The Jacobi Factoring Circuit

Classically Hard Factoring in Sublinear Quantum Space and Depth

Gregory D. Kahanamoku-Meyer*, Seyoon Ragavan*, Vinod Vaikuntanathan*, Katherine Van Kirk†
*MIT, †Harvard




Integer Factorisation

Given an n-bit integer $N < 2^n$, find its prime factorisation in $\mathsf{poly}(n)$ time.

Classical Factoring: A Crash Course

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For more: see Pomerance's survey "A Tale of Two Sieves"!

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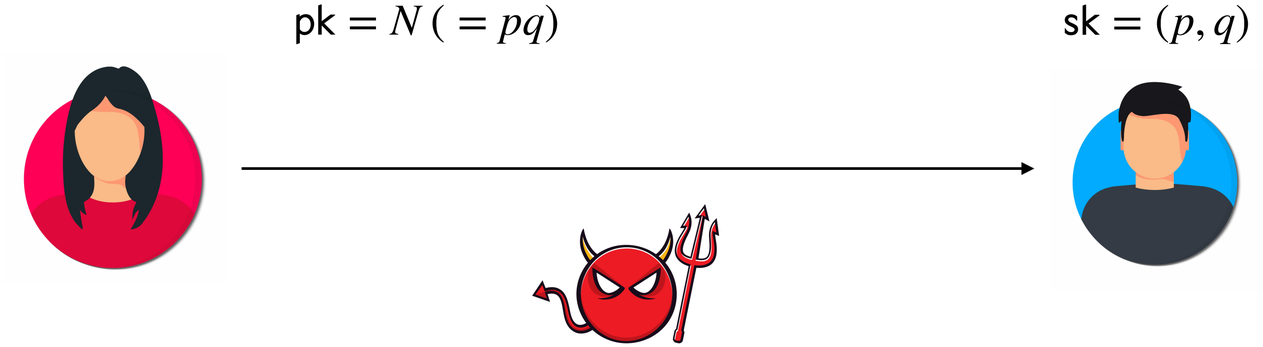
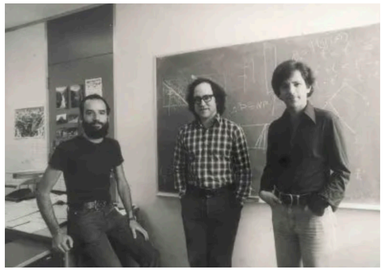
- **Quantum algorithms:** poly(n) **time!** (Shor 1994)

Implication 1: Breaking Cryptography

- RSA public-key cryptography:





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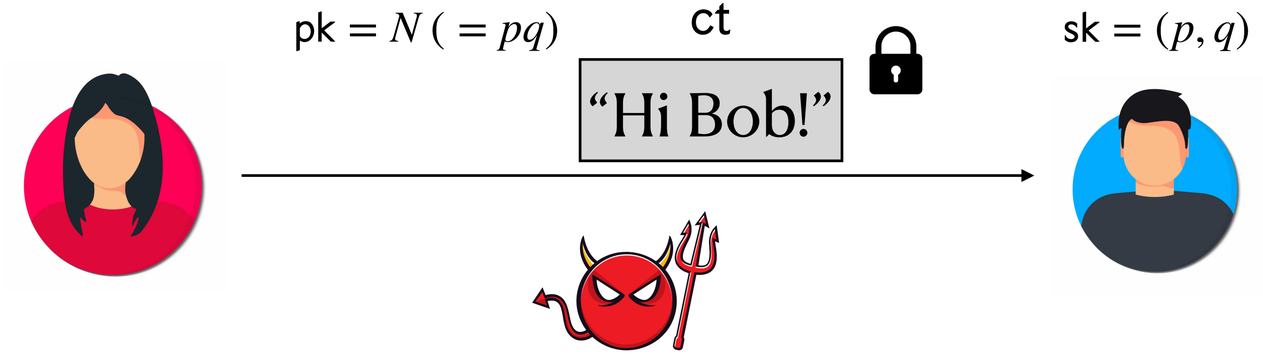

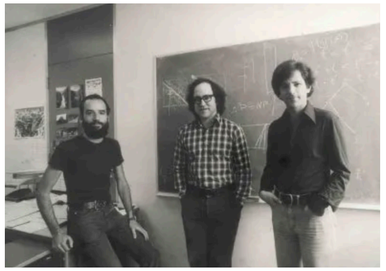

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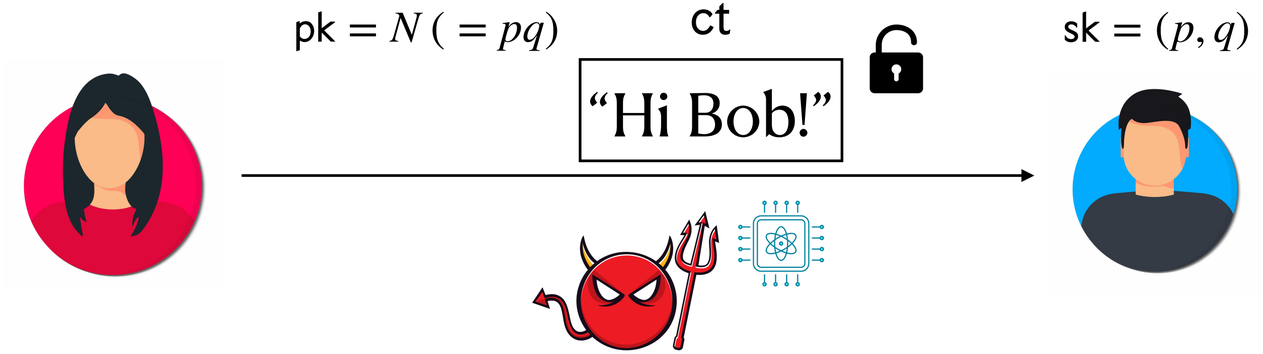

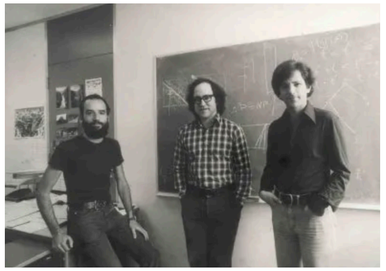

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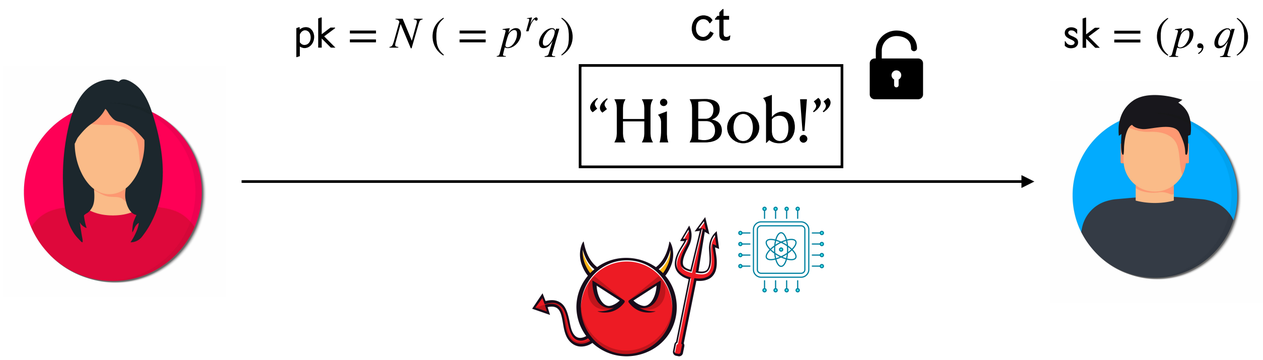
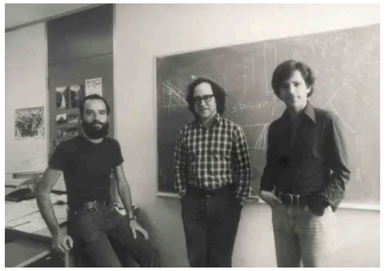
Goal: faster decryption than RSA





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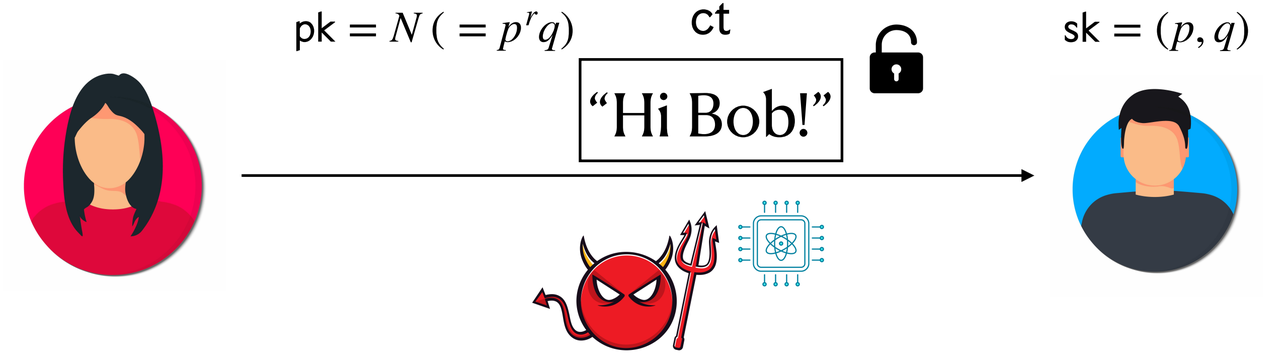

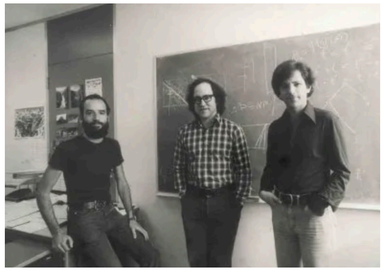

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Coming up: even better quantum circuits for factoring $p^r q$ ($r > 1$)

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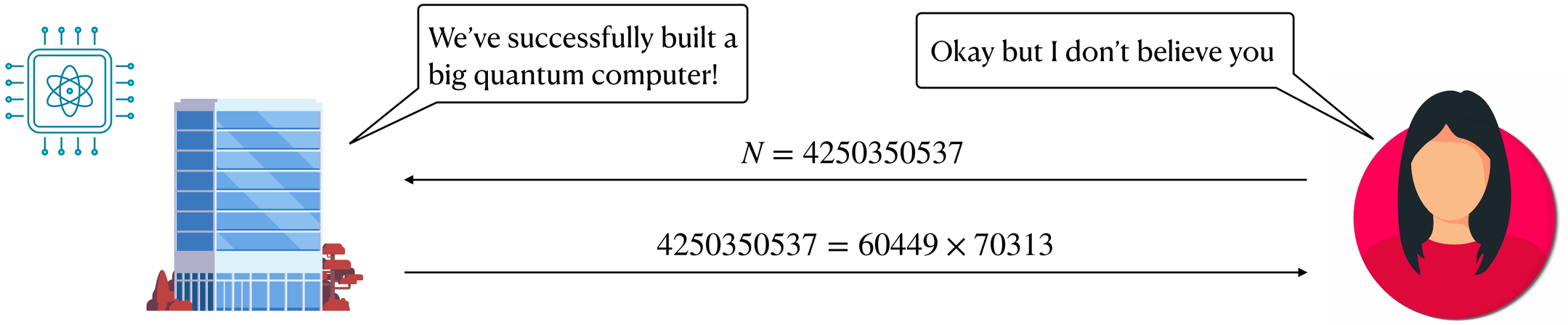
Or: Efficiently Verifiable Quantum Advantage



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One answer: By factoring a large integer of Alice's choice!

