Cloning Games, Black Holes, and Cryptography

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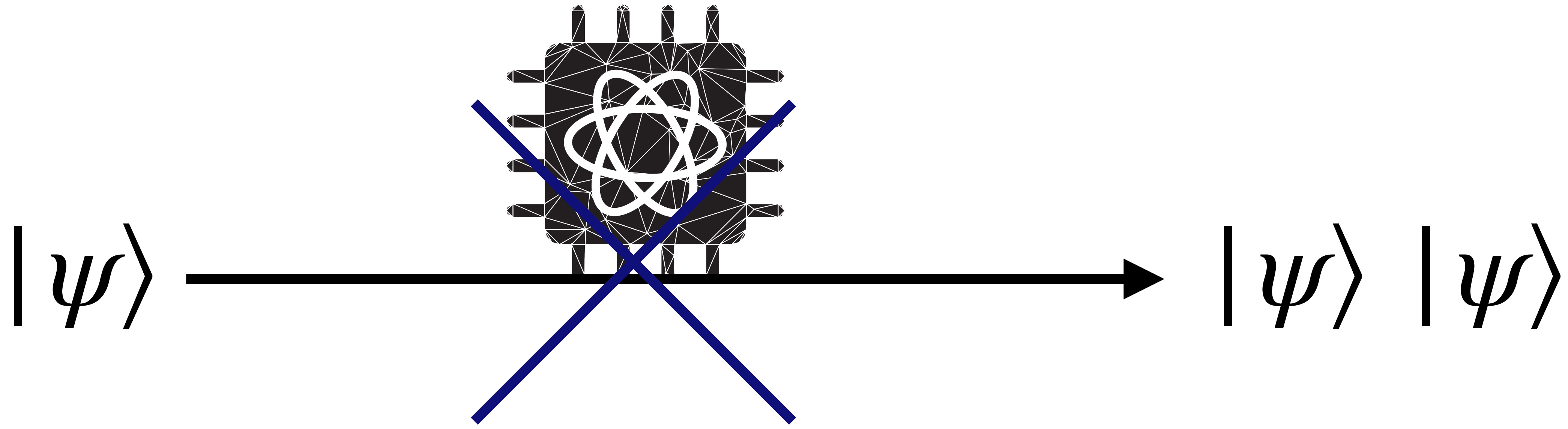
MIT and Boston University



The No-Cloning Theorem

Wootters-Zurek '82

- Informally: quantum information cannot be copied and pasted



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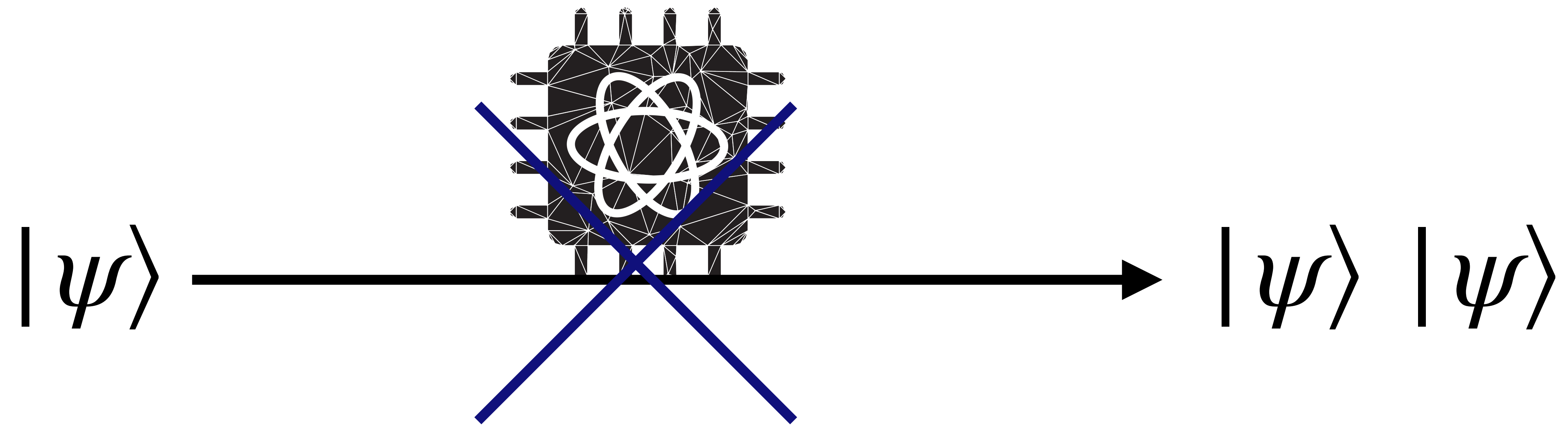

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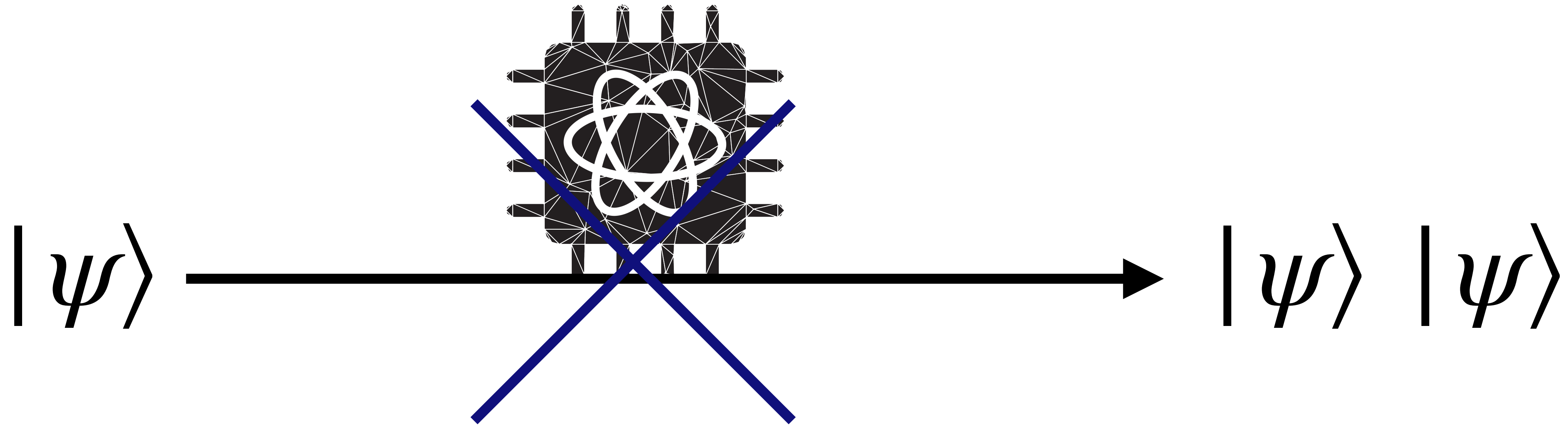

