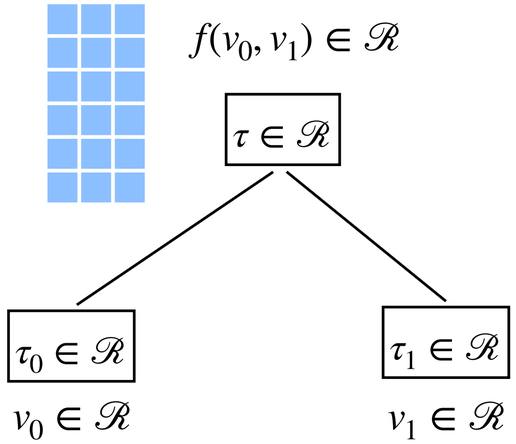

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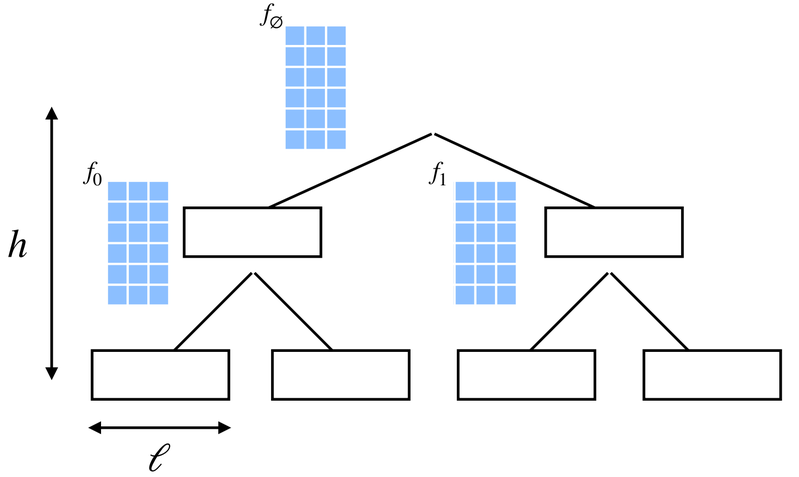

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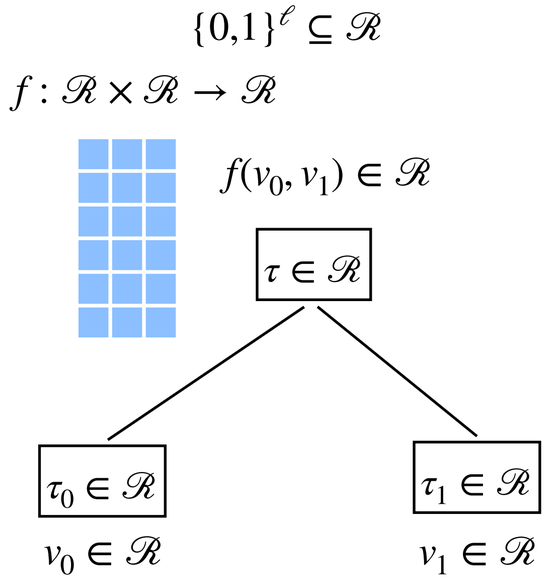

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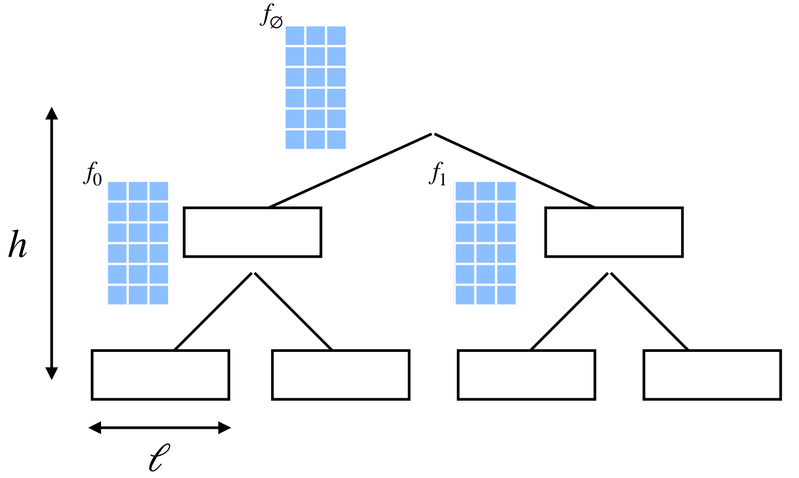

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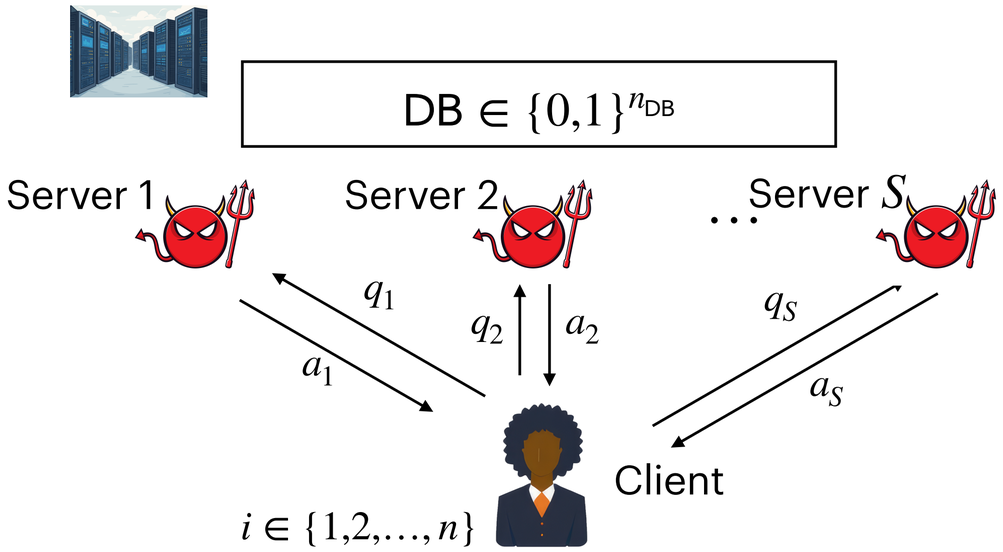

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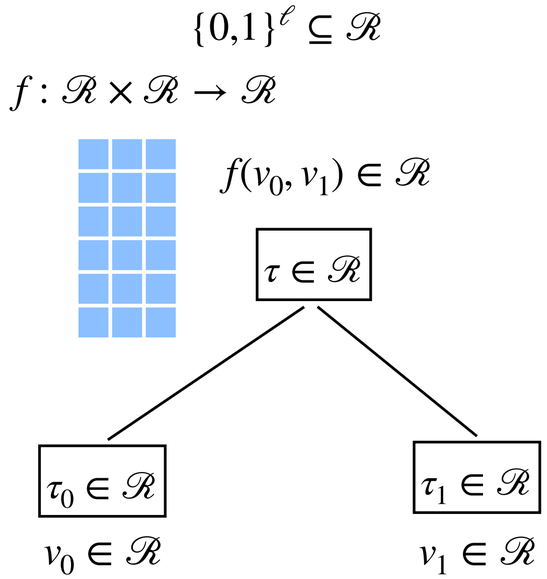

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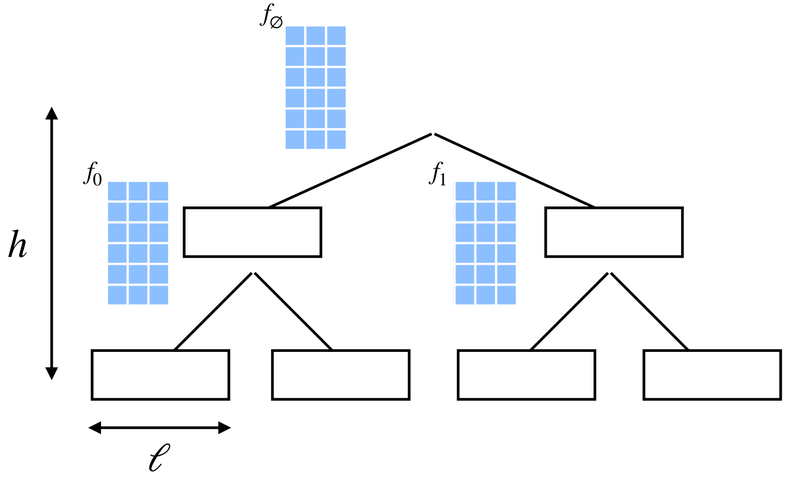

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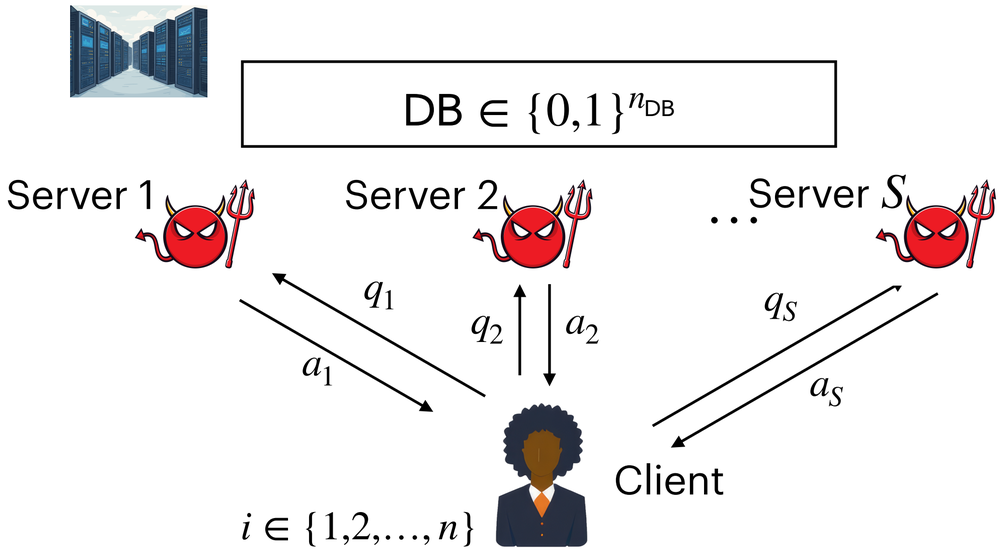