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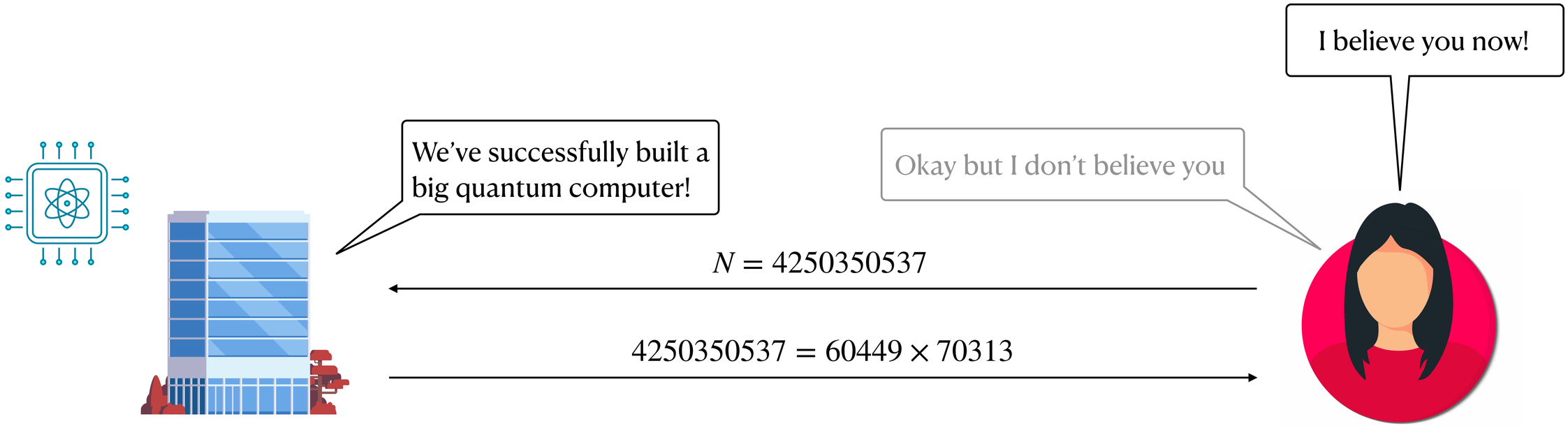

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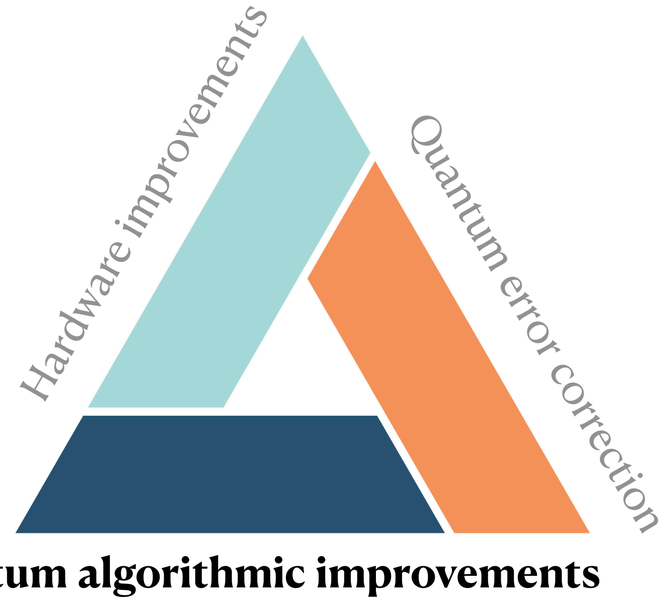
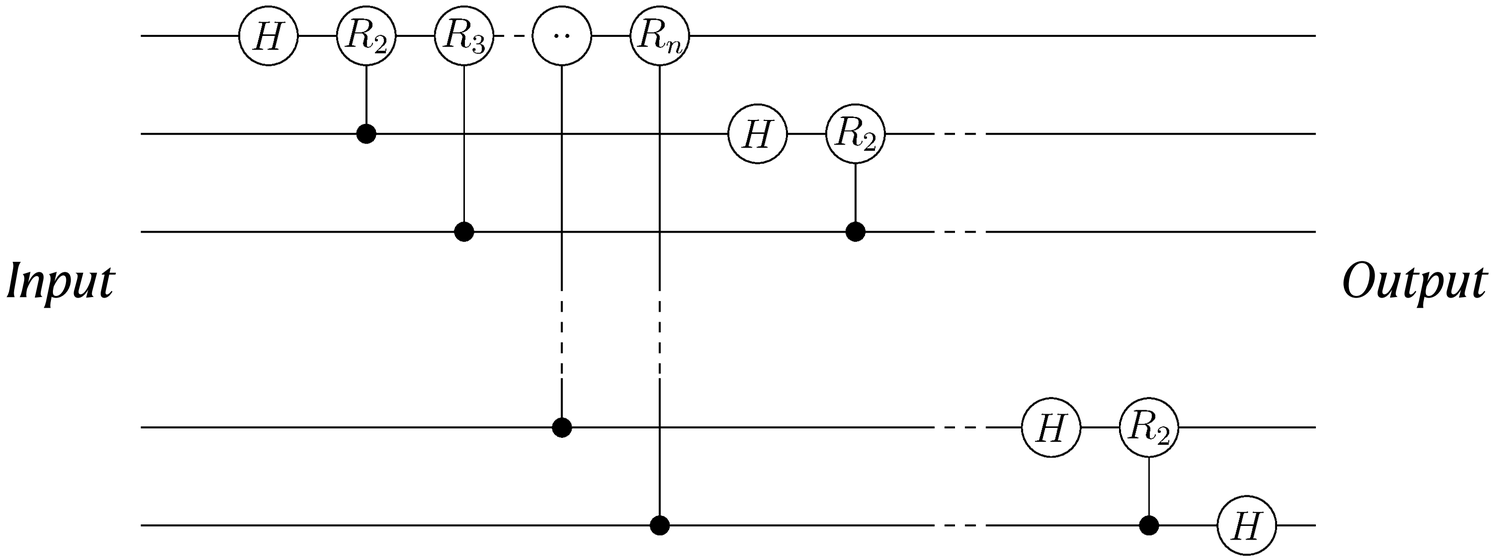


Efficiency Metrics for Quantum Circuits

- Similar to a classical circuit, but uses reversible logic gates on a fixed number of wires



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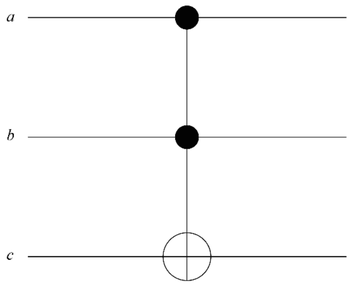

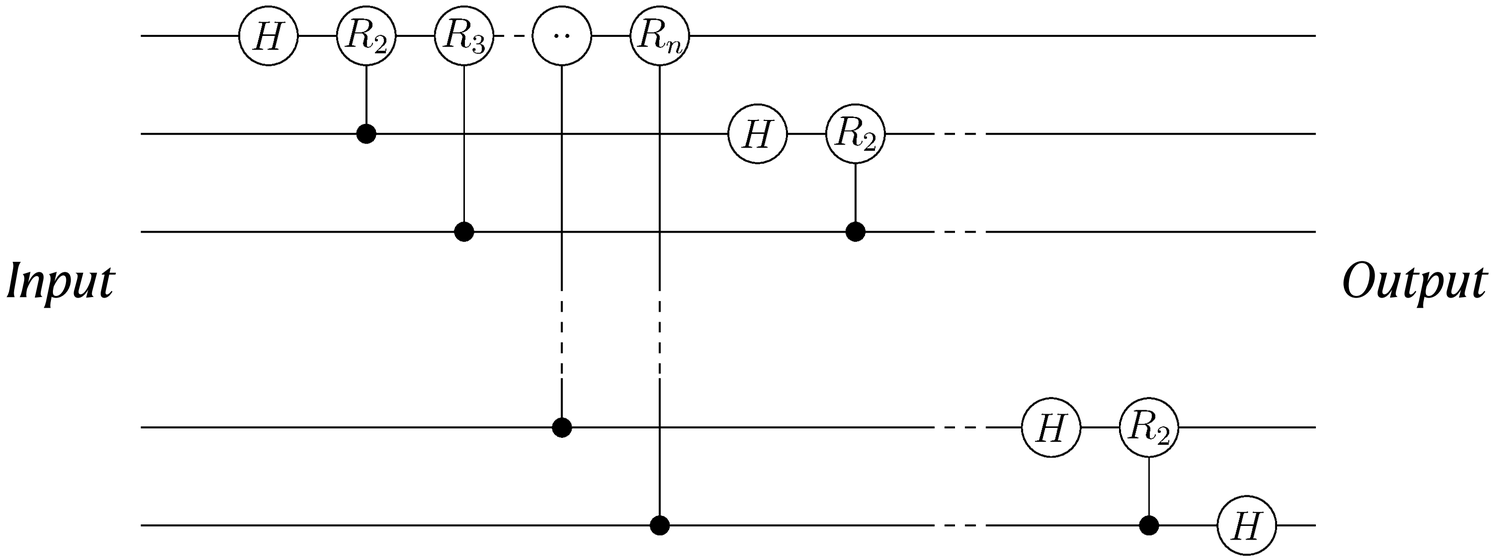

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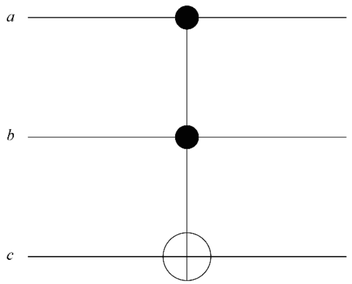