- Robust variant (Werner '98): it is not even possible to generate something close to $|\psi\rangle|\psi\rangle$ (in trace distance)
- Provides hope for **unclonable cryptography:** applications where we want to safeguard against duplication of information
    - However, it is rarely sufficient…

Goal: "Useful" No-Cloning Theorems

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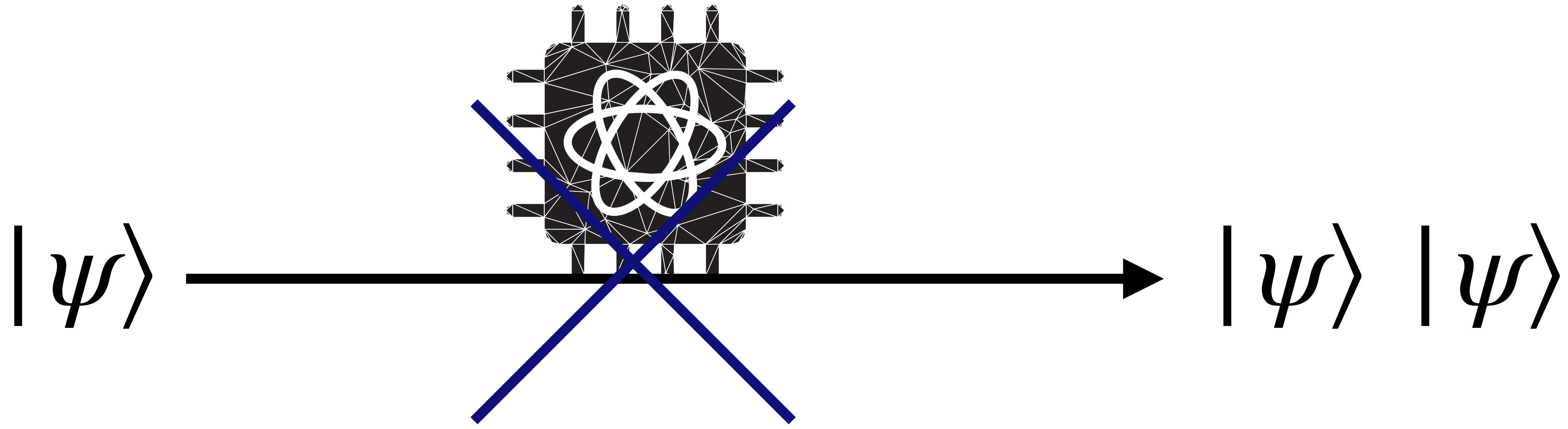

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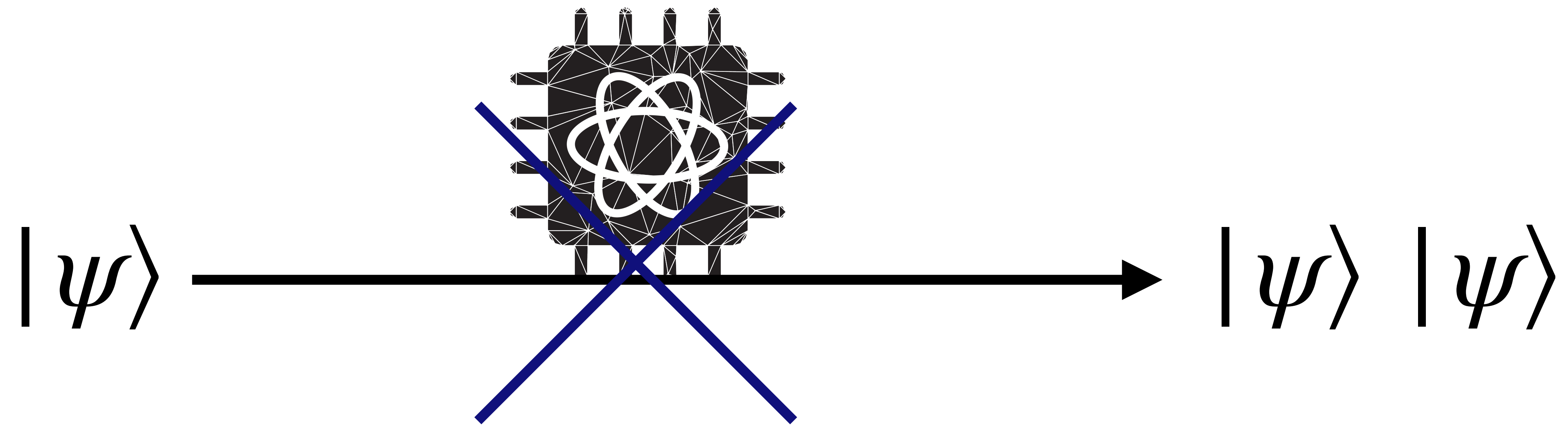

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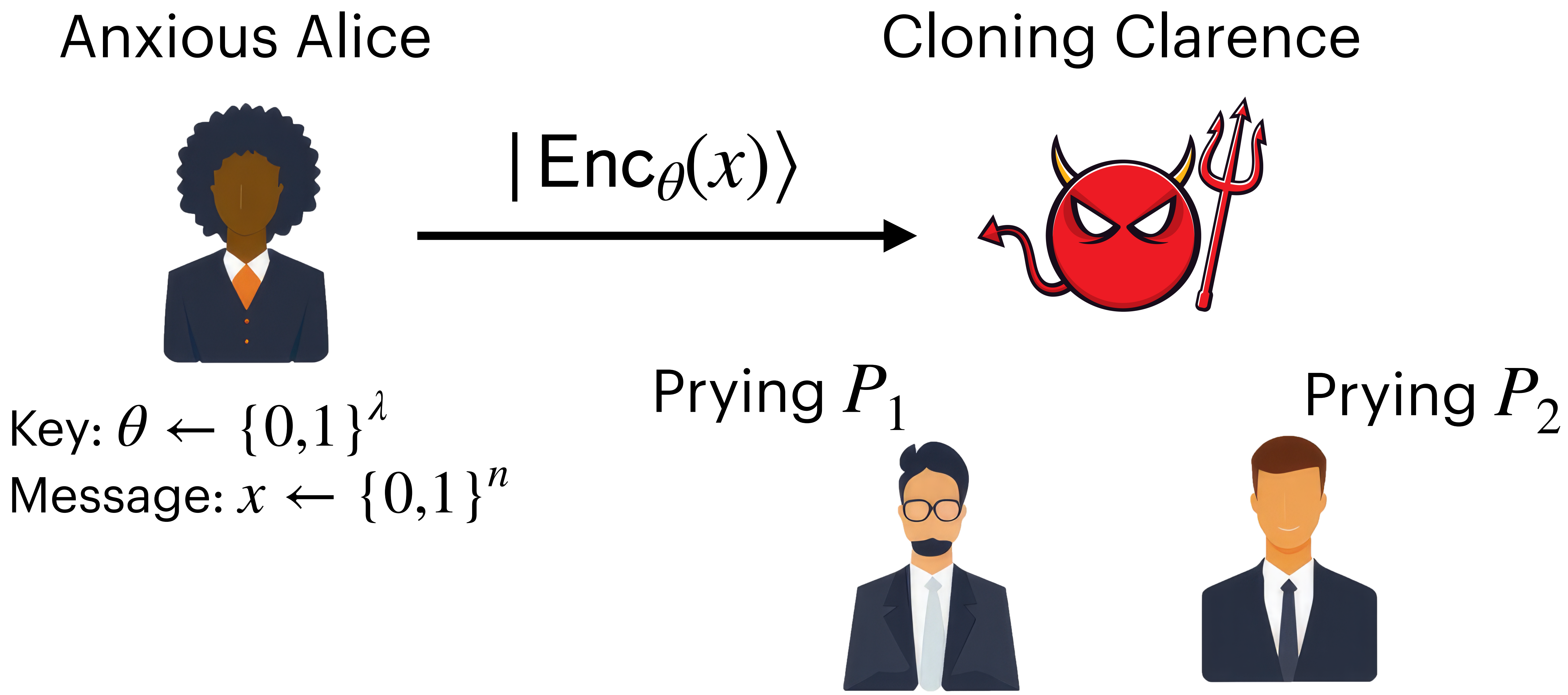
- **"Useful" variant (even stronger than robust):** it is not even possible for both states to have "some property" in common with $|\psi\rangle$
  - "Some property" depends on the particular application