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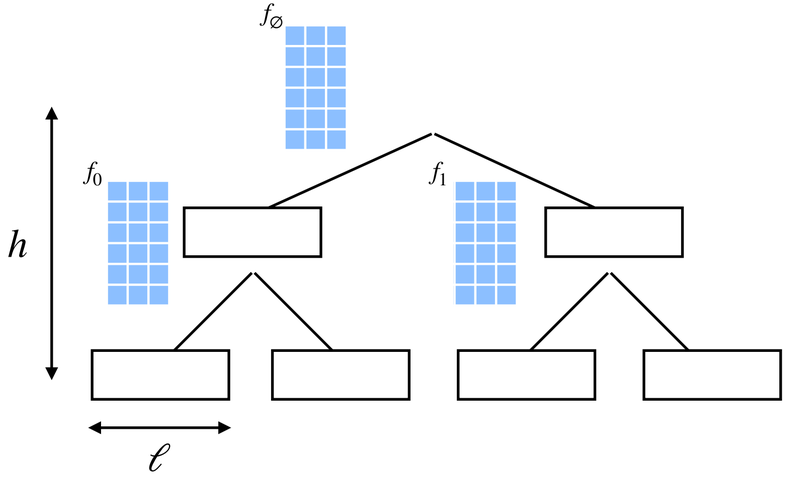

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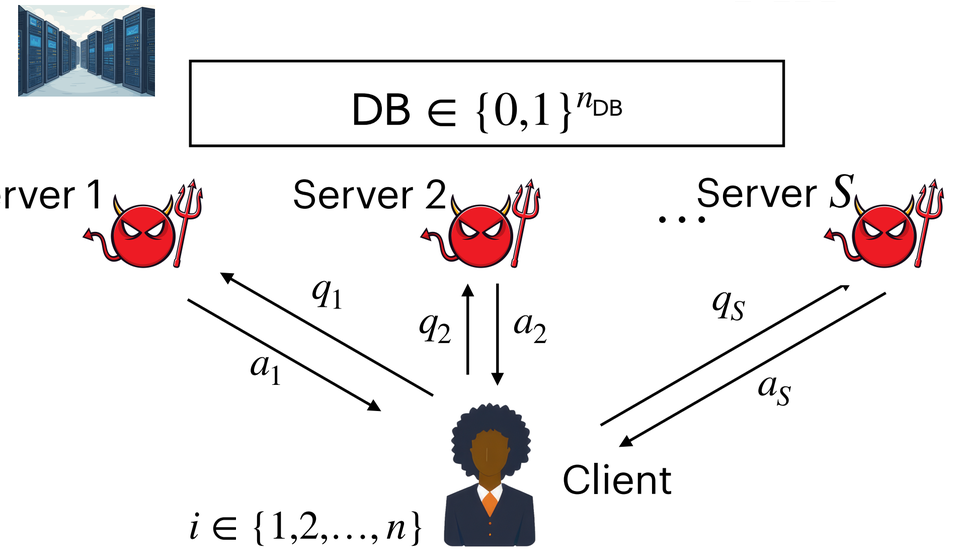

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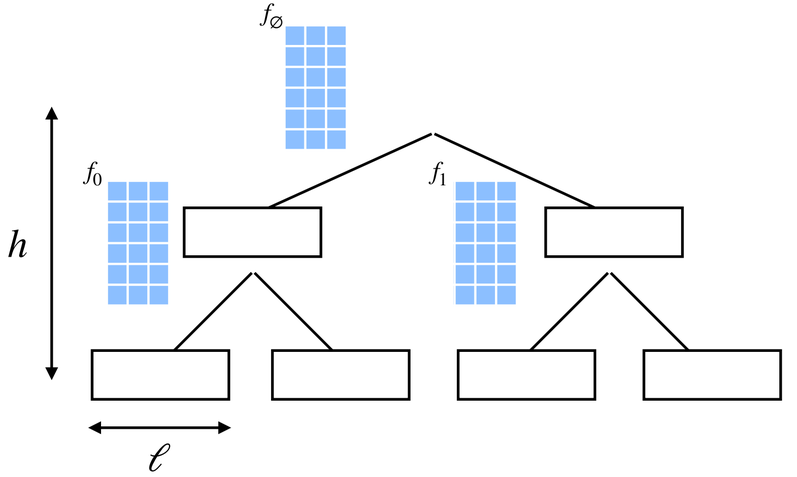

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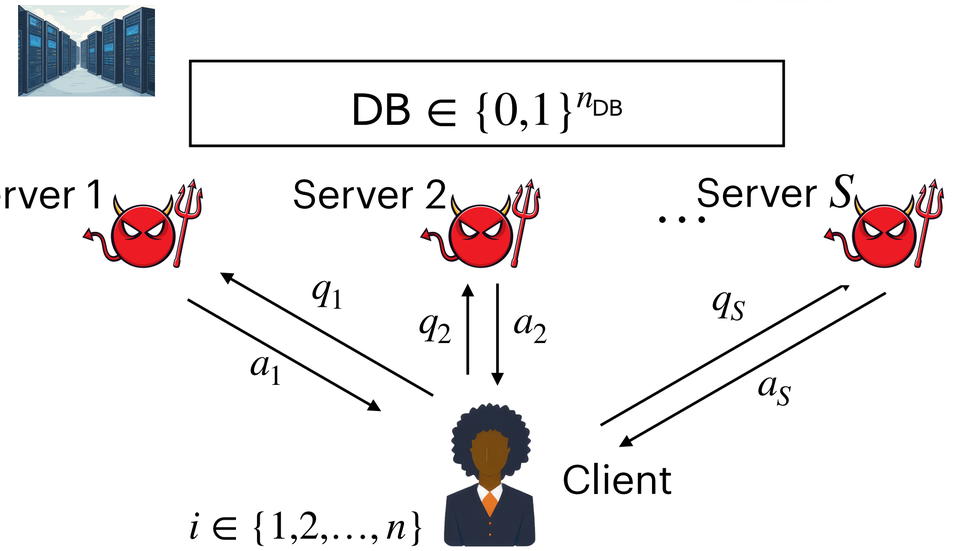

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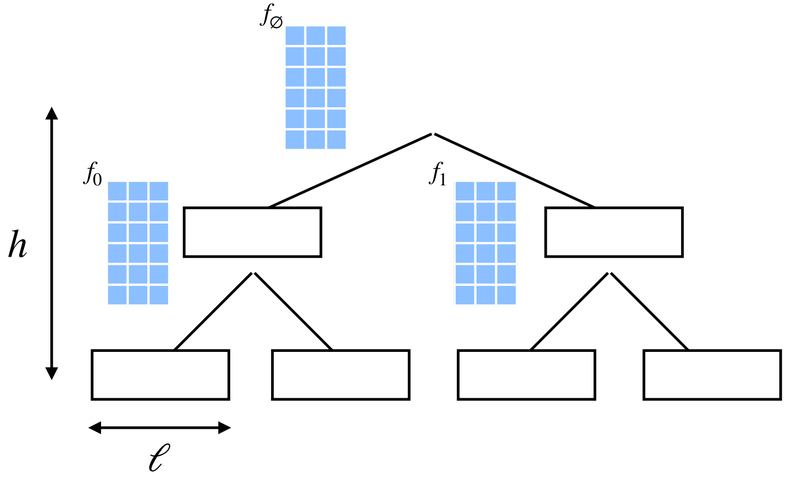

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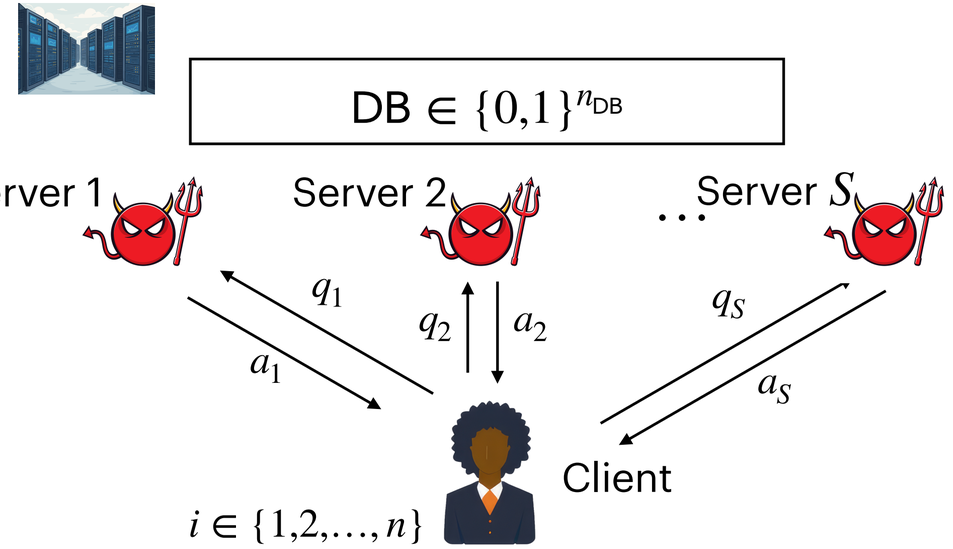

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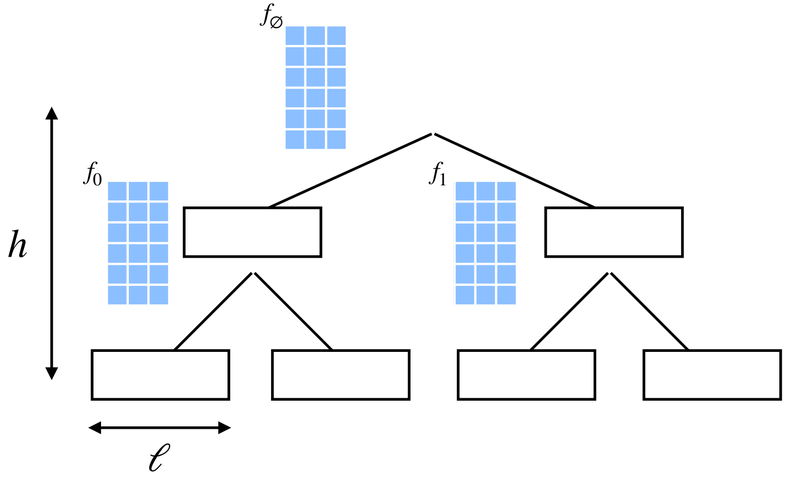