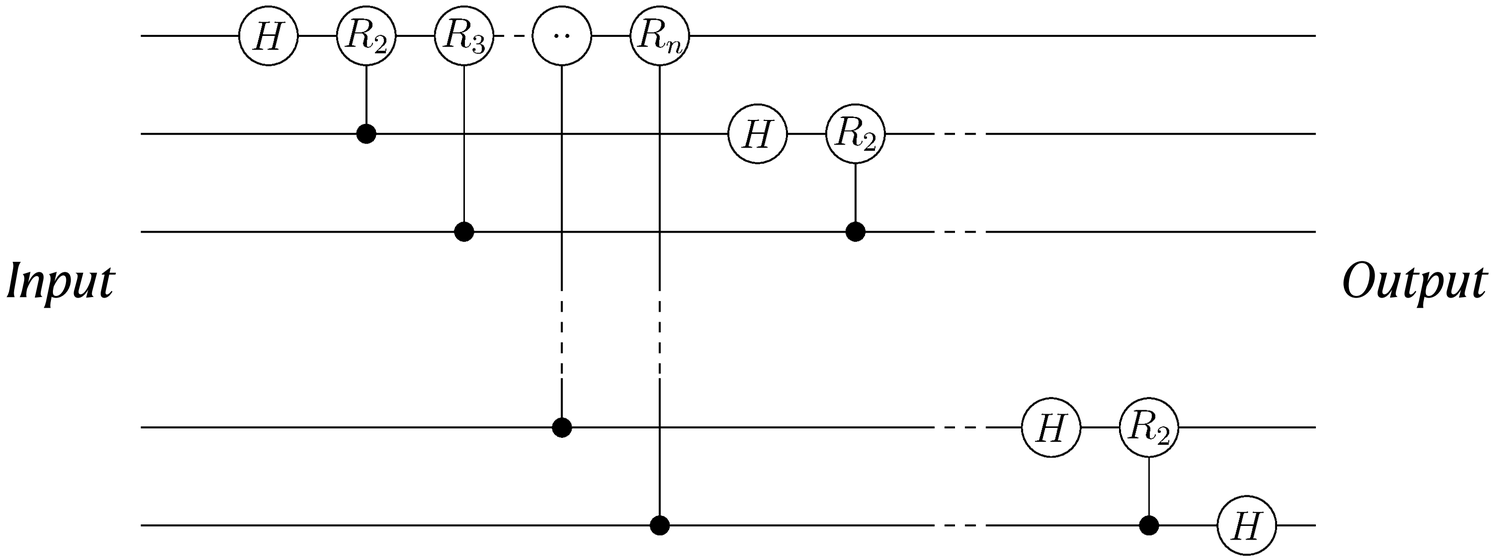

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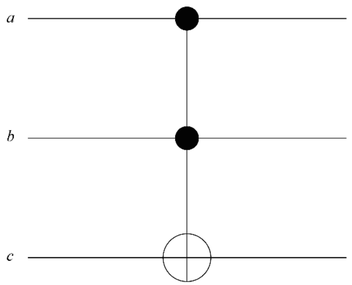

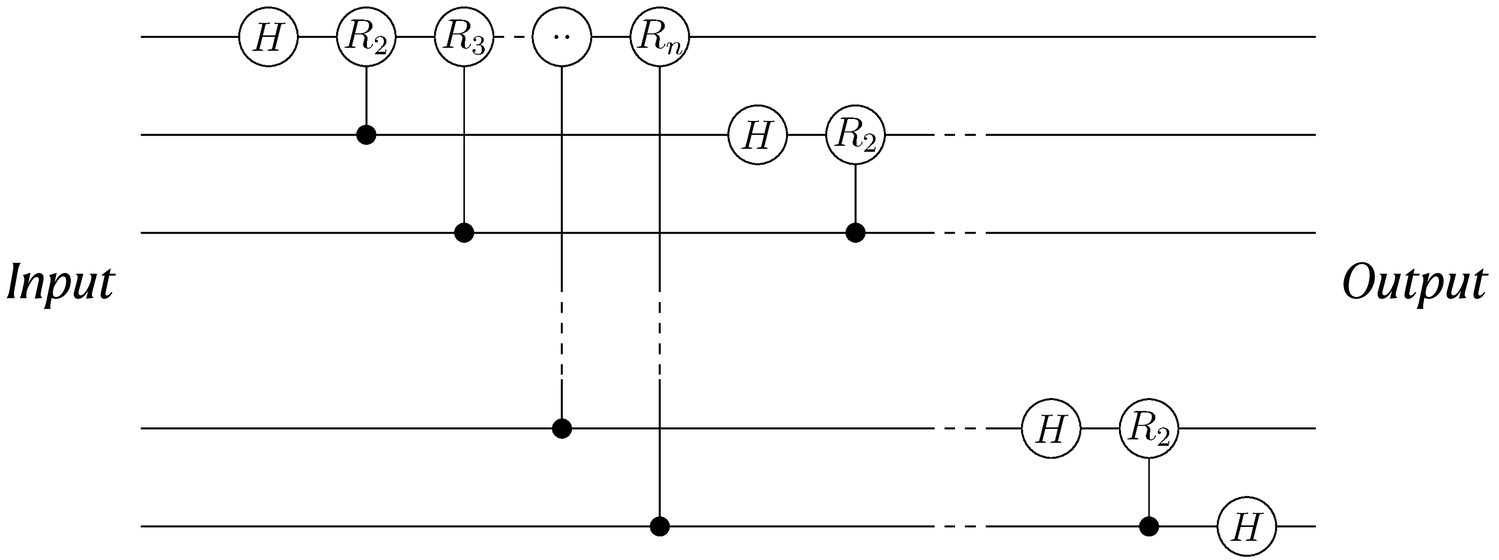

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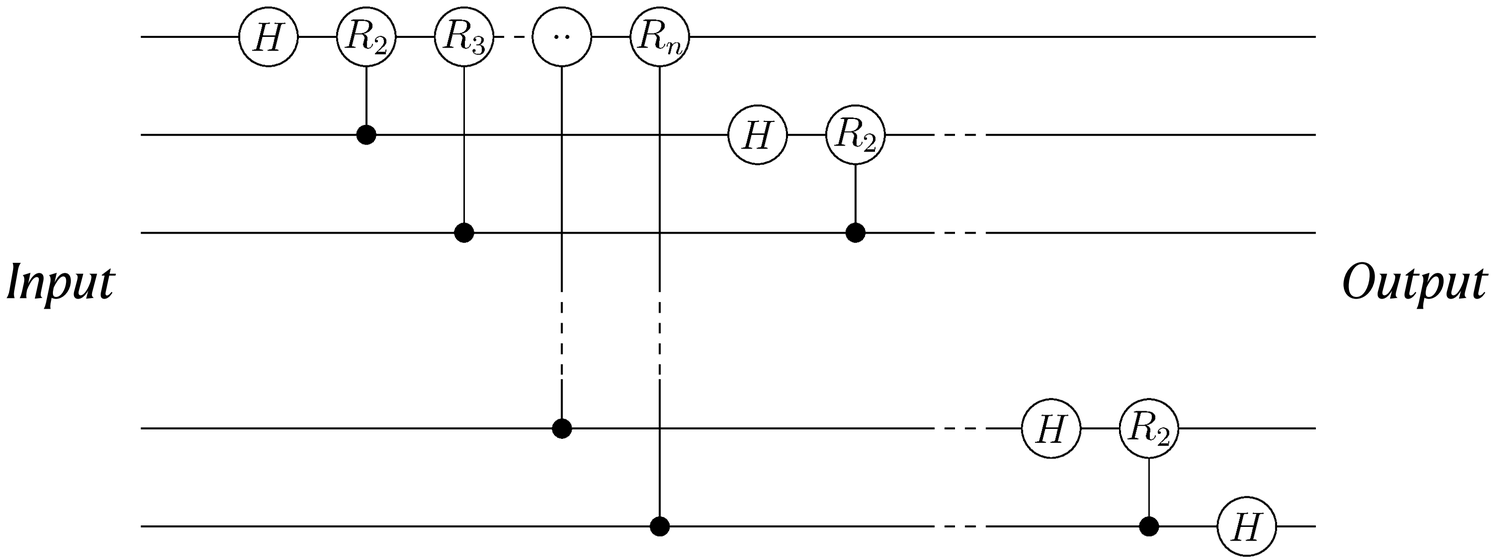
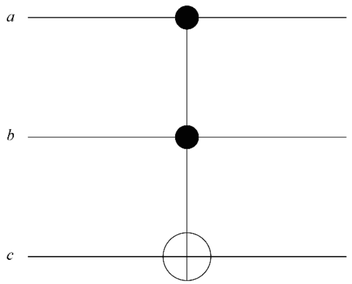
Depth: number of time steps (if gates acting on disjoint wires can be run in parallel)





Proofs of Quantumness from Factoring

Authors	Types of inputs	Gates	Space	Depth
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$

n is the number of bits in the input N

$\tilde{O}(\cdot)$ hides constant and $\mathrm{poly}(\log n)$ factors

All results in this talk are using fast integer multiplication (multiply n-bit integers in $\tilde{O}(n)$ time)

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Paper 2024/222

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[Submitted on 21 May 2025]

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- This history overview: focuses on *asymptotic* improvements

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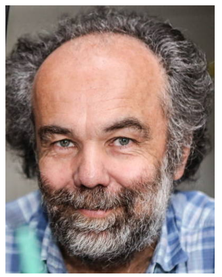

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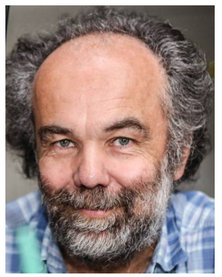
Any input: would break RSA cryptography, and suffice as a proof of quantumness

$N = P^2Q$: only suffices as a proof of quantumness



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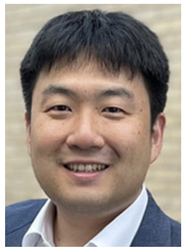

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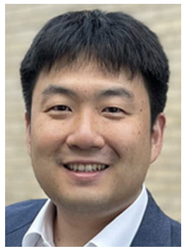

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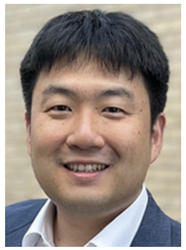

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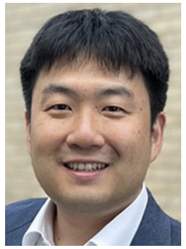
- Con: quantum prover needs to store state between rounds

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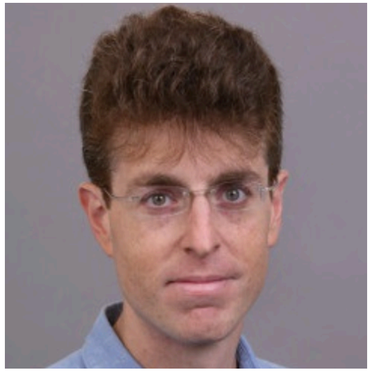

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Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$
LPDS (2012)	$N = P^2Q$	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(n)$
KCVY (2021)	N/A	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(1)$
Regev (2023)	Any	$\tilde{O}(n^{1.5})$	$O(n^{1.5})$	$\tilde{O}(n^{0.5})$
RV (2024)	Any	$\tilde{O}(n^{1.5})$	$\tilde{O}(n)$	$\tilde{O}(n^{0.5})$
K**R**VV (2024)	$N = P^2Q$ $(Q < 2^m)$	$\tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m + m)$

This work: LPDS with space and depth proportional to $\log Q$ rather than $\log N$




Aside: Setting Parameters

How should we set $m = \log Q$ relative to n?