(Search-Secure) Unclonable Encryption

Broadbent and Lord ‘20



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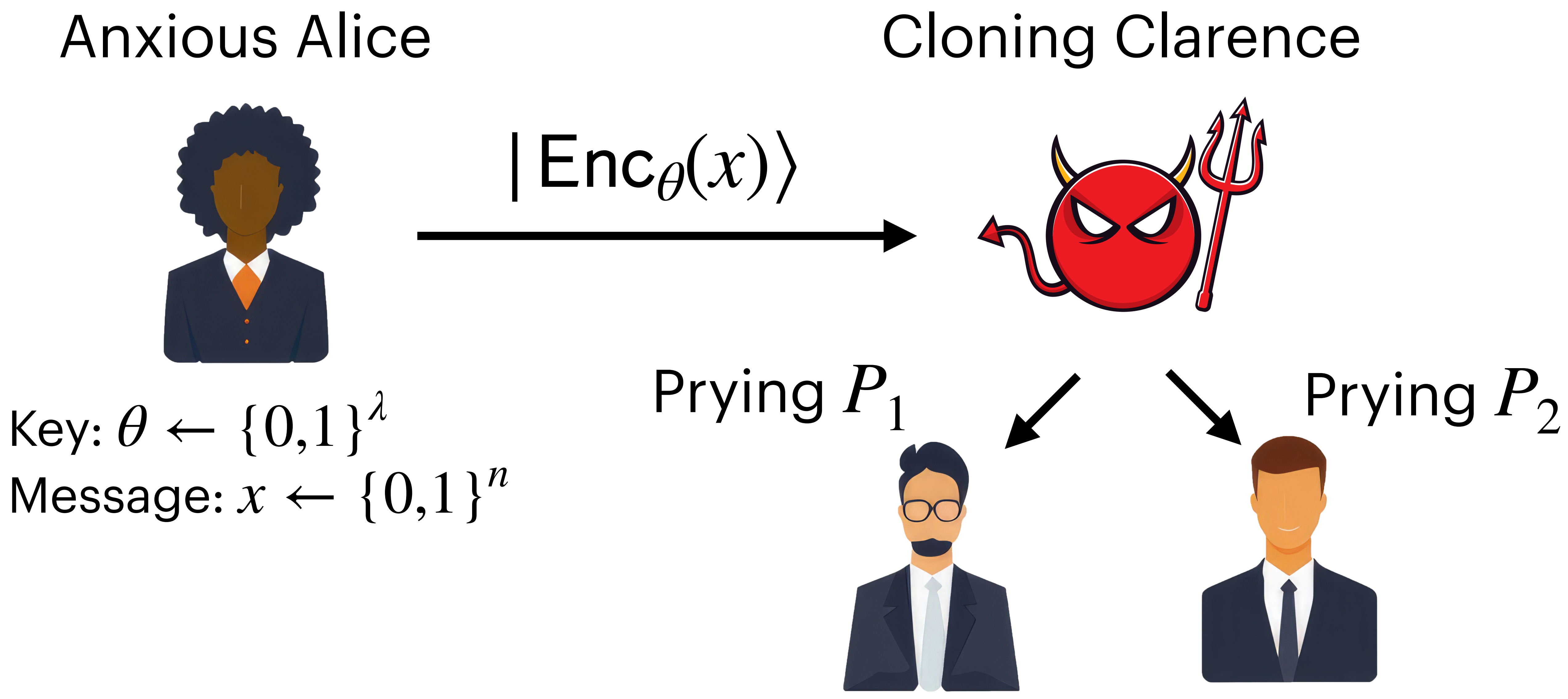