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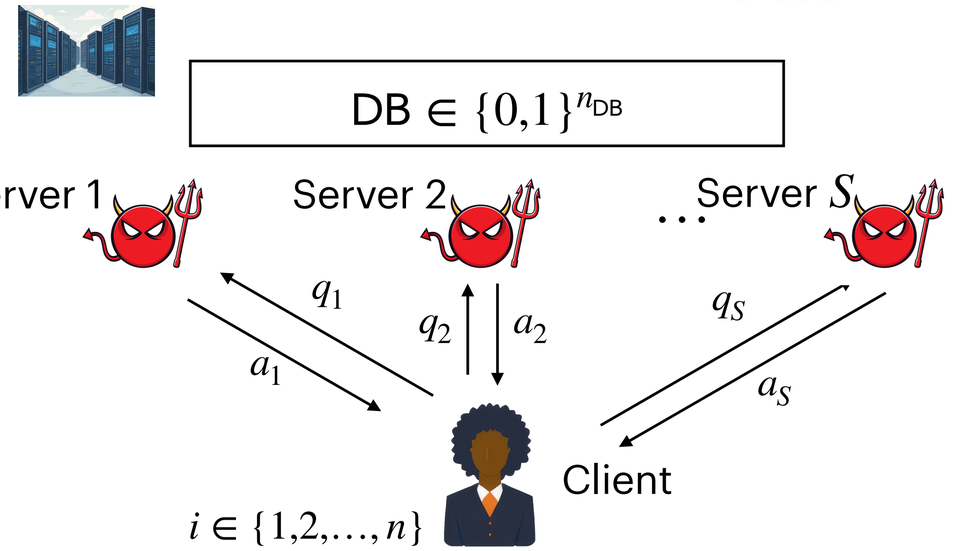

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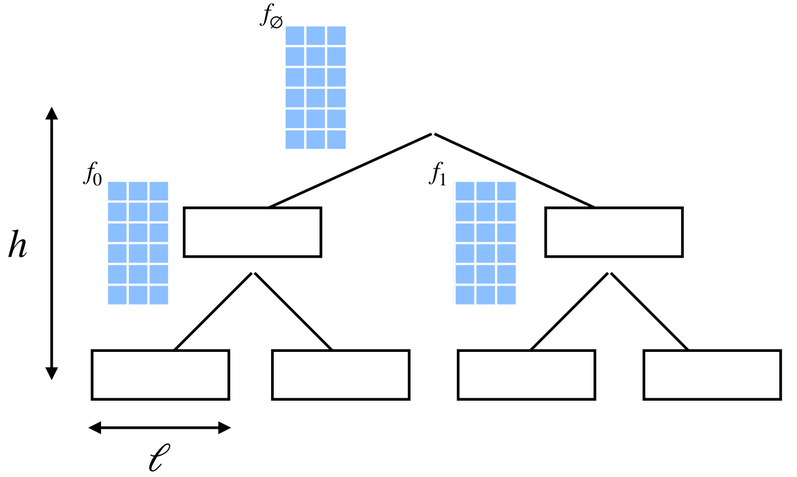

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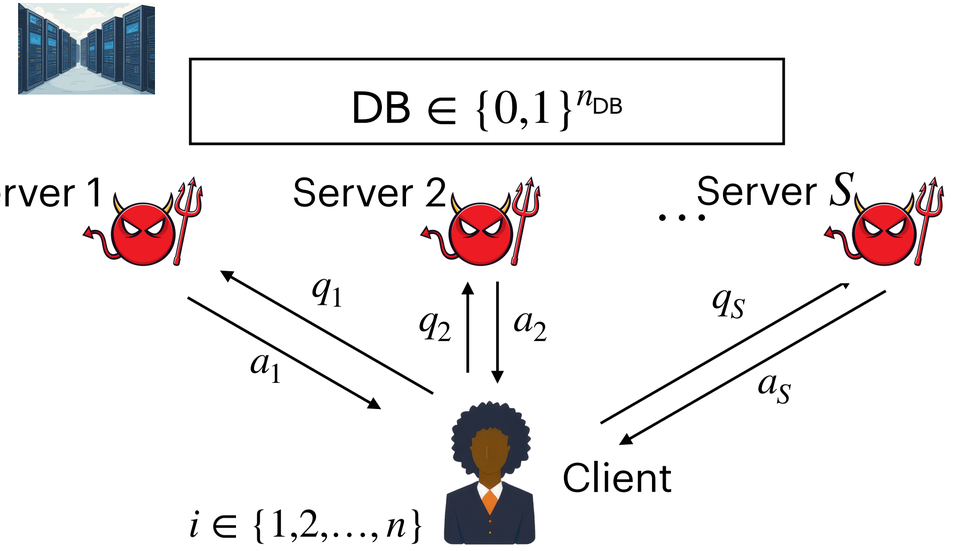

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Matching Vector PIR → Our Algorithm

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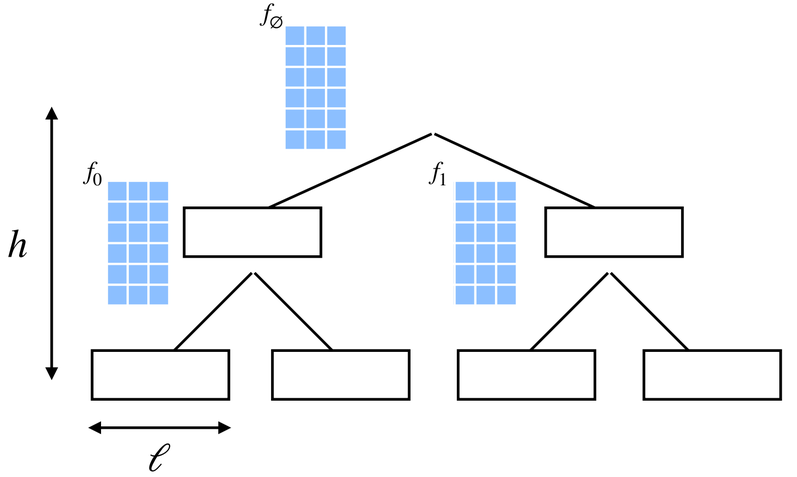

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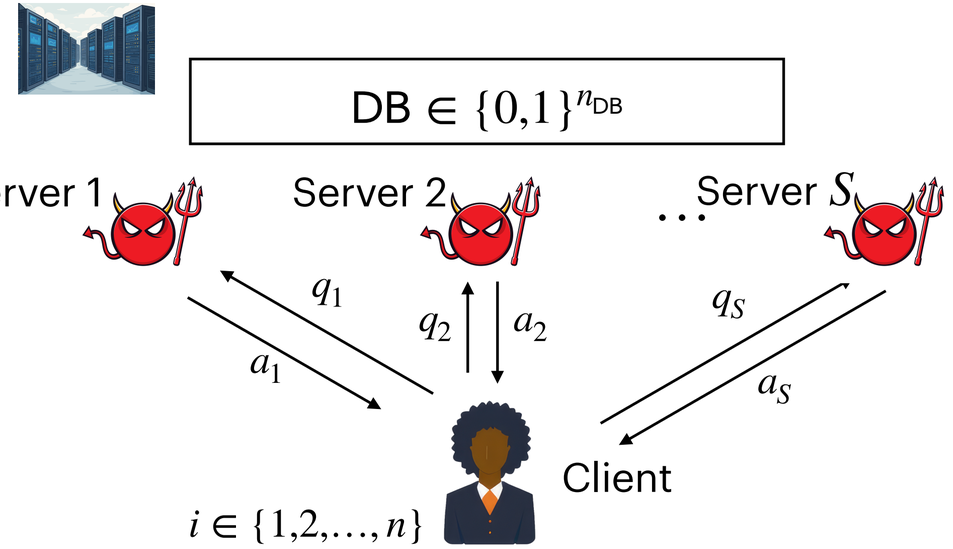

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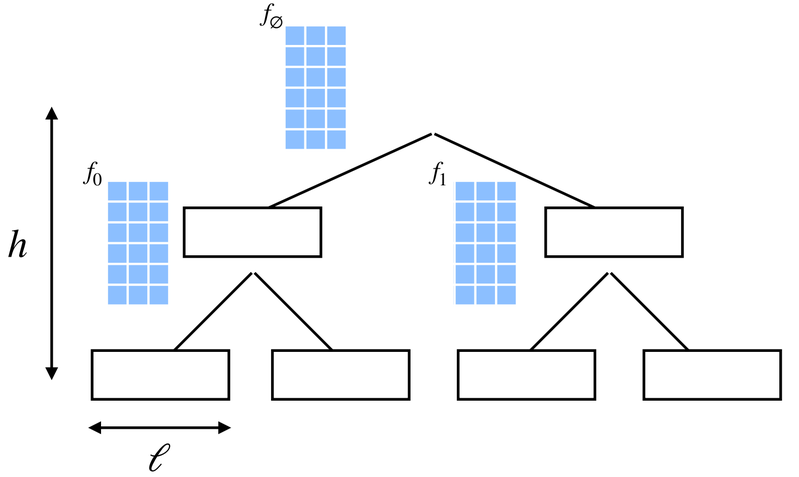

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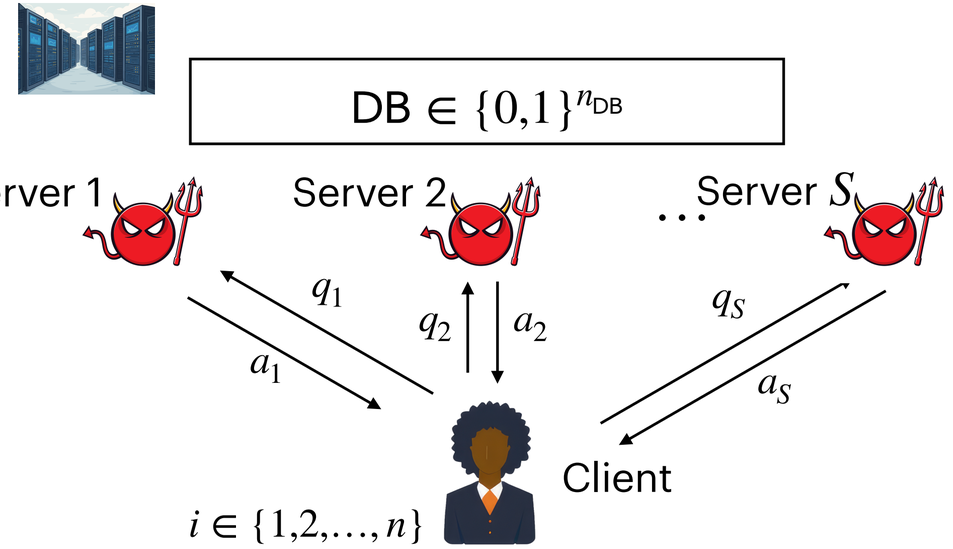

