On the Cut-Query Complexity of Approximating Max-Cut

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Max-Cut

- Undirected weighted graph G on $[n]$, edge weights $w_{i,j} \geq 0$
- Task: (c-approximately) find

$$\max_S v(S) = \max_S \sum_{i \in S} \sum_{j \in [n] \setminus S} w_{i,j}$$

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- Instead: an oracle that takes $S \subseteq [n]$ as queries and informs the algorithm of $v(S)$
- Algorithm's cost is the number of queries made; *it can perform unlimited local computation without penalty*

Comparison with the Standard Model

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  - [GW95]: poly(n) time algorithm achieving a 0.878-approximation
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  - Observation:

$$w_{i,j} = \frac{1}{2}(v(\{i\}) + v(\{j\}) - v(\{i,j\})).$$

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  - Algorithm: query all singletons and sets of size 2 to learn the graph, then brute force
- [GW95] does not imply any algorithm with $o(n^2)$ queries; needs access to the entire graph

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- Many other graph problems have been studied in this model and other limited-access query models:
  - Problems: connectivity, bipartiteness, triangle detection ...
  - Models: linear measurements [AGM12, ACK21], matrix-vector products [SWYZ19], OR/XOR/AND queries [BvdBE+22]

Previous Work in the Cut-Query Model

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    - Proceeds by constructing a cut sparsifier of G, which our work extends to weighted graphs

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  - [MN20]: $\widetilde{O}(n)$ queries suffice in weighted graphs

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  - [AEG+22]: improved this to $O(n)$
  - [MN20]: $\widetilde{O}(n)$ queries suffice in weighted graphs
  - [GPRW20]: $\Omega(n)$ queries are necessary for weighted graphs
    - Uses a linear algebraic *cut dimension* argument, which we extend in one of our lower bounds

Our Results for Max-Cut

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- Main results:
  1. $\Omega(n)$-query deterministic lower bound for $c > 1/2$
  2. $\widetilde{O}(n)$-query randomised algorithm for $c < 1$

Warm-Up: Cut Dimension Lower Bound for $c = 1$

Theorem

A deterministic algorithm requires at least $\Omega(n^2)$ queries to exactly find the value of the max cut.

- Directly adapts the cut dimension lower bound for min-cut in the query model by [GPRW20]

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- Directly adapts the cut dimension lower bound for min-cut in the query model by [GPRW20]
- Adversary answers all queries as if the graph is K_n (all edge weights 1)
- Idea: at the end, there will be another graph that is consistent with all queries but has different max cut value to K_n

Cut Dimension Lower Bound: Setup

- Assign each cut S a *query vector* $v_S \in \mathbb{R}^{\binom{n}{2}}$ indexed over unordered pairs (i, j) of distinct vertices:

$$(v_S)_{i,j} = \begin{cases} 1, & (i,j) \text{ crosses the cut defined by } S, \\ 0, & \text{otherwise.} \end{cases}$$

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- Let $w \in \mathbb{R}^{\binom{n}{2}}$ be the vector containing all edge weights. Then:

$$v(S) = w^T v_S.$$

Cut Dimension Lower Bound: Setup

- $\mathbf{1} \in \mathbb{R}^{\binom{n}{2}}$ with all entries 1: the weight vector of K_n
- Let the other graph have weights $\mathbf{1} + z \in \mathbb{R}^{\binom{n}{2}}$. Non-negative weights $\Rightarrow$ require $z \geq -\mathbf{1}$.

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- If there exists $z \in \mathbb{R}^{\binom{n}{2}}$ satisfying all highlighted constraints, a deterministic algorithm cannot exactly identify the value of the max cut.

Defining the Cut Dimension

Definition

Define the *max cut space* of a graph G to be:

$$\text{span}(\{v_U : U \text{ a max cut of } G\})$$

The *max cut dimension* of G is the dimension of its max cut space.

Cut Dimension Lower Bound: Proof

Lemma ([GPRW20])

A deterministic algorithm that exactly finds the max cut needs at least as many queries as the max cut dimension d of K_n.

Proof.

- Assume $q < d$. Then we can find nonzero z in the max cut space of K_n that is orthogonal to $\text{span}(v_{Q_1}, \ldots, v_{Q_q})$. There must exist a max cut C such that $z^T v_C \neq 0$.

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□

Max cut dimension of K_n is $\Omega(n^2) \Rightarrow$ lower bound follows.

Extending Cut Dimension Argument to $c > 1/2$

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For $c > 1/2$, a deterministic algorithm that estimates the value of the max cut within a factor of c requires at least $\Omega(n)$ queries.

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- Again, adversary answers according to K_n
- Want to find a perturbation z such that $\mathbf{1} + z$ has max cut value at least a factor of $1/c$ more than K_n.

$c > 1/2$ Deterministic Lower Bound: Constraints on z

- $\mathbf{1} + z$ must have non-negative edge weights:

$$z \geq -\mathbf{1}.$$

- $\mathbf{1} + z$ must agree with $\mathbf{1}$ on all queries:

$$z^T v_{Q_i} = 0 \forall i.$$

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- There must be a max cut (or just a *near max cut* where each side has $n/2 + o(n)$ vertices) C such that $(\mathbf{1} + z)^T v_C$ is much larger than $\mathbf{1}^T v_C$:

$$z^T v_C \geq \frac{\delta^2 n^2}{4},$$

where $\delta \in (0, 1)$ depends on c.