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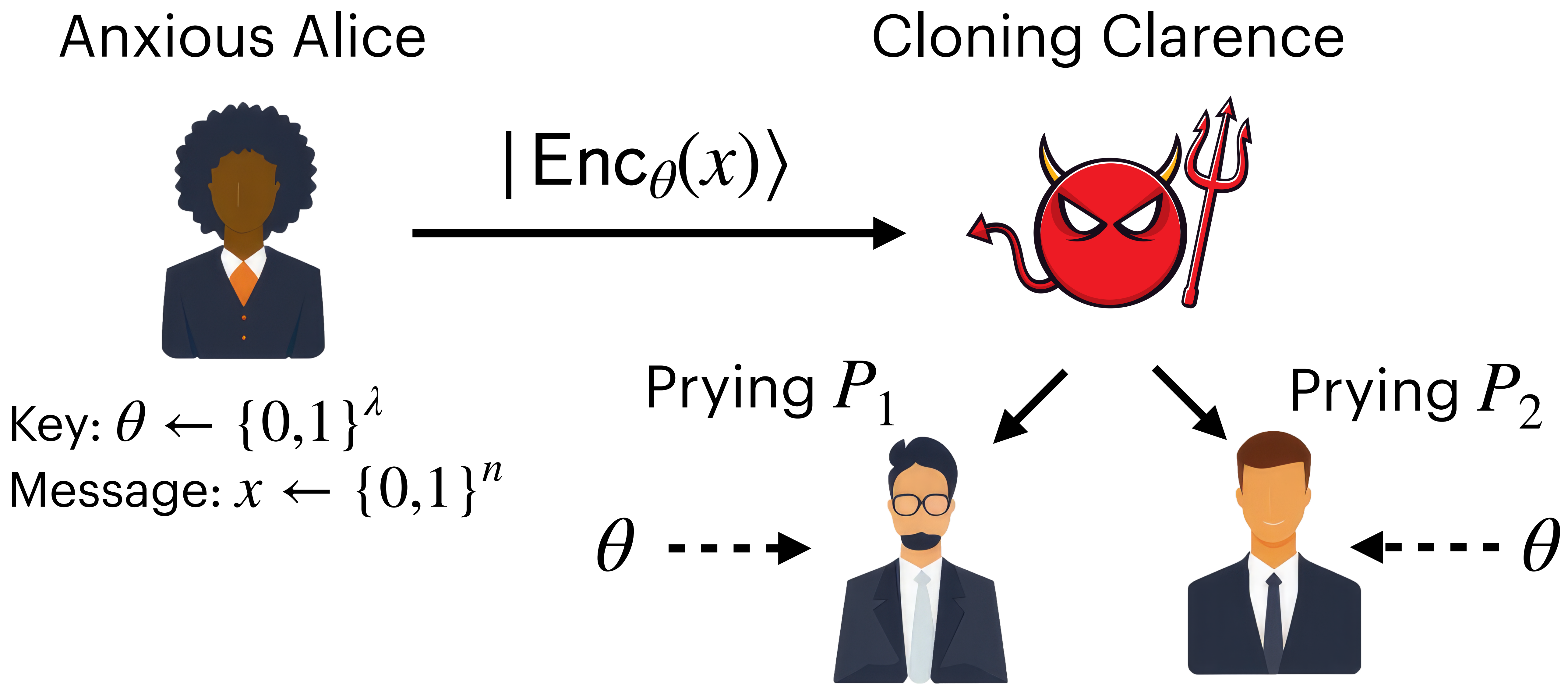

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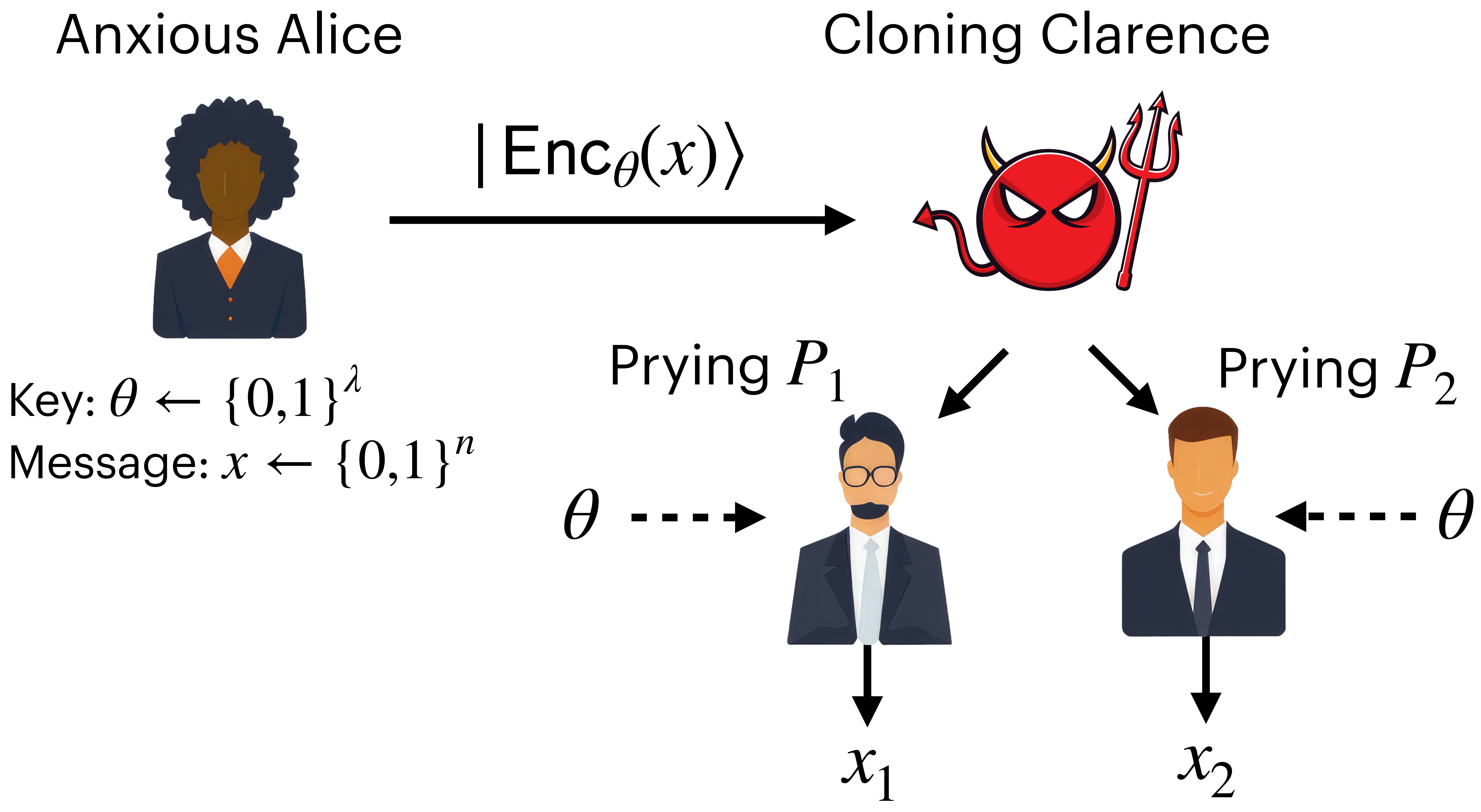