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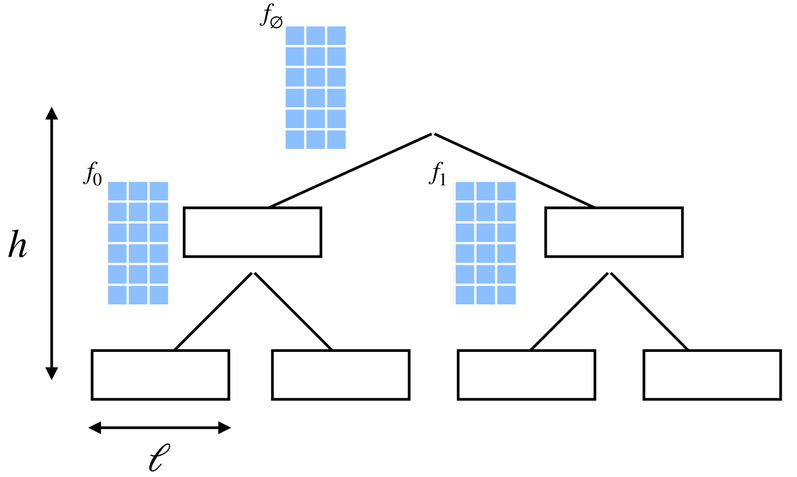

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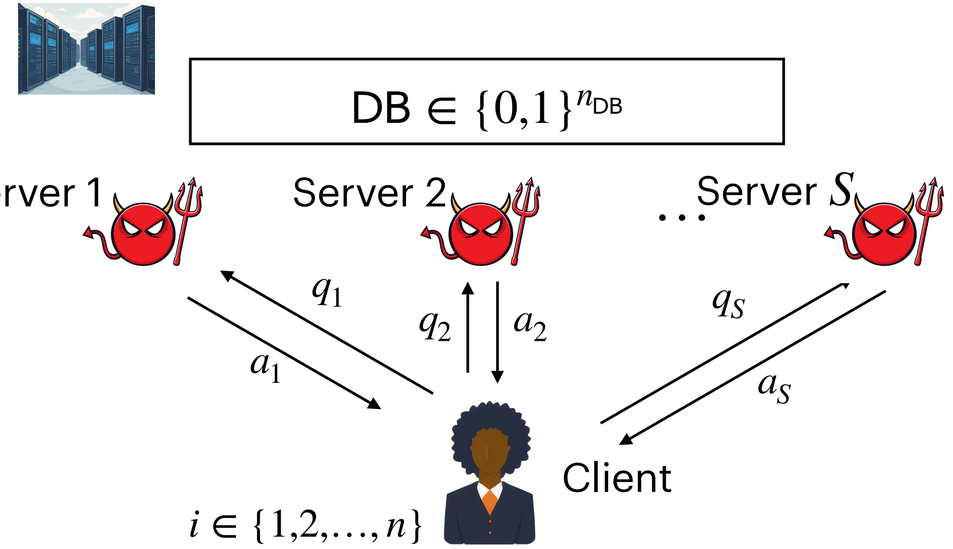

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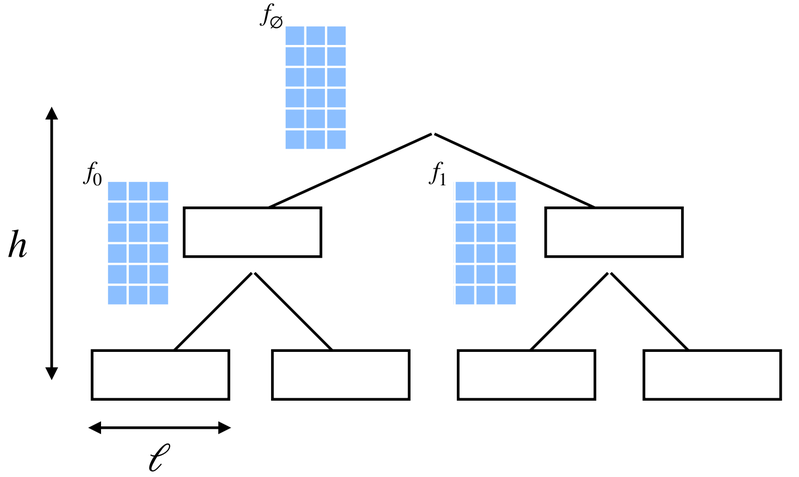

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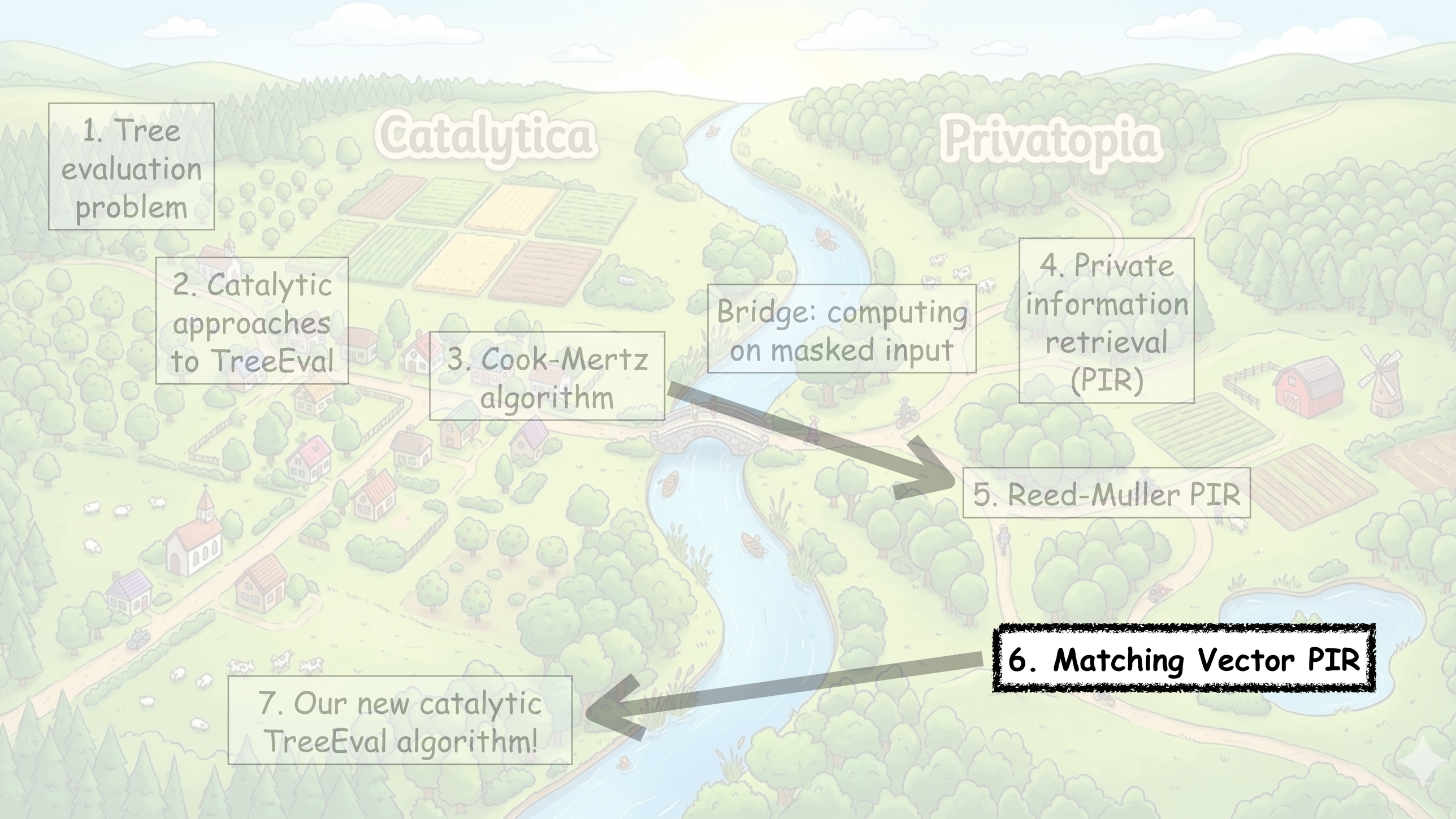
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Catalytica
Privatopia
2. Catalytic approaches to TreeEval
3. Cook-Mertz algorithm
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4. Private information retrieval (PIR)
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Goal: PIR on a database of size n with $O(1)$ servers

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Today: 4 servers, CC $= 2^{\tilde{O}(\sqrt{\log n})}$

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  - Even more: $\langle \mathbf{u}_i, \mathbf{u}_j \rangle \in \{0,1\} \pmod{p}$, and similarly mod q

Efremenko's 4-Server Construction

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Efremenko's 4-Server Construction

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- Each server, given its query $\mathbf{v} \in \mathbb{Z}_m^d$: replies with

$$\sum_{j=1}^{n} \mathsf{DB}_j \cdot g^{\langle \mathbf{v}, \mathbf{u}_j \rangle} \cdot h^{\langle \mathbf{v}, \mathbf{u}_j \rangle} \in R$$

Efremenko's 4-Server Construction

How the Client Reconstructs DB_i

Cheatsheet

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle \in \{0,1\} \pmod{p \text{ and } q}$$

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- If $j = i$, then $g^{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} = g$ and $h^{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} = h$

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- Final result:

$$\mathsf{DB}_i \cdot \underbrace{g^{\langle \mathbf{r}, \mathbf{u}_i \rangle} \cdot h^{\langle \mathbf{r}, \mathbf{u}_i \rangle} \cdot (1-g)(1-h)}_{\text{nonzero}} \rightarrow \mathsf{DB}_i \in \{0,1\}$$ 🎉