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Our Result

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An algorithm that factors special-form integers (that are still classically as hard as RSA integers to factor) in <u>sublinear</u> space and depth!




Roadmap

- Shor's algorithm: quantum period finding and application to factoring
- LPDS12 algorithm: factoring P^2Q in $\tilde{O}(\log N)$ gates
- Our work: pushing the space of LPDS12 down to $\tilde{O}(\log Q)$
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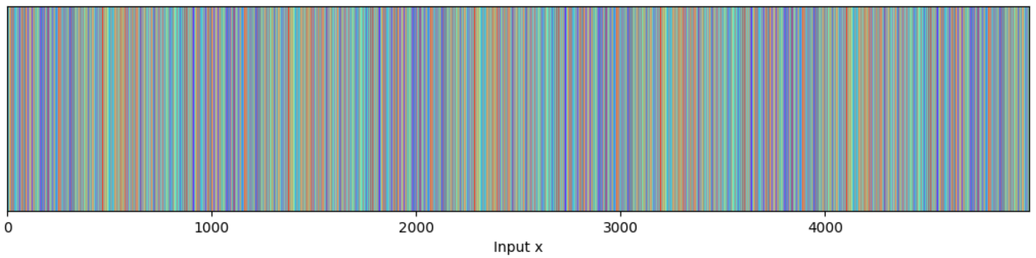
- Strictly periodic function $f : \mathbb{Z} \to \mathbb{Z}$ with unknown period T
  - $x \equiv y \pmod{T} \Leftrightarrow f(x) = f(y)$



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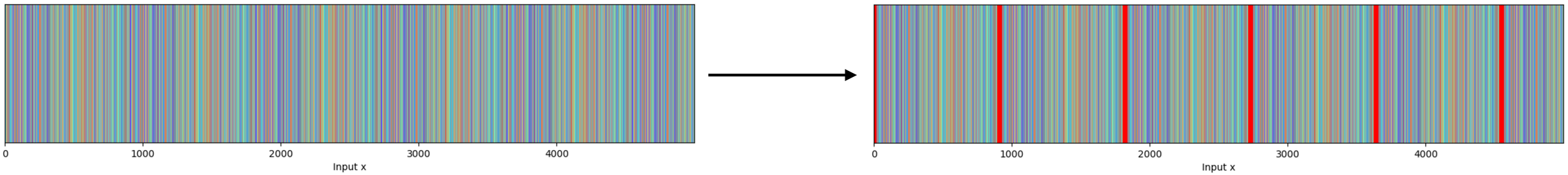

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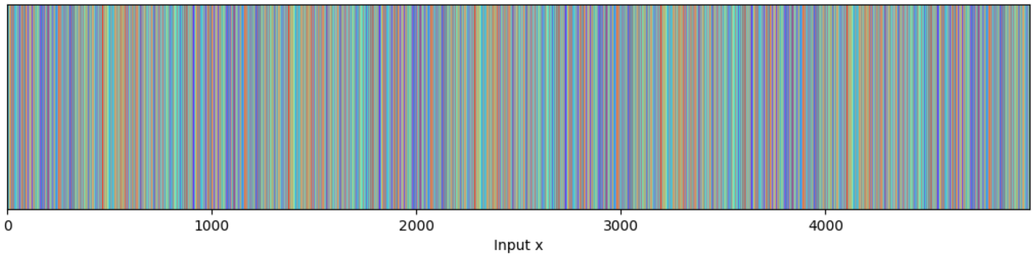

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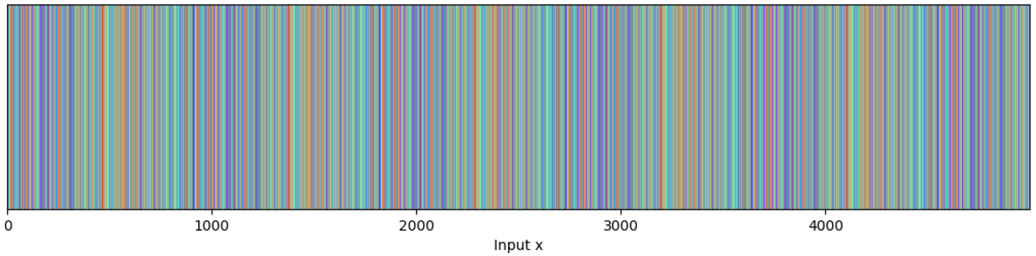

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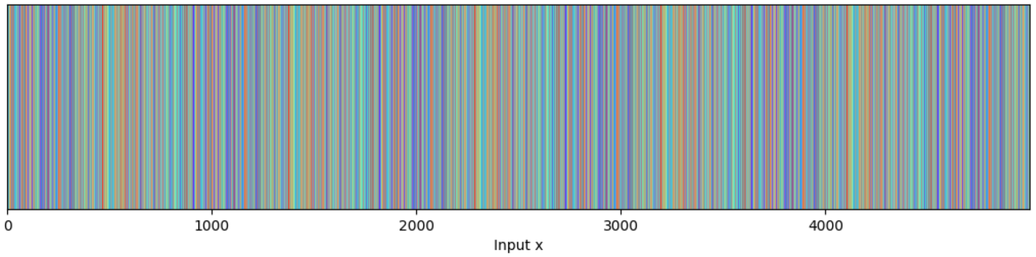

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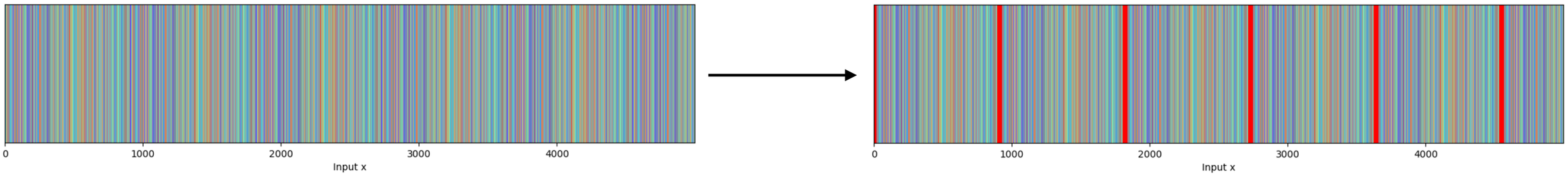
- With decent probability: $\gcd(a, T) = 1 \rightarrow$ learn T



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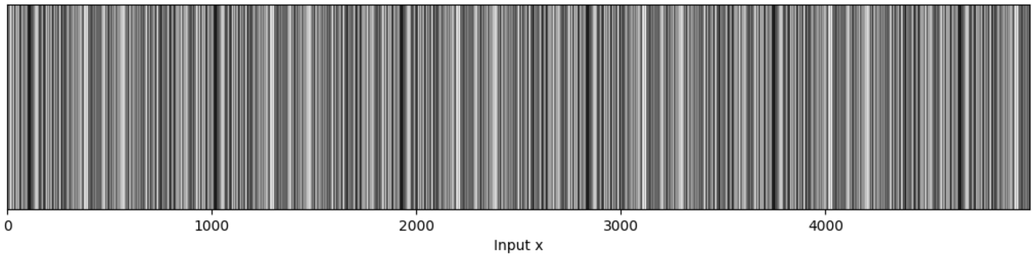

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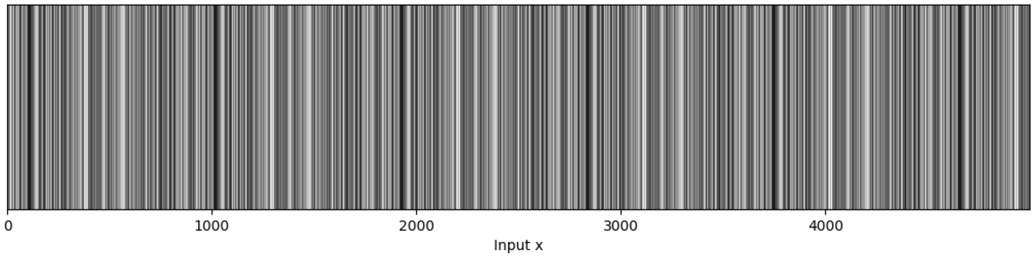

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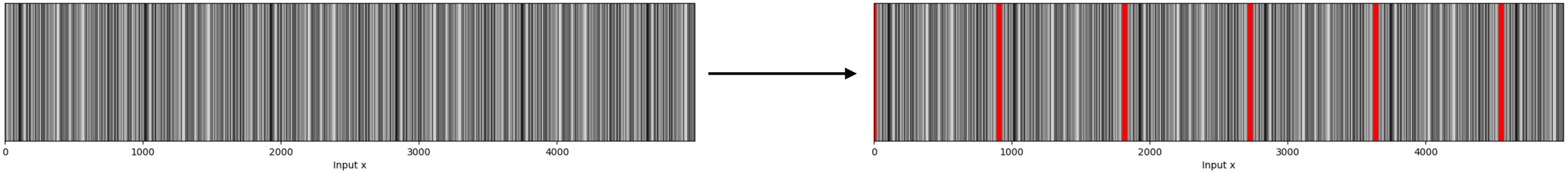
- Informal theorem statement: for "reasonable" f, this is still sufficient to recover T



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- As long as this is not the case, we can succeed after taking enough samples!
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- Even better, we can recover T from one sample if $\hat{f}$ concentrates on values a such that $\gcd(a, T) = 1$

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    - *"Luck" is because the base is chosen to be a random square rather than always 4*

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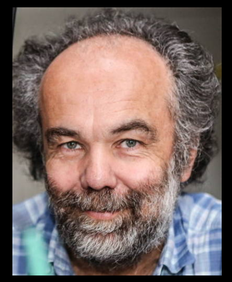

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- Legendre symbol essentially indicates whether this is the case:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a nonzero quadratic residue modulo } p; \text{ and} \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p; \text{ and} \\ 0, \text{ if } a \text{ divisible by } p\,. \end{cases}$$

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- Useful property: $a \equiv b \pmod{N} \Rightarrow \left(\frac{a}{N}\right) = \left(\frac{b}{N}\right)$
- Theorem (from Euclid to Schönhage 1971): can compute $\left(\frac{a}{N}\right)$ efficiently without knowing the factorisation of N — in fact, in time $\tilde{O}(\log N)$

Period Finding Beyond Shor!