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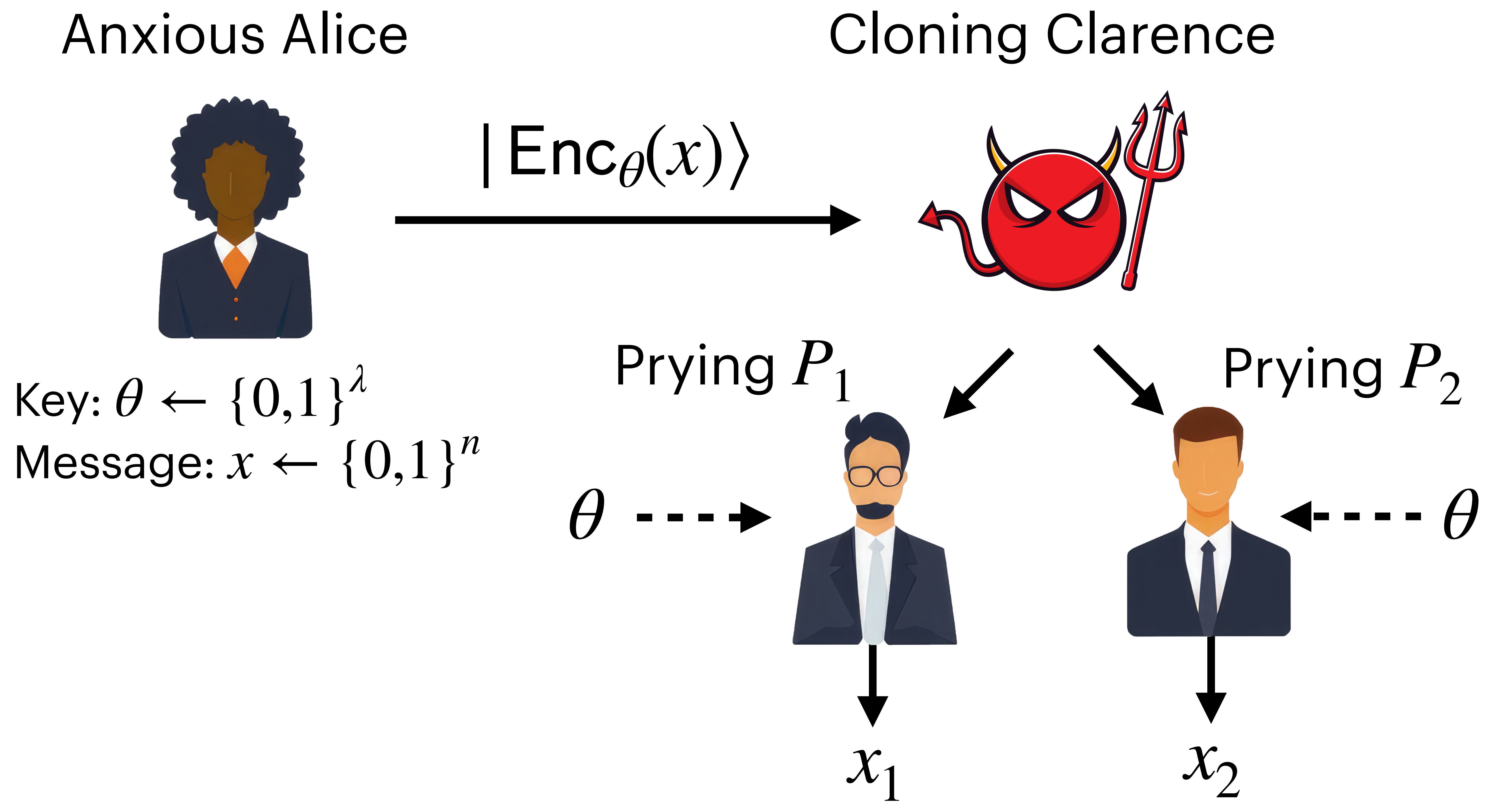
4. They guess x_1, x_2 and win if both guesses are correct ($x_1 = x_2 = x$)