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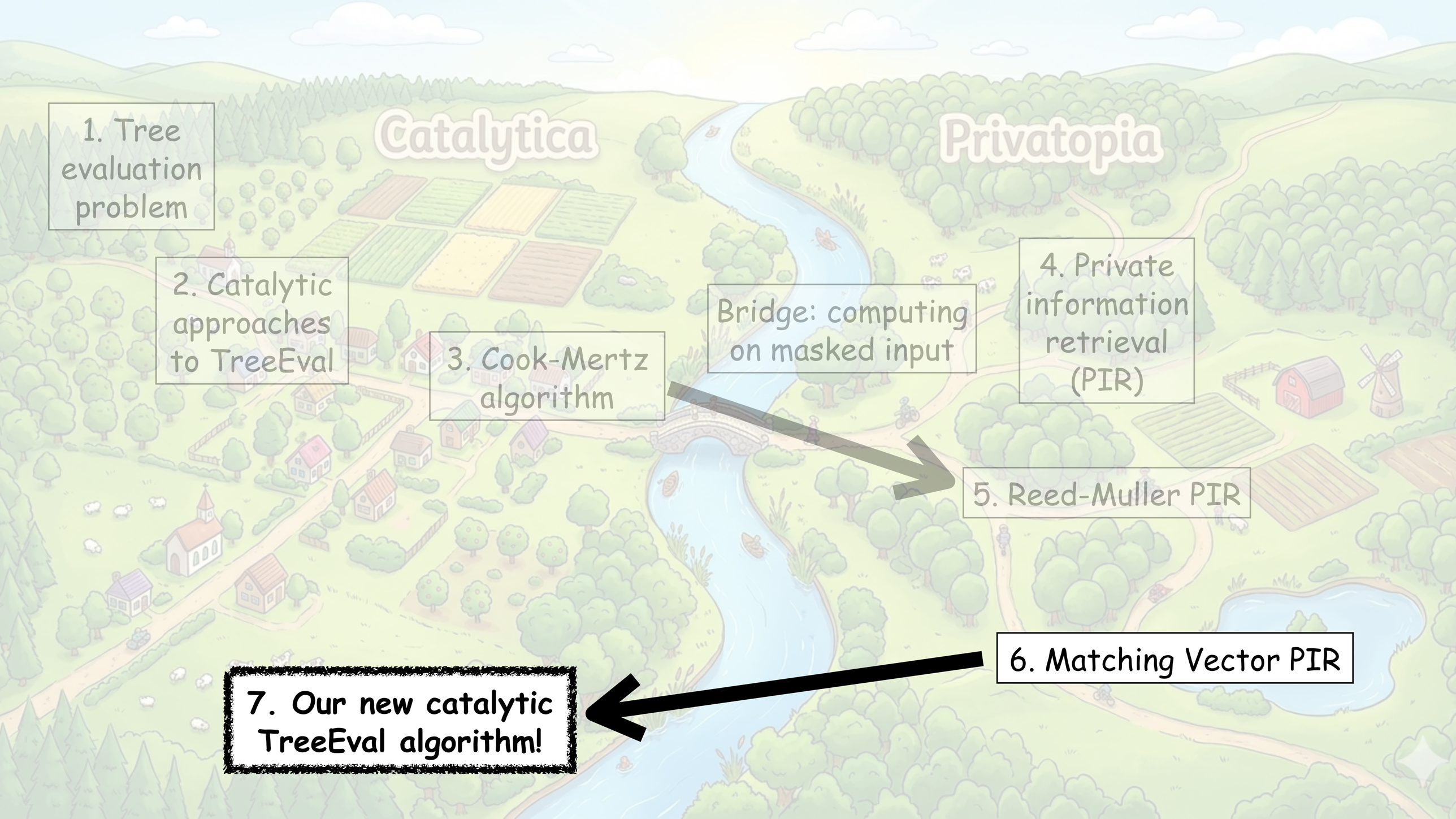

Catalytica
Privatopia
1. Tree evaluation problem
2. Catalytic approaches to TreeEval
3. Cook-Mertz algorithm
Bridge: computing on masked input
4. Private information retrieval (PIR)
5. Reed-Muller PIR
6. Matching Vector PIR
7. Our new catalytic TreeEval algorithm!

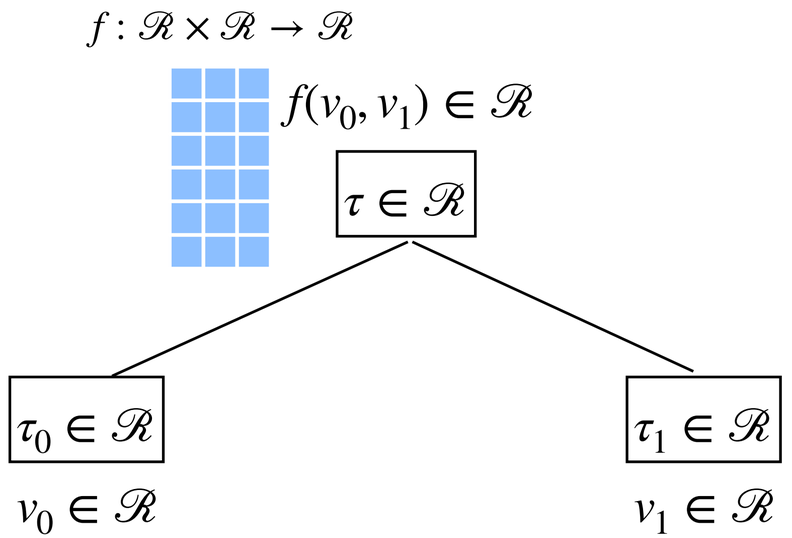
Refresher: The One-Level Gadget

- Embed $\{0,1\}^{\ell}$ in some ring $\mathscr{R}$
- Inputs:
  - Control bit $c' \in \{0,1\}$
  - Arbitrary catalytic registers $\tau_0, \tau_1, \tau \in \mathscr{R}$
  - Oracle $\mathcal{O}(b, c \in \{0,1\})$: update

$$\tau_b \leftarrow \tau_b + (-1)^c v_b$$

- Output: want the registers to end up in the state

$$(\tau_0, \tau_1, \tau) = (\tau_{0,\text{init}}, \tau_{1,\text{init}}, \tau_{\text{init}} + (-1)^{c'} f(v_0, v_1))$$



Metrics

- $\log |\mathscr{R}|$: final catalytic space
- $h \cdot$ [free space]: final free space
- [oracle queries]^h: final runtime

Our One-Level Gadget Using Matching Vectors

- Let $\{\mathbf{u}_v \in \mathbb{Z}_m^d\}_{v \in \{0,1\}^\ell}$ be a MV family
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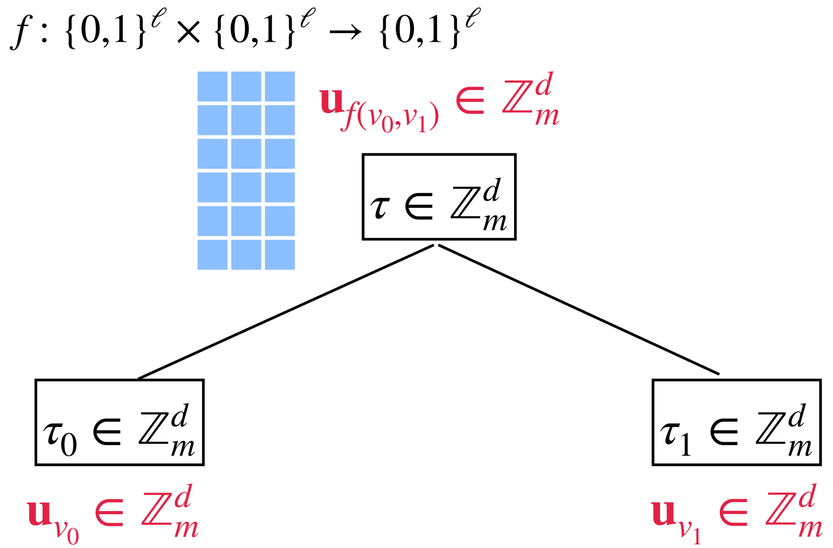

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- Generalisation to 2^t servers: use t primes instead of two
  - Gives $2^{\log^\epsilon n}$ catalytic space, $O_\epsilon(\log n)$ free space, and $\mathsf{poly}_\epsilon(n)$ runtime

Conclusion

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We leverage this connection and use matching-vector PIR to get new algorithms for TreeEval.

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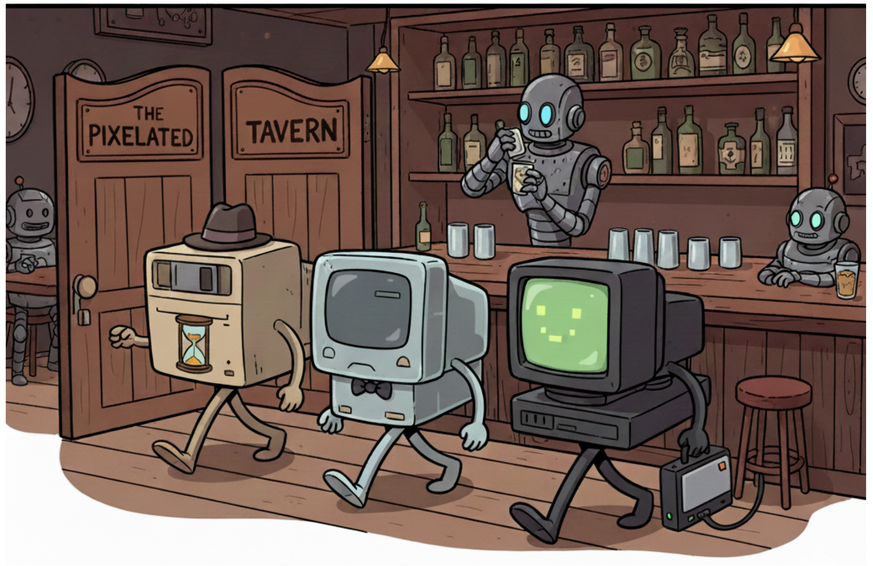
- Other applications of information-theoretic cryptography to catalytic computing?