Goal to improve on Shor: find a different function $j(x)$ which:

- Also has periodicity that enables us to factor N;
- Has a shorter period; and
- Is easier to implement than $4^x \bmod N$ ✔

The function $j(x)$ that we're looking for is the Jacobi symbol $j(x) = \left(\frac{x}{N}\right)$!

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$$\left(\frac{a}{N}\right) = \left(\frac{a}{P}\right)^2\left(\frac{a}{Q}\right) = \left(\frac{a}{Q}\right), \text{ which is periodic}^* \text{ with period } Q!$$

* modulo minor technical caveats; could have $\left(\frac{a}{P}\right) = 0$ for a tiny fraction of inputs a

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Quantumly Factoring $N = P^2Q$

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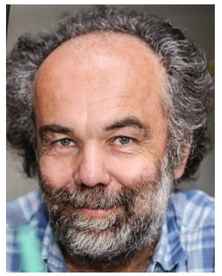

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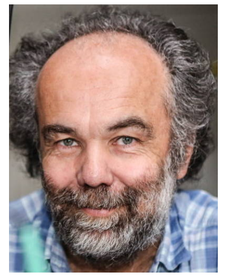

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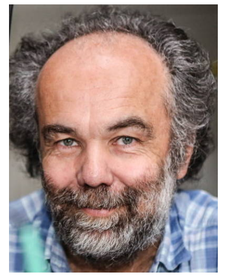

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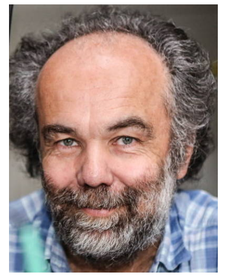
- Space and depth (if naively implemented): also $\tilde{O}(\log N)$



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  - Proof: Gauss sums $\rightarrow \hat{f}(a) = 0$ whenever $\gcd(a, Q) > 1$

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- Generalised quantum period finding: we recover $\frac{a}{Q}$ with probability

$$\left|\hat{f}(a)\right|^2 = \frac{1}{Q}\left|\sum_{x=0}^{Q-1}\left(\frac{x}{Q}\right)\cdot\exp\left(\frac{2\pi i a x}{Q}\right)\right|^2$$

- Typically (and in Shor's factoring algorithm): may need > 1 samples (runs of the quantum circuit) to recover the period
  - Troublesome case: we sample a such that $\gcd(a, Q) > 1$
- **Theorem (follows from Gauss sums): 1 sample <u>always</u> suffices for Jacobi!**
  - Proof: Gauss sums $\rightarrow \hat{f}(a) = 0$ whenever $\gcd(a, Q) > 1$
  - Special case for intuition: if Q is prime, we have $\hat{f}(0) = \mathbb{E}\left[\left(\frac{x}{Q}\right)\right] = 0$

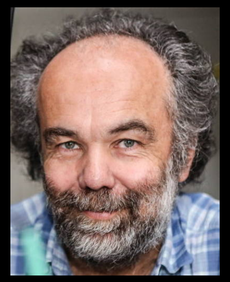
Is the Jacobi Function "Reasonable"?

The Jacobi function isn't just "reasonably good" for general quantum period finding, it's actually magically well-suited to it — even more so than the periodic function used by Shor to factor!

Roadmap

- Shor's algorithm: quantum period finding and application to factoring
- LPDS12 algorithm: factoring P^2Q in $\tilde{O}(\log N)$ gates
- Our work: pushing the space of LPDS12 down to $\tilde{O}(\log Q)$
- Summary and open questions



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Why is There Any Hope for Sublinear Space?

Goal to improve on Shor: find a different function $j(x)$ which:

- Also has periodicity that enables us to factor $N = P^2Q$; ✔
- **Has a shorter period; and** ✔ **(just Q rather than N)**
- Is easier to implement than $4^x \bmod N$ ✔

The function $j(x)$ that we're looking for is the Jacobi symbol $j(x) = \left(\frac{x}{N}\right)$!

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- **Hope:** when factoring P^2Q with Jacobi: the period is just $Q \rightarrow O(\log Q)$ qubits could suffice!

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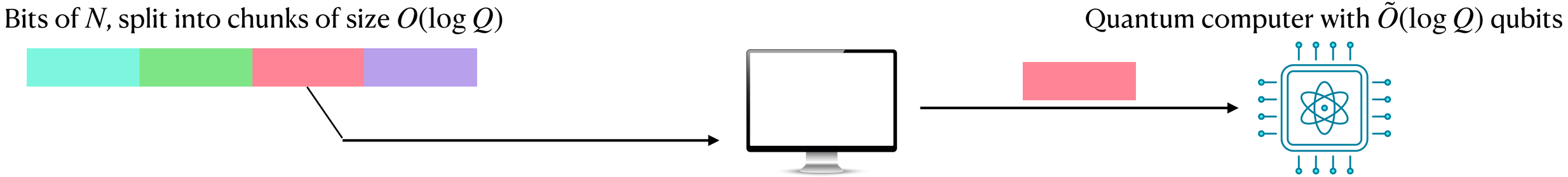

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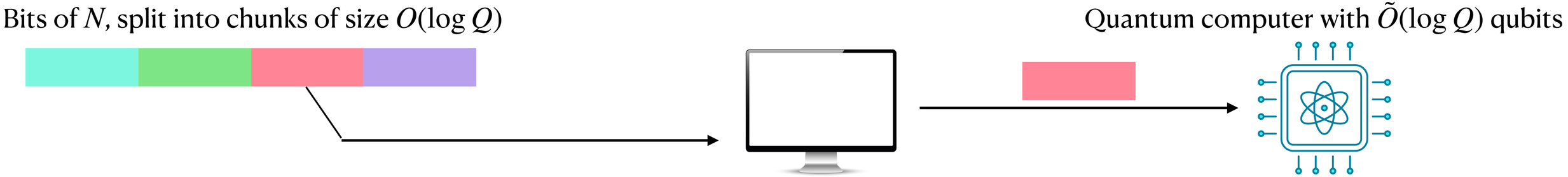
But quantum streaming is just a hope.

Why does the Jacobi symbol lend itself to streaming?



Aside: Algorithms Computing the Jacobi Symbol

ft. Euclid, 2000 years ago

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Jacobi Properties

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- Reciprocity:* $\left(\frac{a}{b}\right) = (-1)^{f(a,b)}\left(\frac{b}{a}\right)$

  for a very simple f

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$$f(a,b) = \begin{cases} 0, \text{ if } a \equiv 1 \pmod 4 \text{ or } b \equiv 1 \pmod 4 \\ 1, \text{ if } a \equiv b \equiv 3 \pmod 4 \end{cases}$$

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Acta Informatica 1, 139—144 (1971)

Schnelle Berechnung von Kettenbruchentwicklungen

A. Schönhage

Eingegangen am 16. September 1970

Summary. A method, given by D. E. Knuth for the computation of the greatest common divisor of two integers u, v and of the continued fraction for u/v is modified in such a way that only $O(n(\lg n)^2(\lg\lg n))$ elementary steps are used for $u, v < 2^n$.

Zusammenfassung. Ein von D. E. Knuth angegebenes Verfahren, für ganze Zahlen u, v den größten gemeinsamen Teiler und den Kettenbruch für u/v zu berechnen, wird so modifiziert, daß für n-stellige Zahlen nur $O(n(\lg n)^2(\lg\lg n))$ elementare Schritte gebraucht werden.

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A Unified Approach to HGCD Algorithms for polynomials and integers

Klaus Thull and Chee K. Yap*

Freie Universität Berlin
Fachbereich Mathematik
Arnimallee 2-6
D-1000 Berlin 33
West Germany

March, 1990

Abstract

We present a unified framework for the asymptotically fast Half-GCD (HGCD) algorithms, based on properties of the norm. Two other benefits of our approach are (a) a simplified correctness proof of the polynomial HGCD algorithm and (b) the first explicit integer HGCD algorithm. The integer HGCD algorithm turns out to be rather intricate.

Keywords: Integer GCD, Euclidean algorithm, Polynomial GCD, Half GCD algorithm, efficient algorithm.

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ON SCHÖNHAGE'S ALGORITHM AND SUBQUADRATIC INTEGER GCD COMPUTATION

NIELS MÖLLER

Abstract. We describe a new subquadratic left-to-right GCD algorithm, inspired by Schönhage's algorithm for reduction of binary quadratic forms, and compare it to the first subquadratic GCD algorithm discovered by Knuth and Schönhage, and to the binary recursive GCD algorithm of Stehlé and Zimmermann. The new GCD algorithm runs slightly faster than earlier algorithms, and it is much simpler to implement. The key idea is to use a stop condition for HGCD that is based not on the size of the remainders, but on the size of the next difference. This subtle change is sufficient to eliminate the back-up steps that are necessary in all previous subquadratic left-to-right GCD algorithms. The subquadratic GCD algorithms all have the same asymptotic running time, $O(n(\log n)^2 \log\log n)$.

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$$\left(\frac{0101}{1001101010100010010110010111010011010}\right) \overset{swap}{\rightarrow} \left(\frac{1001101010100010010110010111010011010}{0101}\right) \overset{mod}{=} \left(\frac{0010}{0101}\right)$$

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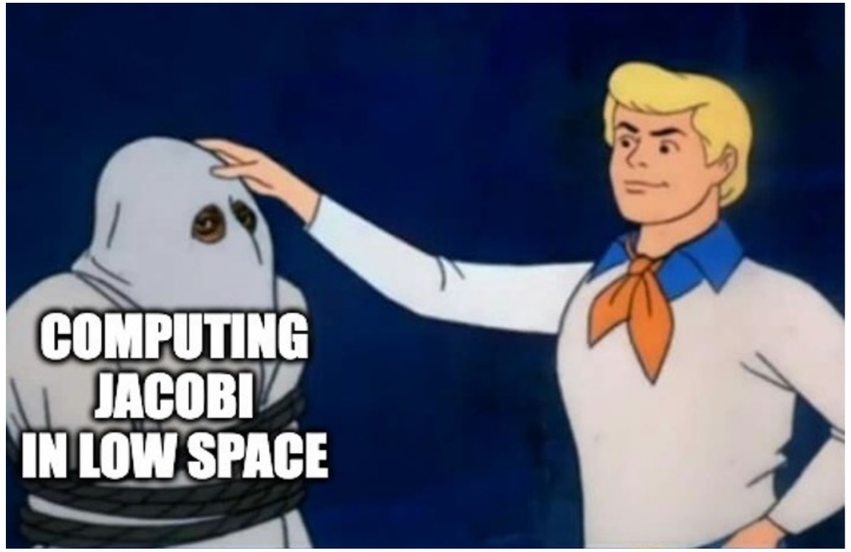
  2. Now can finish in $\tilde{O}(\log Q)$ gates/space using Schönhage

Streaming for Jacobi



Streaming for Jacobi

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← The only place where we need quantum streaming!