(Search-Secure) Unclonable Encryption

The Need for <u>Useful</u> No-Cloning



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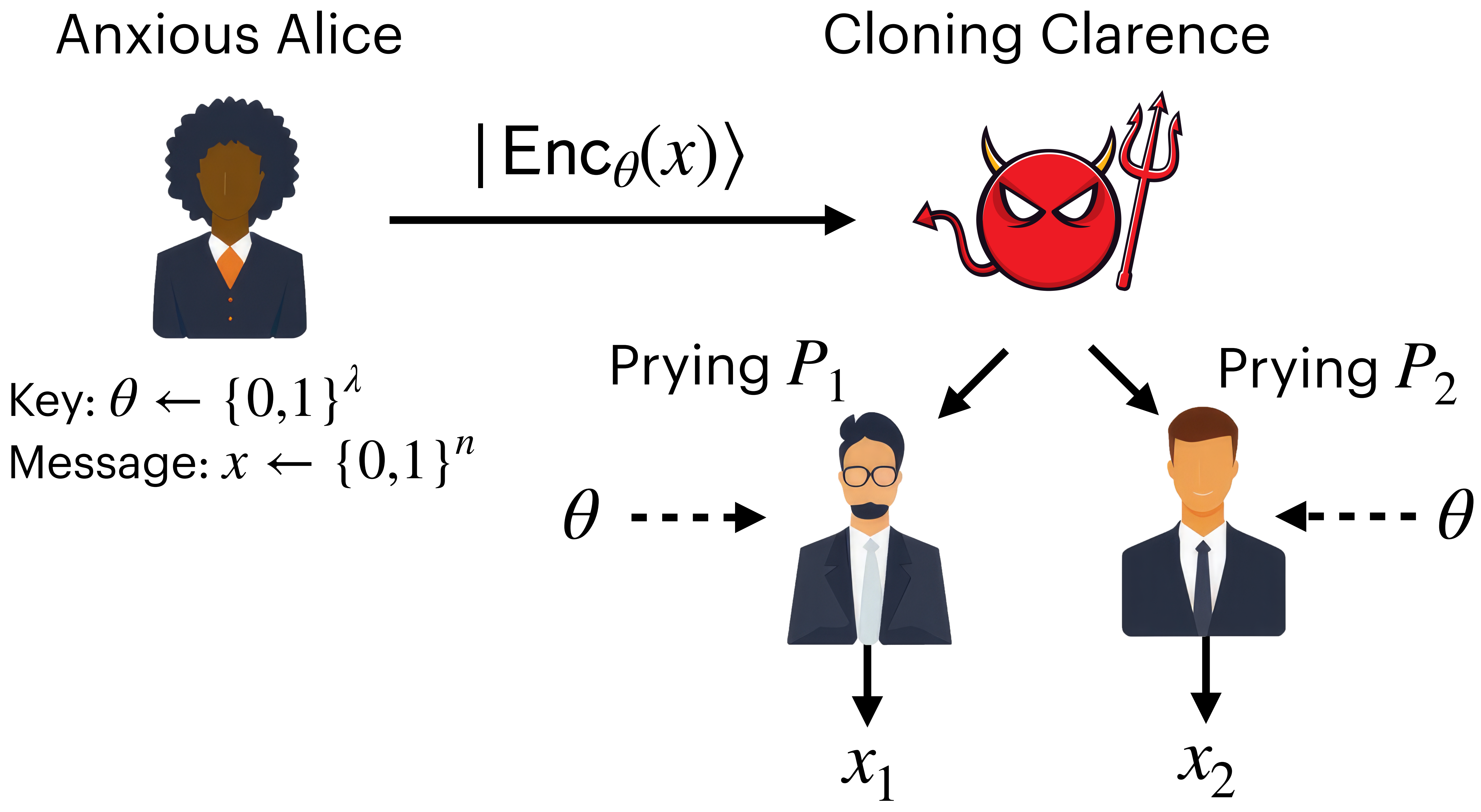
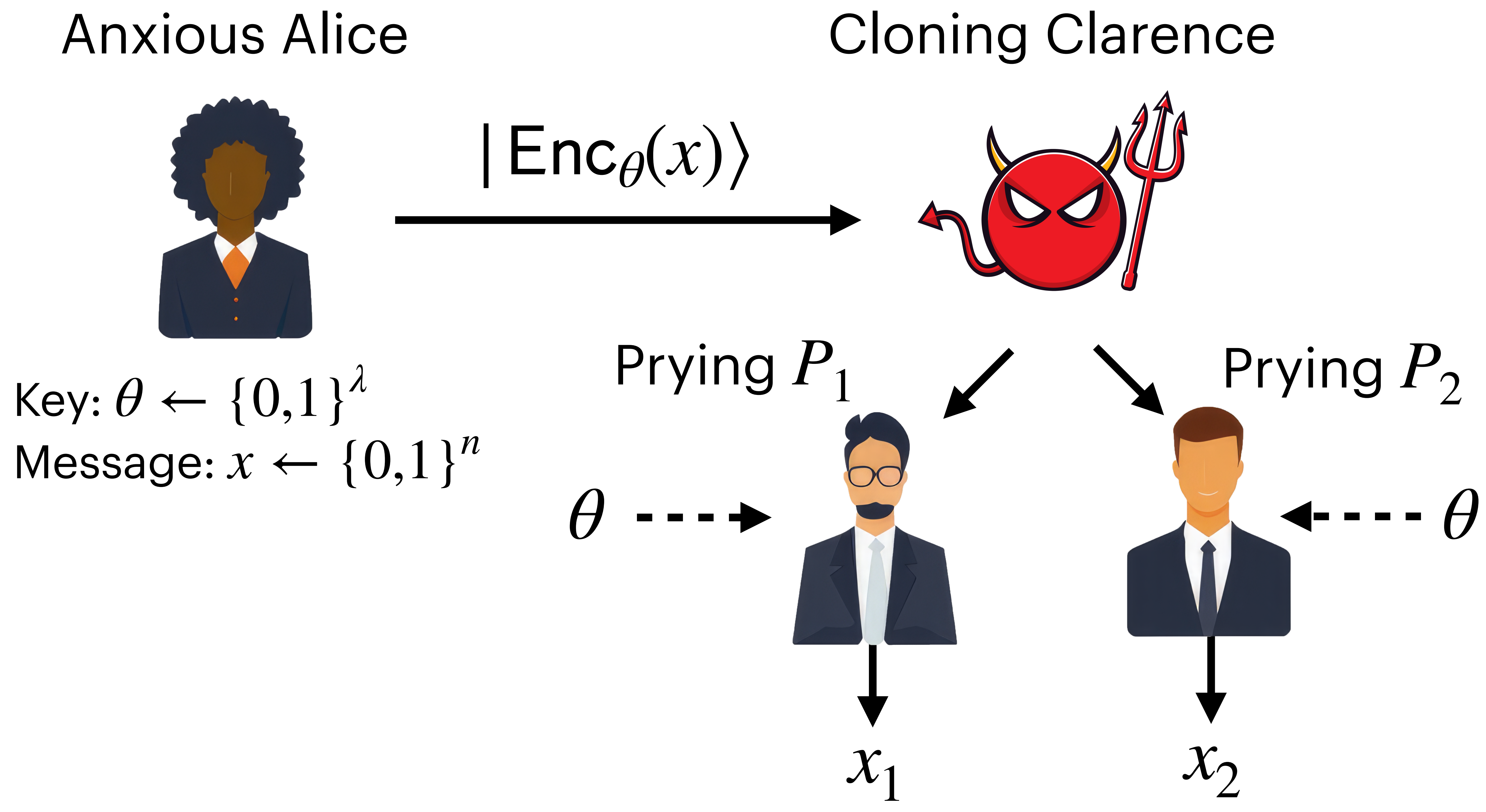


- If $|\mathsf{Enc}_\theta(x)\rangle$ is classical (or clonable): Clarence can just clone it and send copies to P_1, P_2!
- Useful no-cloning: Clarence should not be able to produce *any* two states that reveal x (given θ)

Definitions and Our Results

Previous Slides: Search Security



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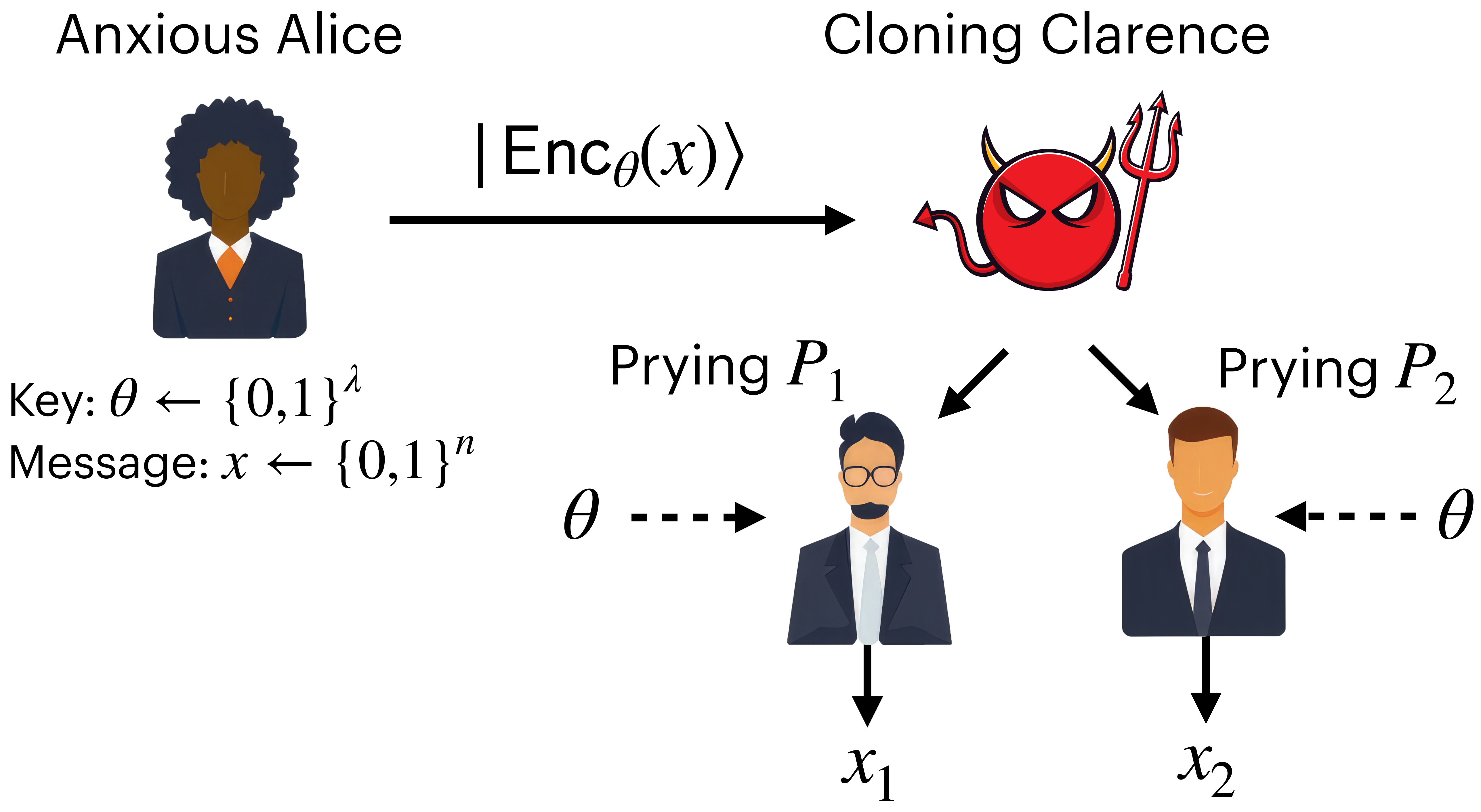