Three Turing Machines Walk Into a Bar

[Williams, STOC 2025]



Three Turing Machines Walk Into a Bar

[Williams, STOC 2025]

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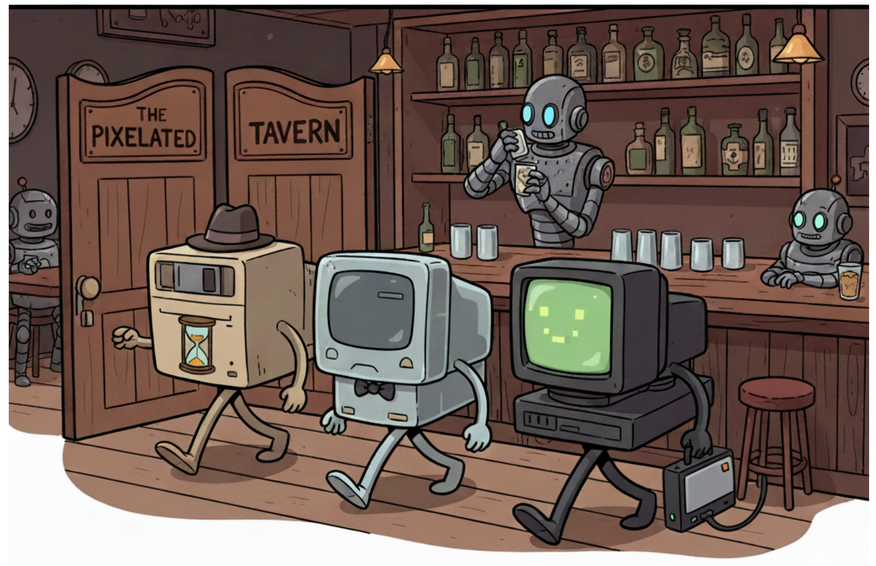

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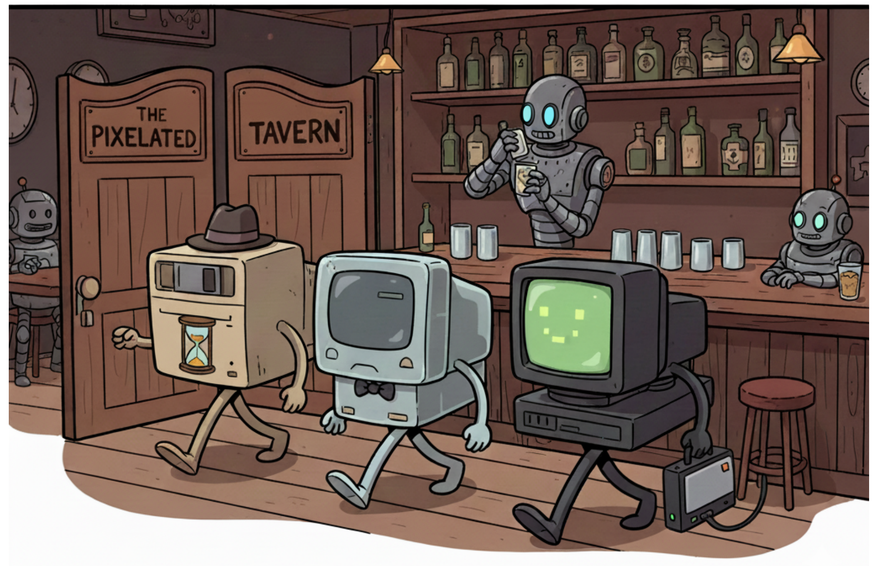

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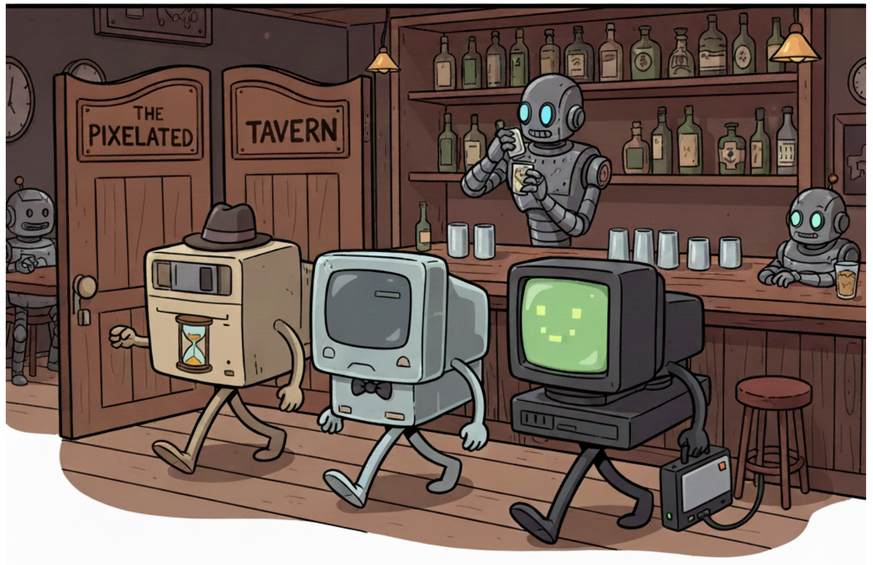

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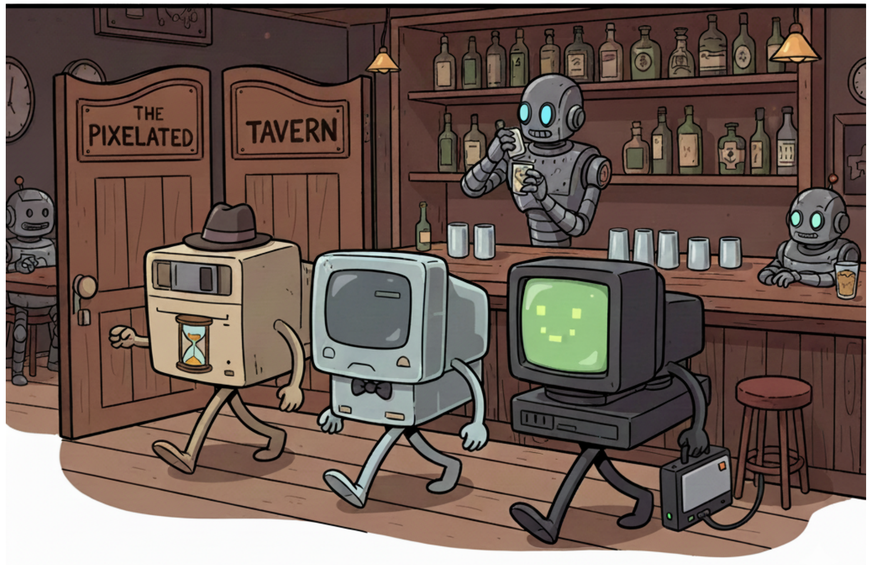

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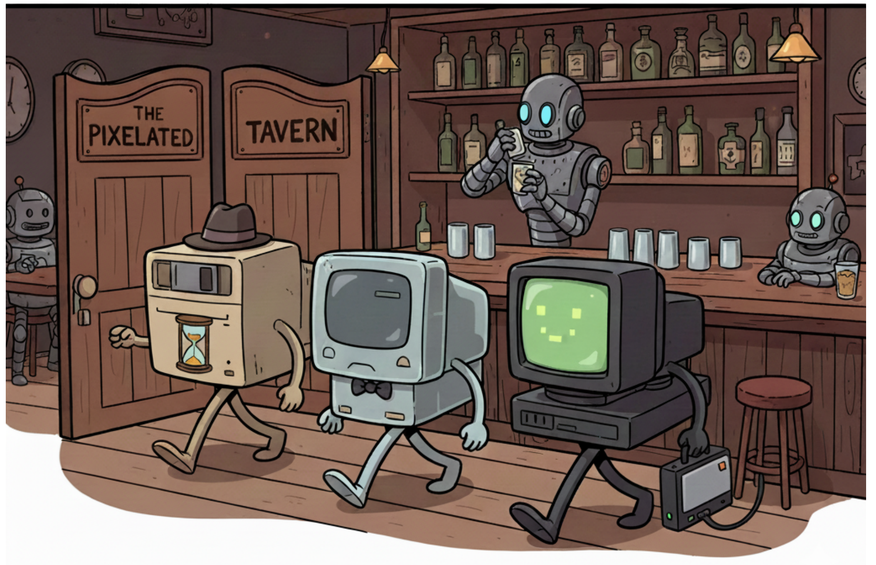

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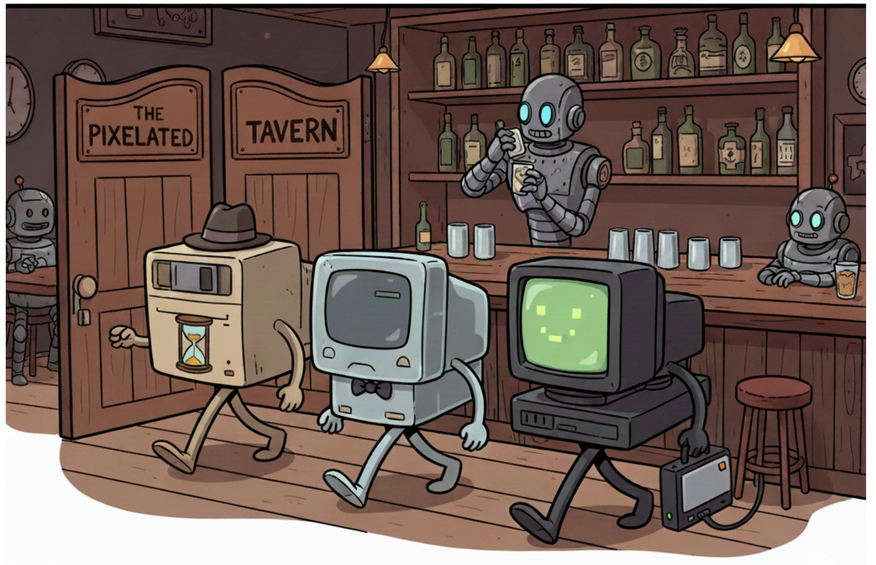

Q: How do the powers of these computers compare?