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- Theorem (K**R**VV24): for quantum a and classically known N, we can compute

$$|a\rangle \mapsto |a\rangle\,|N \bmod a\rangle$$

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Open question: other applications of these results?

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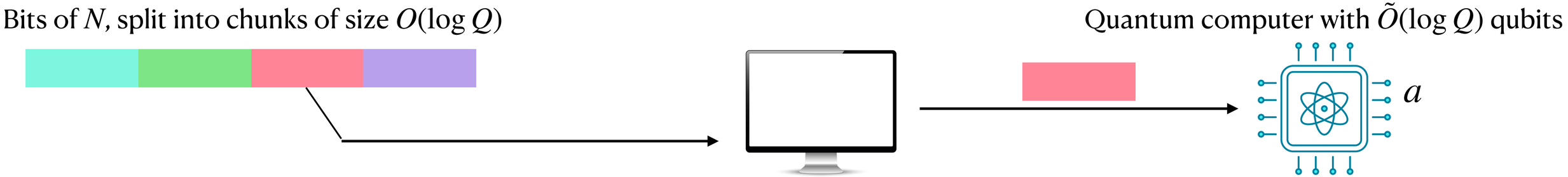
- **Corollary 2: we can factor $N = P^2Q$ in $\tilde{O}(\log N)$ gates and $\tilde{O}(\log Q)$ qubits**
  - **Just need the above theorem for $a \leq \mathrm{poly}(Q)$**

Computing N mod a with Quantum Streaming

Notation: N has n bits, a has $m = O(\log Q)$ bits



Classical computer sending instructions to the quantum computer

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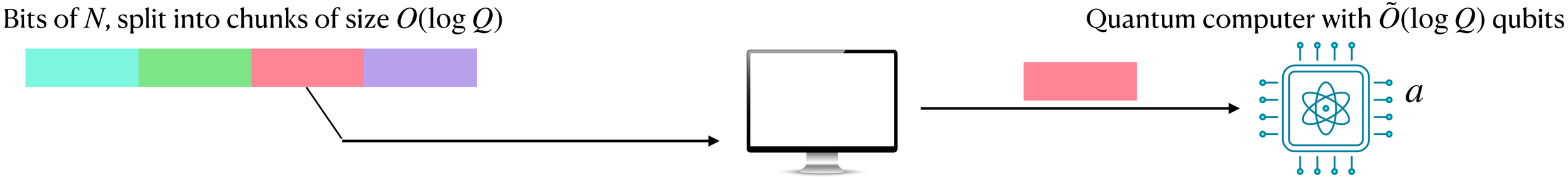

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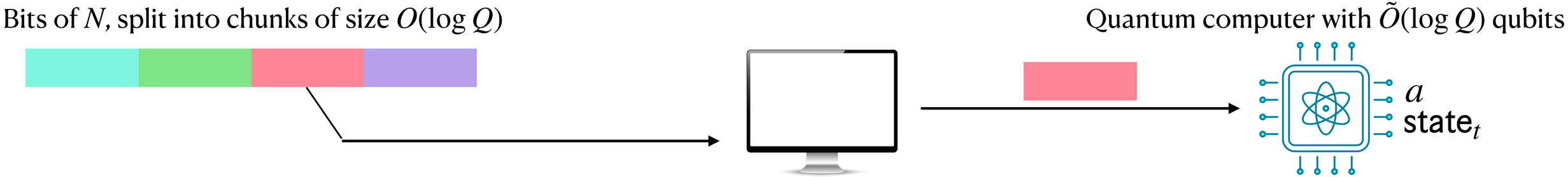

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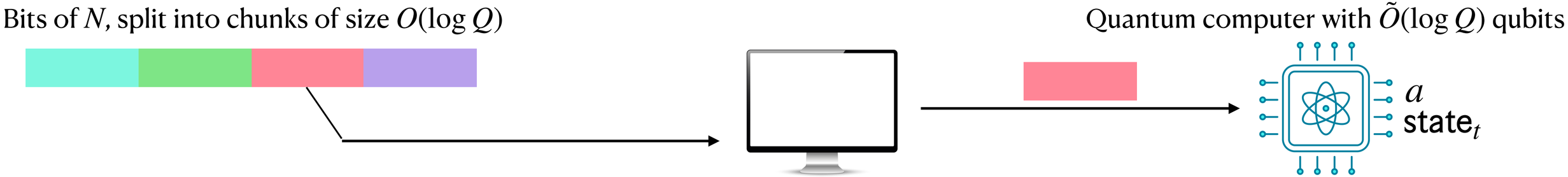

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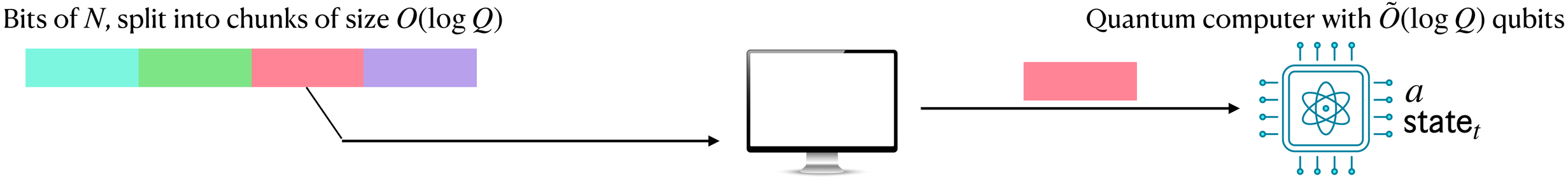

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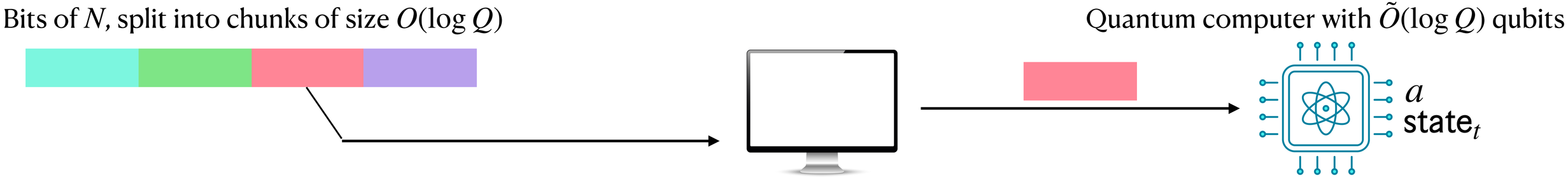

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- It turns out that the final state state_{n-m} suffices to reconstruct $N \bmod a$

Efficiency of Our Algorithm

Recall: $N = P^2 Q$ **with** $N < 2^n, Q < 2^m$

Step	Gates	Space	Depth
Quantum streaming to compute N mod a *(roughly: O(n/m) multiplications of m-bit integers)*	$\frac{n}{m} \times \tilde{O}(m) = \tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m)$
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- Overall: gates $\tilde{O}(n)$, space $\tilde{O}(m)$, depth $\tilde{O}(n/m + m)$
- Recall: can set m as low as $\tilde{O}(n^{2/3})$ while preserving classical cost → space and depth $\tilde{O}(n^{2/3})$

Roadmap

- Shor's algorithm: quantum period finding and application to factoring
- LPDS12 algorithm: factoring P^2Q in $\tilde{O}(\log N)$ gates
- Our work: pushing the space of LPDS12 down to $\tilde{O}(\log Q)$
- Summary and open questions




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Two Main Ideas

To improve on Shor: used the Jacobi symbol $\left(\frac{x}{N}\right)$ which:

- Also has periodicity that enables us to factor $N = P^2Q$; ✔
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- Other quantum factoring algorithms exploiting special structure in N? (Many such algorithms in the classical world)

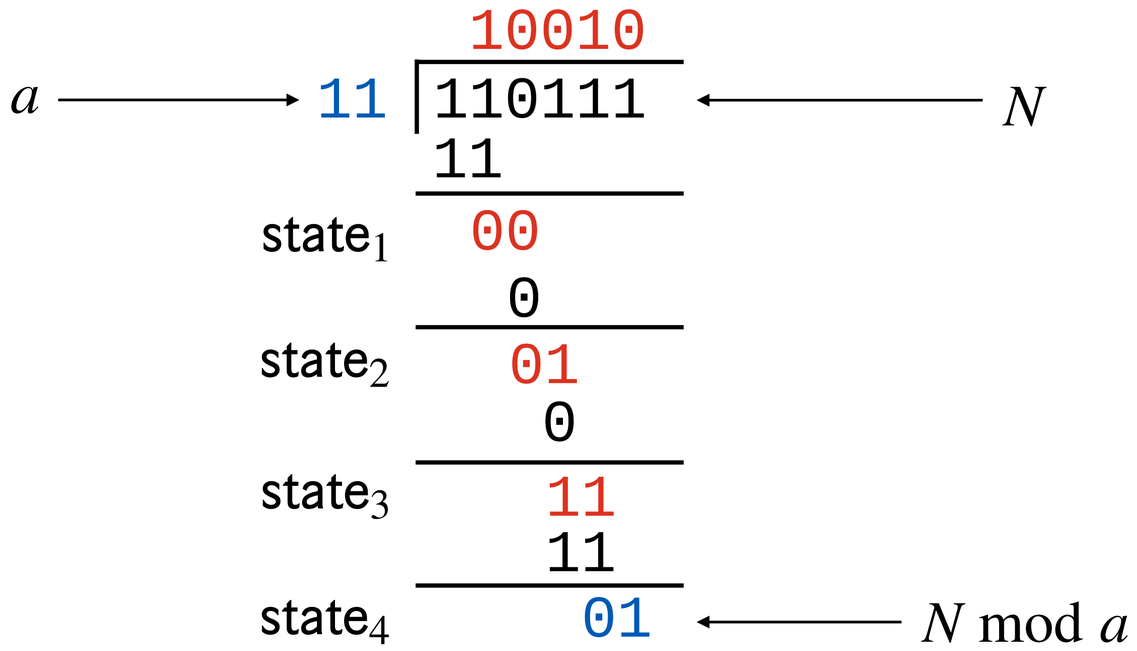
Thank you! Questions?