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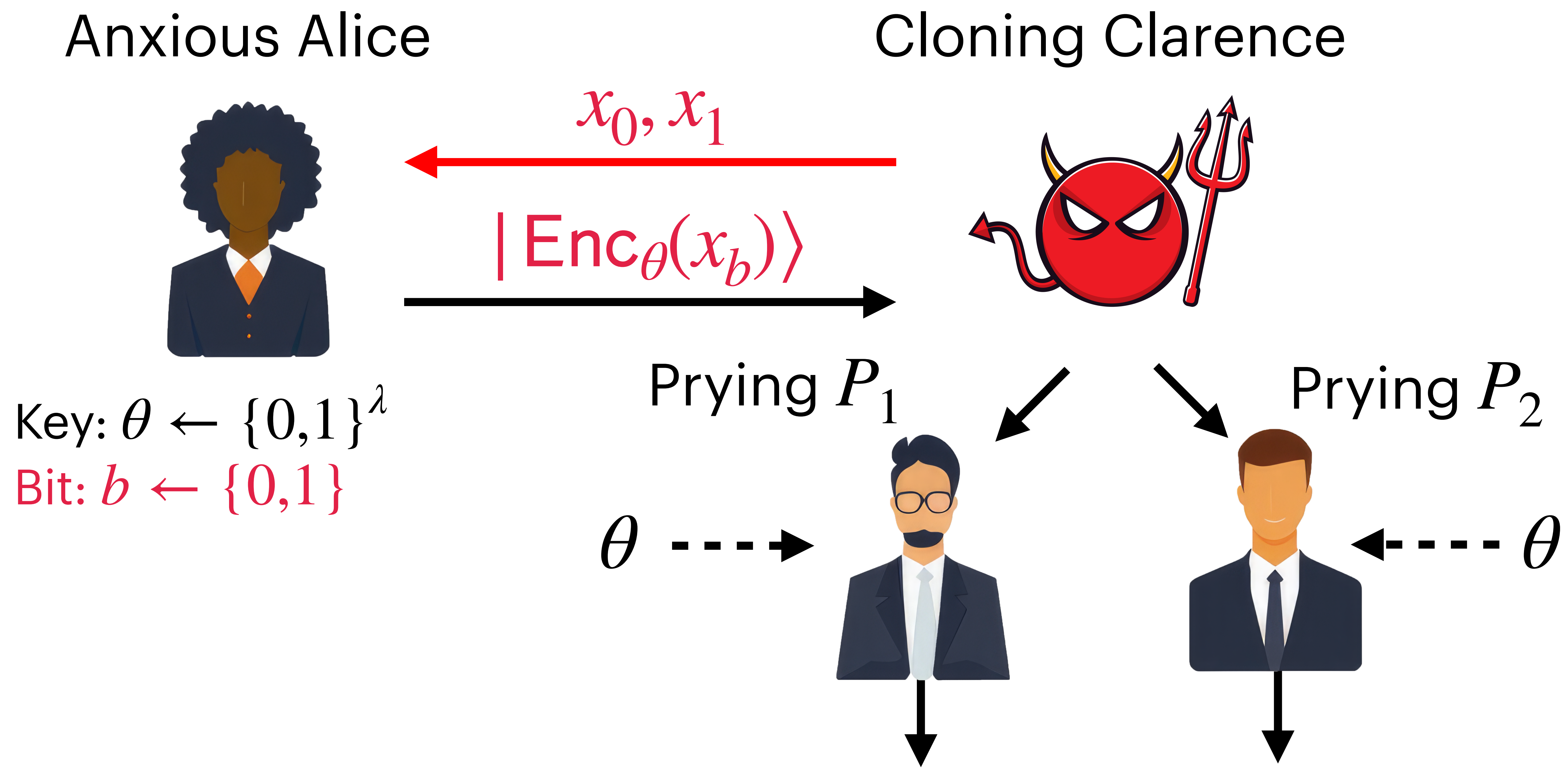
The Ideal Notion: IND-CPA Security