Is one of them "strictly weakest" i.e. the other two computers could do absolutely anything that it could?

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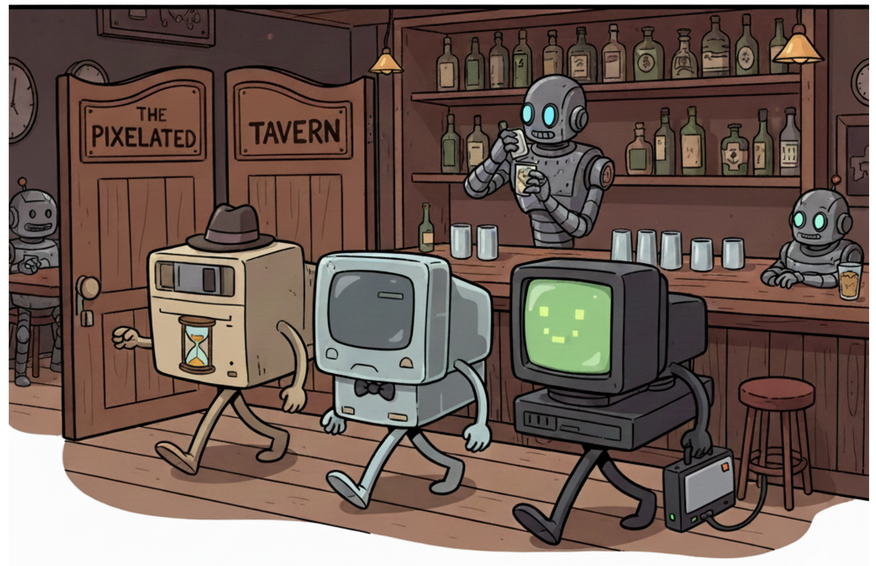

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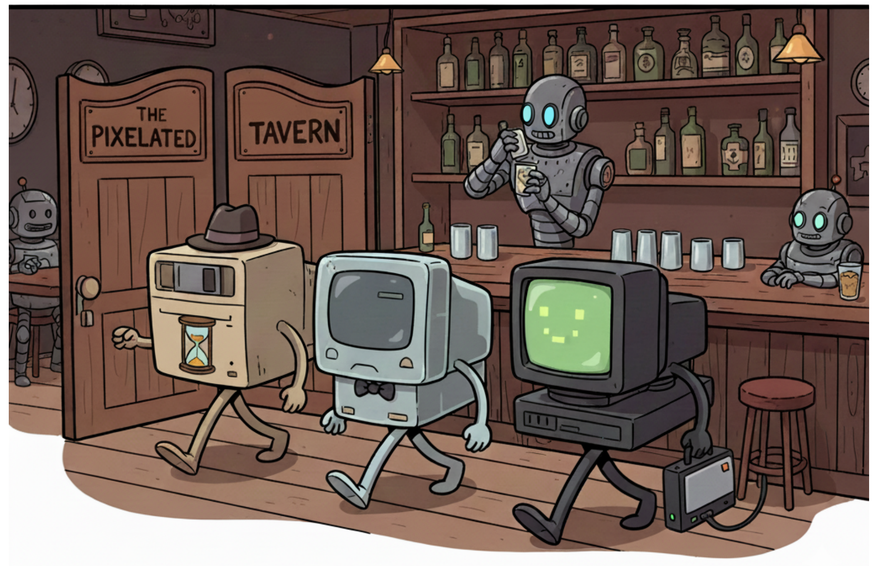

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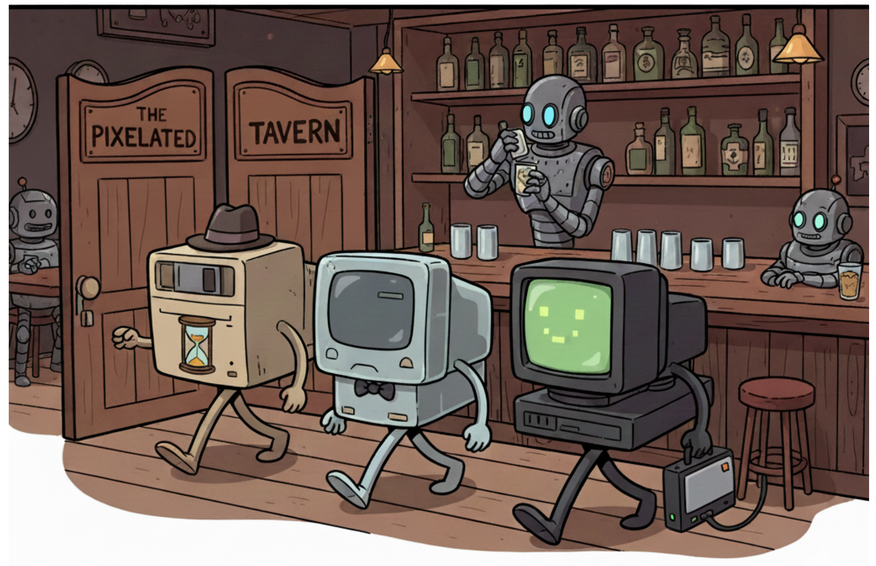

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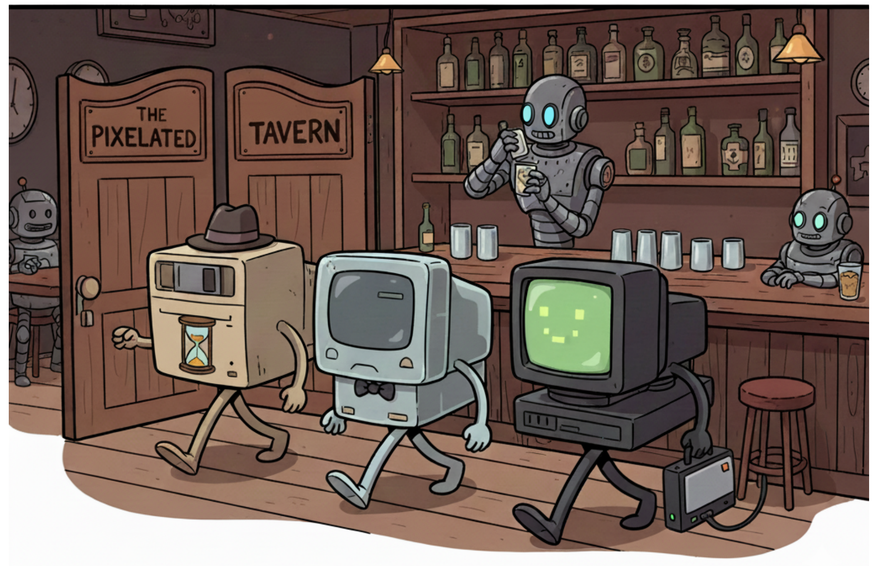

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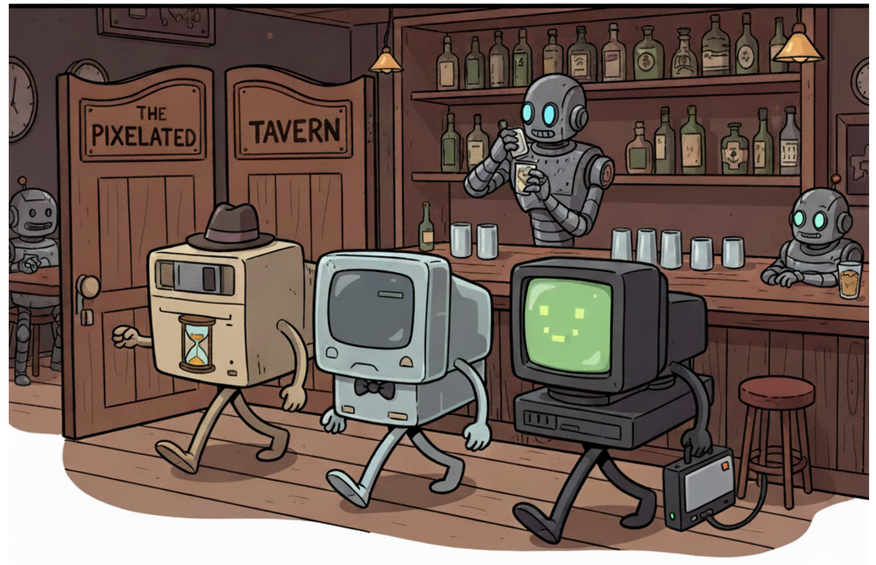

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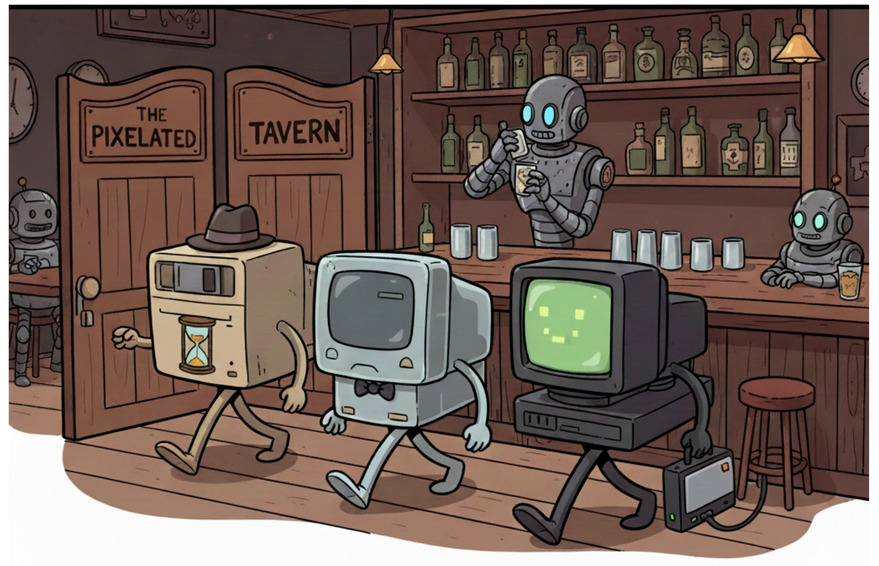