$c > 1/2$ Deterministic Lower Bound: Simplified LP

- Work over $\mathbb{R}^n$, indexed by vertices. Assign to each cut S a vector $u_S \in \mathbb{R}^n$ that is 1 when $i \in S$ and -1 otherwise.

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- It turns out that it suffices to find a near max cut C and $y \in \mathbb{R}^n$ satisfying the following:
  - $-\mathbf{1} \leq y \leq \mathbf{1}$
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- Take the dual of this LP $\Rightarrow$ we need to show that there exists a near max cut C such that

$$\min_{u \in V} ||u - u_C||_1 \geq \delta n$$

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- Proof idea: consider an ϵ-net of points in V with respect to the ℓ_1 metric.

Randomised Algorithm for $c < 1$

We will use *cut sparsifiers:*

Definition

Given weighted graphs G, H on the same set of n vertices with non-negative weights, we say H is an ϵ-sparsifier of G if all of the following conditions hold:

- H has $\widetilde{O}_\epsilon(n)$ edges with nonzero weight.
- For any cut $S \subseteq [n]$, we have $(1-\epsilon)v(S;G) \leq v(S;H) \leq (1+\epsilon)v(S;G)$.

Here $v(S;G)$ denotes the value of the cut defined by S on the graph G.

Algorithm: compute an ϵ-sparsifier, then find the max cut of the sparsifier by running brute force locally.

Sparsification in the Cut Query Model: Results

Theorem ([RSW18], based on ideas by [BK15])

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Theorem (This work, using additional ideas by [BK15])

For weighted G, we can construct an ϵ-sparsifier in $\widetilde{O}(n/\epsilon^2)$ queries (regardless of the value of W).

Sparsification: Naive Attempts

Idea: randomly subsample edges and use a Chernoff bound.

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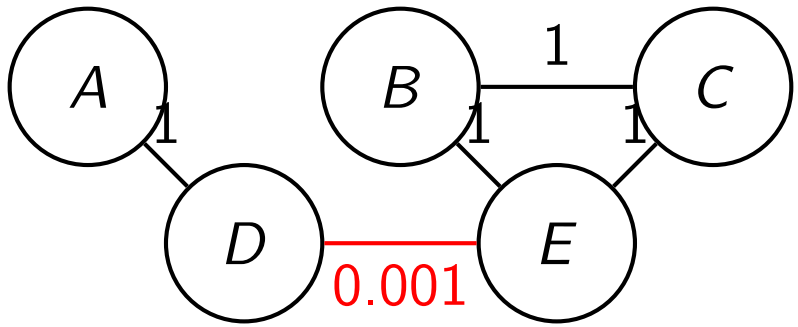
- Attempt 1: randomly sample each edge with probability proportionate to $w_{i,j}$. Issue: could miss bridges of low weight.



Sparsification: Edge Strength-Based Subsampling [BK15]

For an edge $e = (i, j)$, define the *edge strength* of e as the maximum min cut over all vertex-induced subgraphs containing e:

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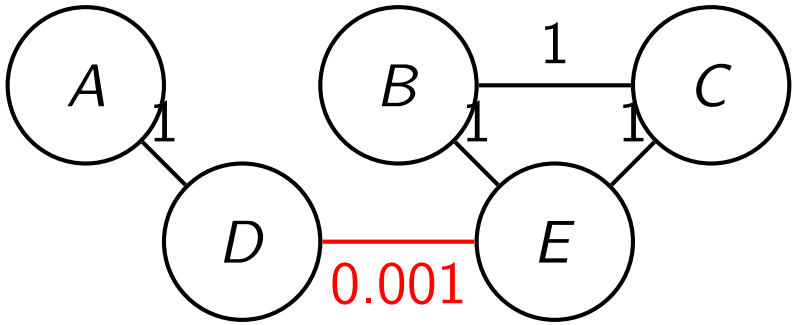

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- This work: only subsample at $O(\log n)$ different resolutions $\Rightarrow$ sparsification in $\widetilde{O}(n/\epsilon^2)$ queries.
  - Estimate edge strengths within a factor of poly(n) using $\widetilde{O}(n)$ queries
    - Uses a very weak Kruskal-style algorithm and results by [BK15] relating edge strengths to maximum spanning trees

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  - Estimate edge strengths within a factor of $\text{poly}(n)$ using $\widetilde{O}(n)$ queries
  - Incorporate this information into the [RSW18] algorithm $\Rightarrow$ no longer need to search through all $O(\log W + \log n)$ resolutions

Open Direction: Deterministic Sparsifiers?

- Success in designing deterministic sparsification algorithms in the classical model $\Rightarrow$ might hope for these to be adaptable to the cut query model
- This appears to be difficult; the deterministic sparsification algorithm by [BSS14] is linear algebraic and relies heavily on having access to the graph's Laplacian matrix

Thank you!

References I

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