Authors	Types of inputs	Gates	Space	Depth
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$
LPDS (2012)	$N = P^2Q$	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(n)$
KCVY (2021)	N/A	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(1)$
Regev (2023)	Any	$\tilde{O}(n^{1.5})$	$O(n^{1.5})$	$\tilde{O}(n^{0.5})$
RV (2024)	Any	$\tilde{O}(n^{1.5})$	$\tilde{O}(n)$	$\tilde{O}(n^{0.5})$
K**R**VV (2024)	$N = P^2Q$ $(Q < 2^m)$	$\tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m + m)$

Bonus Slides

Our Construction, In Detail

A Natural Attempt: Long Division



Required state

Excess state that we cannot clean up

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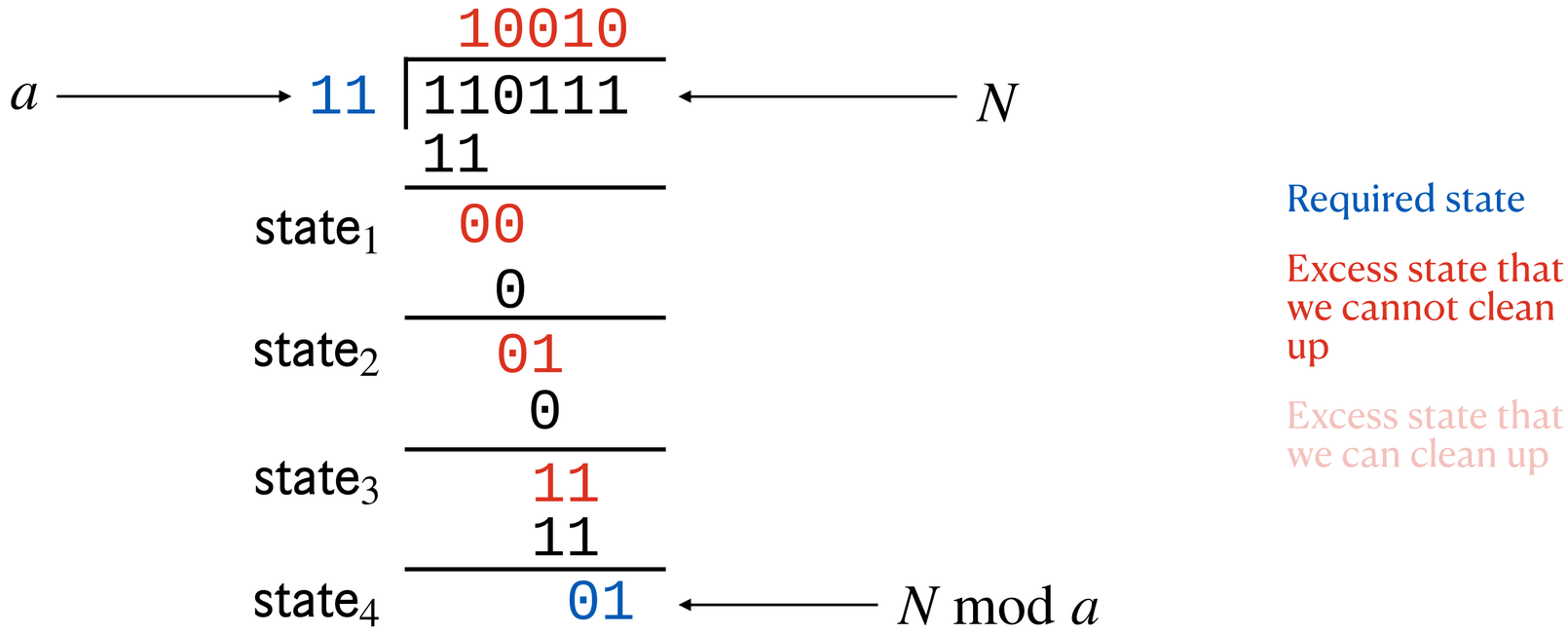

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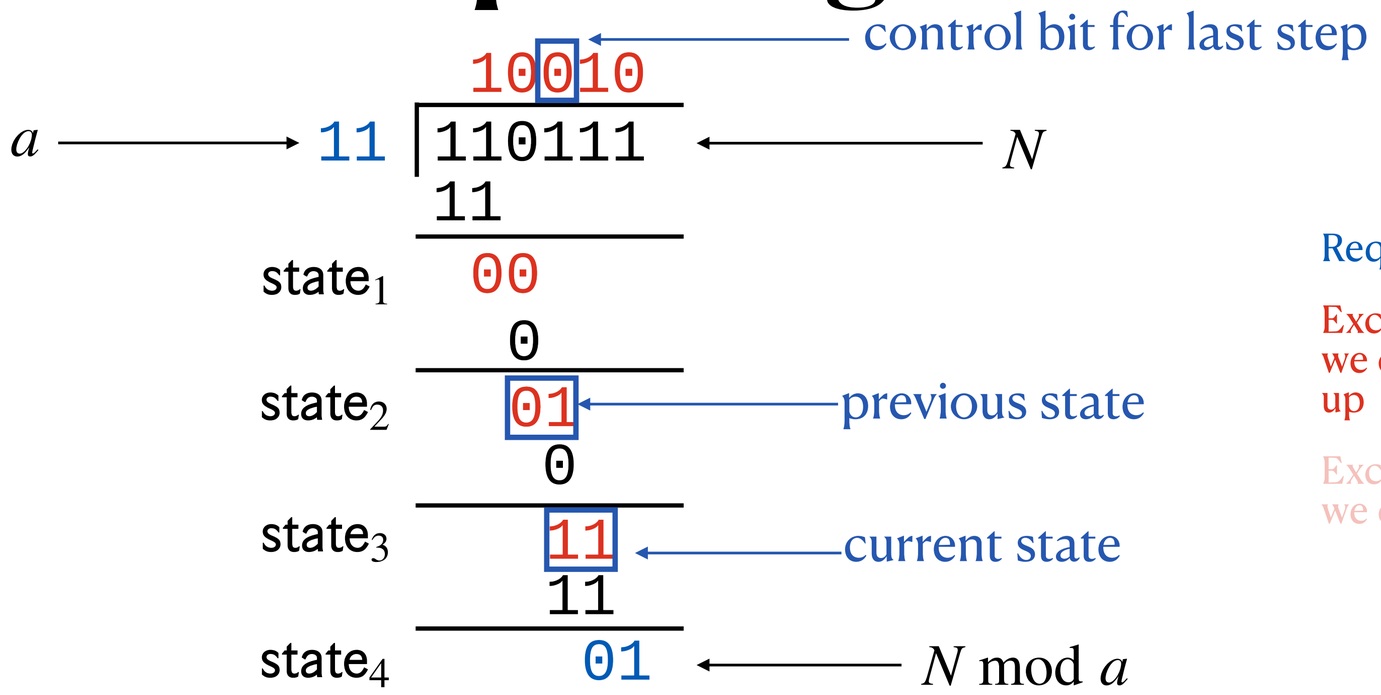
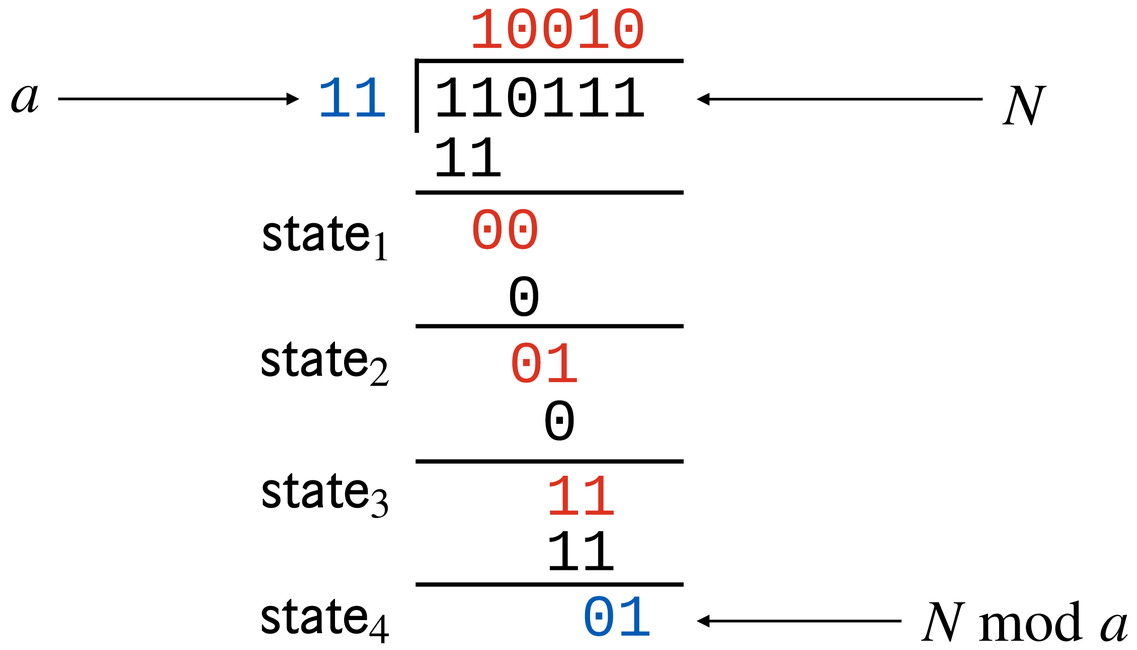


- Pro: only need to look at $O(m)$ bits of N at a time, and each state_t is compact
- Con: no reversibility → end up using $O(n)$ qubits anyway