Answer: Computer A is strictly weakest

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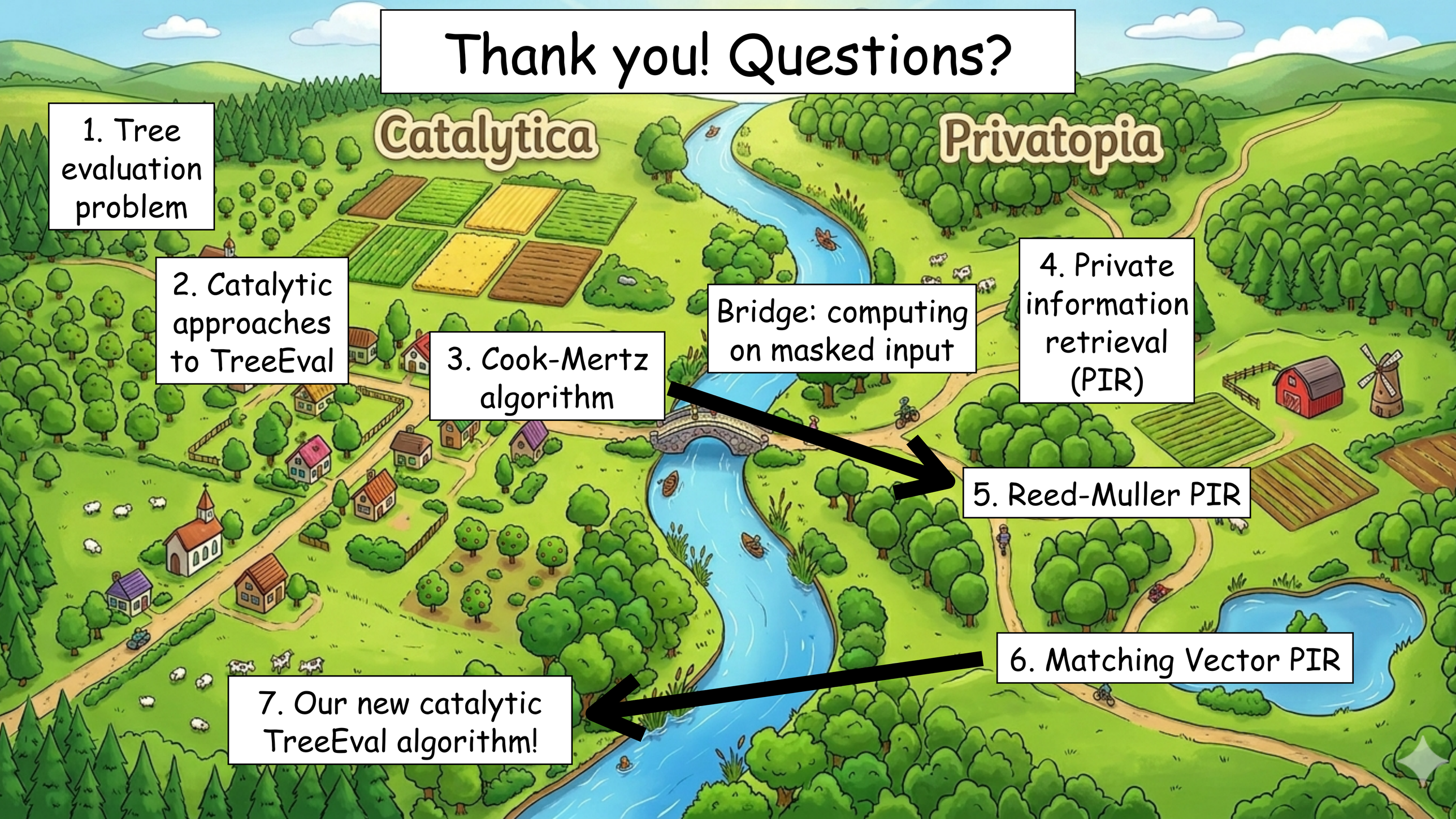
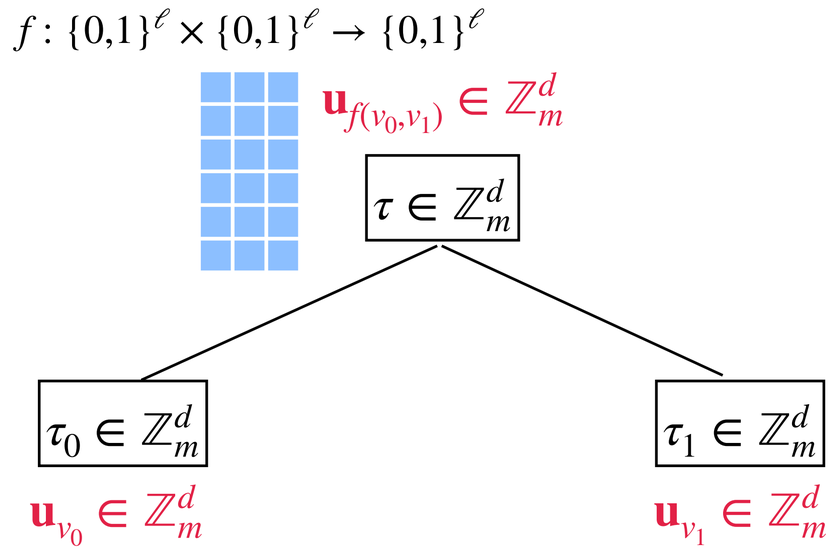

Thank you! Questions?
Catalytica
Privatopia
1. Tree evaluation problem
2. Catalytic approaches to TreeEval
3. Cook-Mertz algorithm
Bridge: computing on masked input
4. Private information retrieval (PIR)
5. Reed-Muller PIR
6. Matching Vector PIR
7. Our new catalytic TreeEval algorithm!

Bonus Slides

The Fine Print



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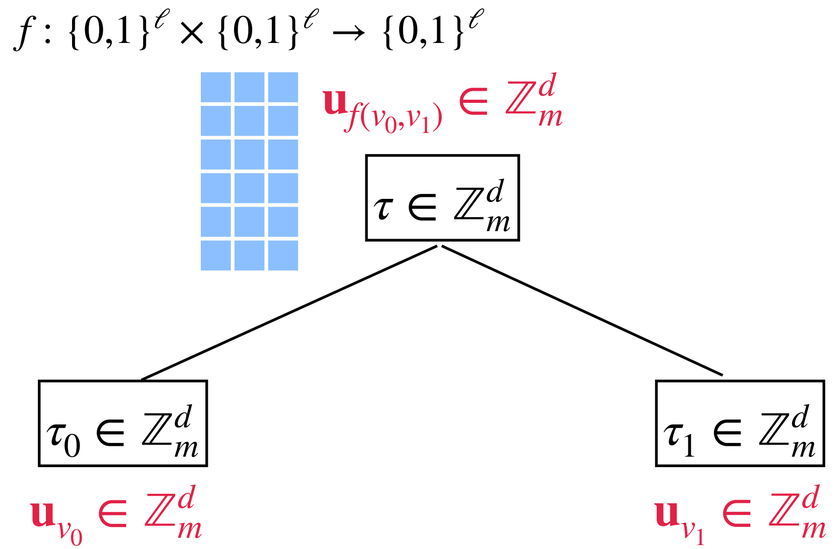
  - True for all PIR protocols described in this talk



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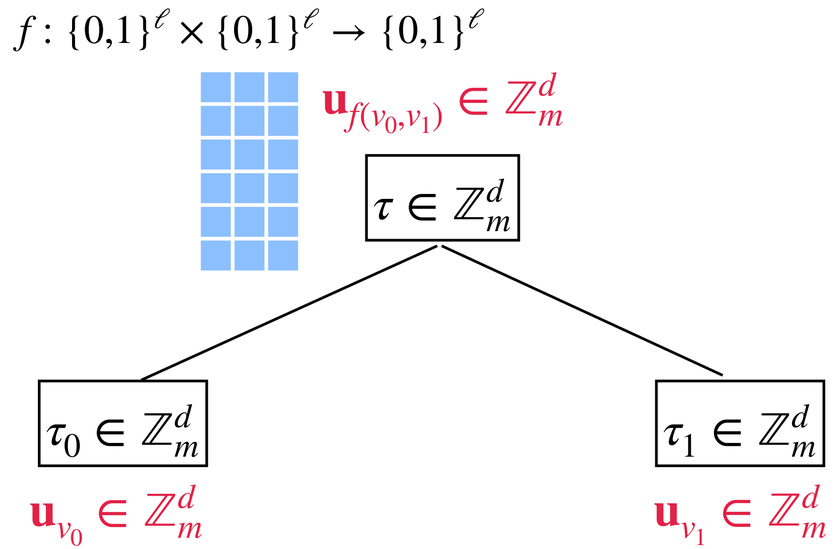

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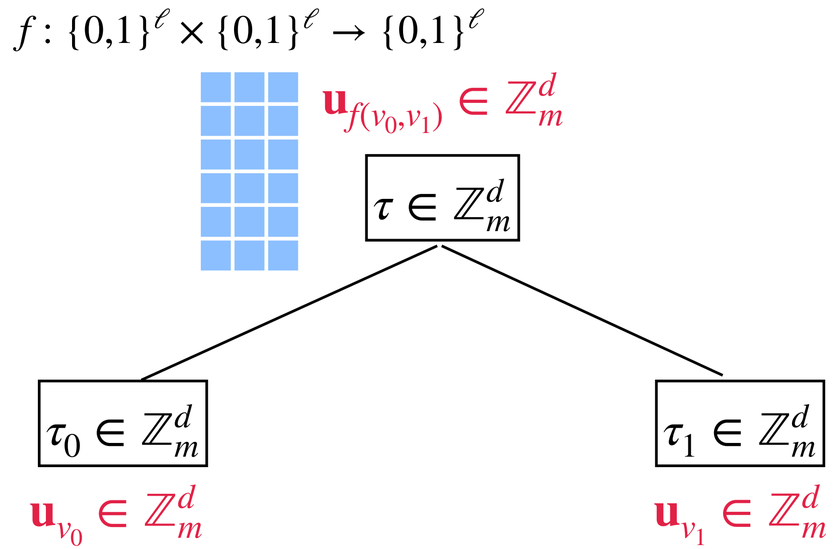

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  - High-level idea: notice that $\{\mathbf{u}_a \otimes \mathbf{u}_b\}_{a,b\in\{0,1\}^\ell}$ is also a MV family that is indexed by $a||b$