Long Division: A Bird's Eye View



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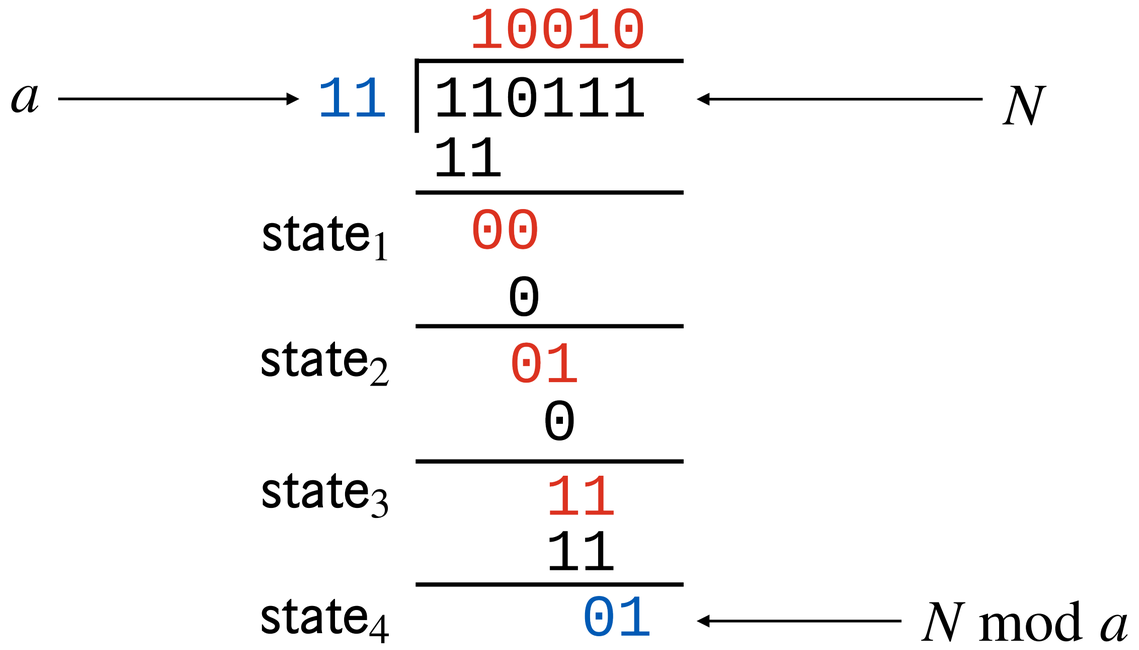

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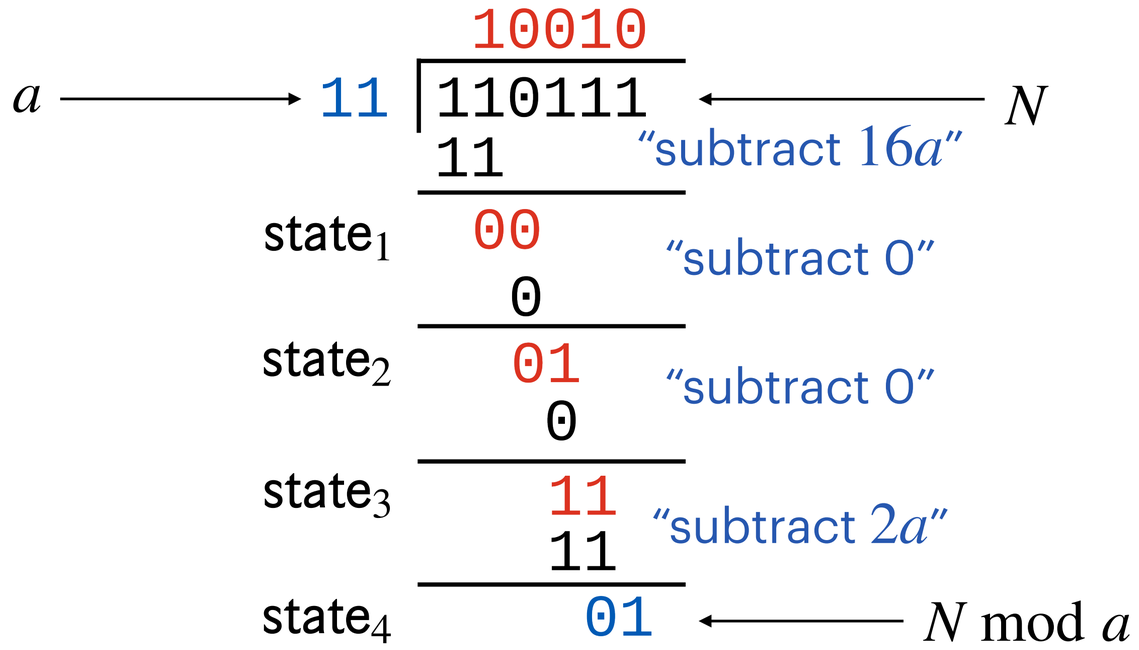

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- Goal of long division: find k such that $N \approx ka$, and output $N - ka$
- Does this by subtracting $2^r a$ from state, starting with large r (most significant bit of k) and going down to $r = 0$ (least significant bit of k)

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    - Equivalently: $N \equiv ka \pmod{2^{n-m}}$

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```
2^0 a =          11
    N =      110111
    k =           1
state =          11
```

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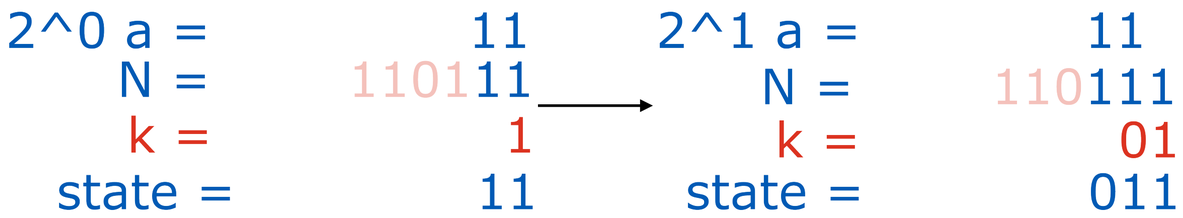

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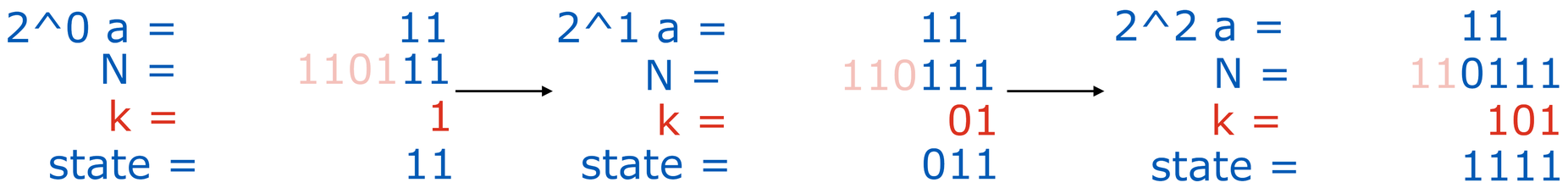

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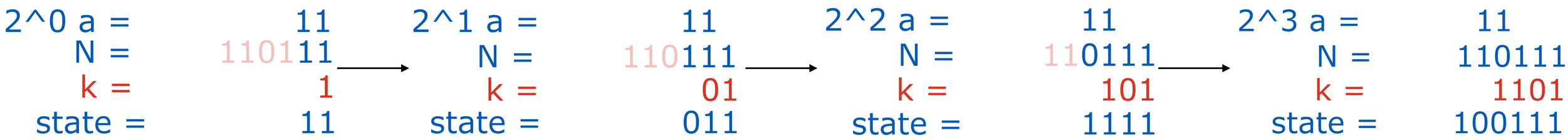

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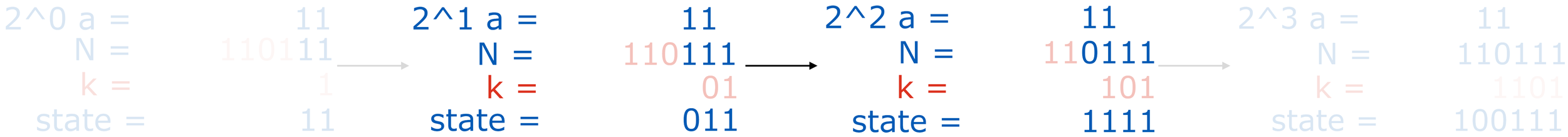

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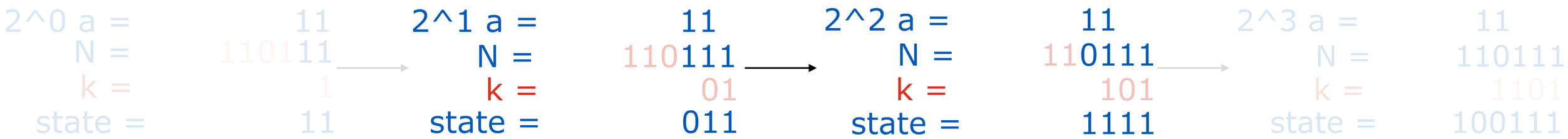

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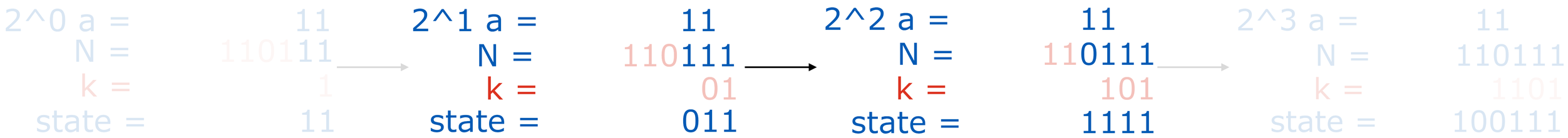

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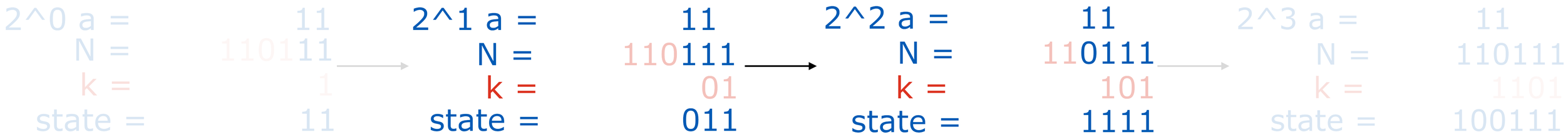

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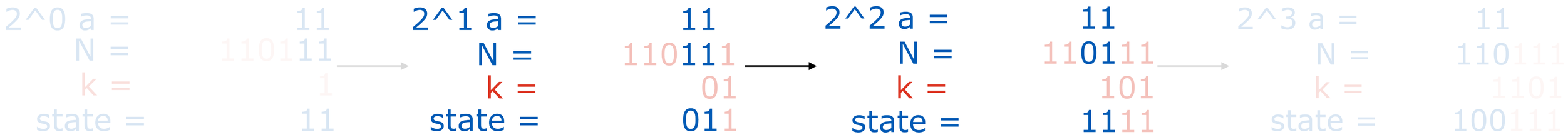

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This now gets us down to $O(m)$ space!

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   a. Recursion step just does some multiplications and bit shifts on m-bit integers

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  - Depth: $O(n/m) \times \tilde{O}(1) = \tilde{O}(n/m)$

From Backwards Long Division to $N \bmod a$

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simplified presentation based on an
observation by Daniel J. Bernstein

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That's it! We computed $N \bmod a$!

simplified presentation based on an observation by Daniel J. Bernstein