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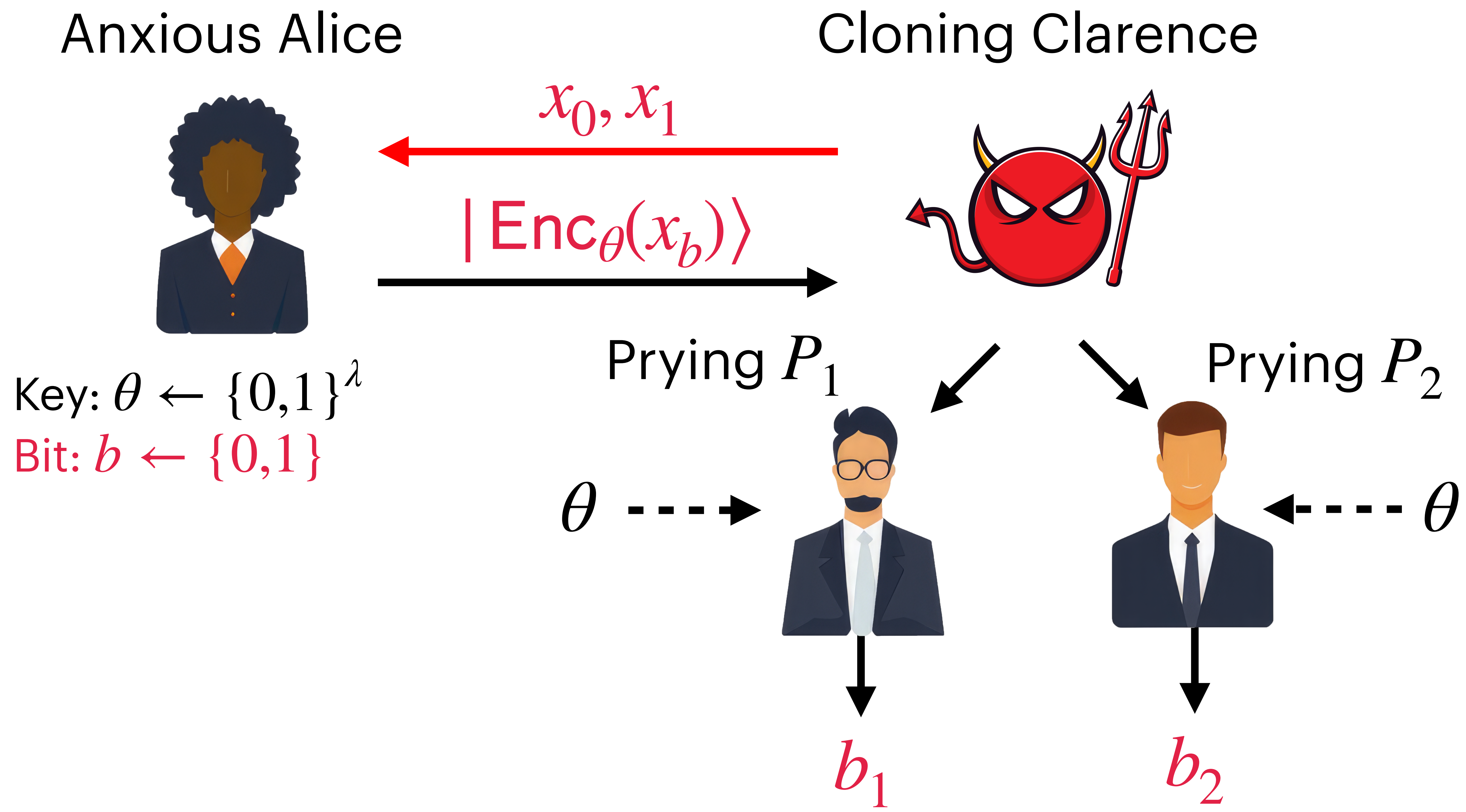
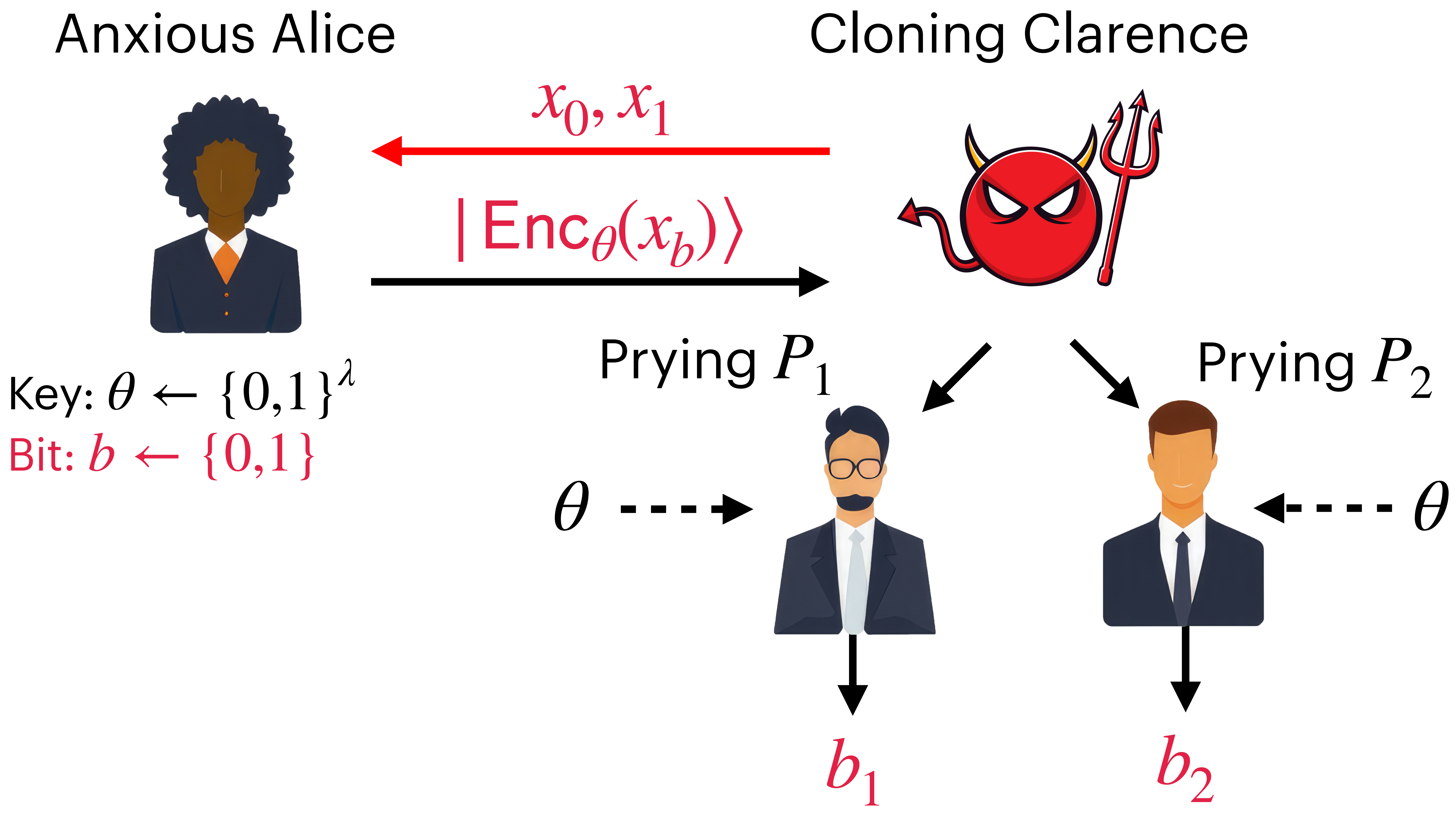


1. Clarence sends two challenge messages $x_0, x_1 \in \{0,1\}^n$ to Alice
2. Alice encrypts one of them at random and sends it to Clarence
3. At the end, P_1, P_2 guess which of the two messages was encrypted
4. IND-CPA security: want

$$\Pr_{\theta \leftarrow \{0,1\}^\lambda, b \leftarrow \{0,1\}} \left[b_1 = b_2 = b \right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

Plausible Bootstrapping from Search to IND Security

Broadbent and Lord '20



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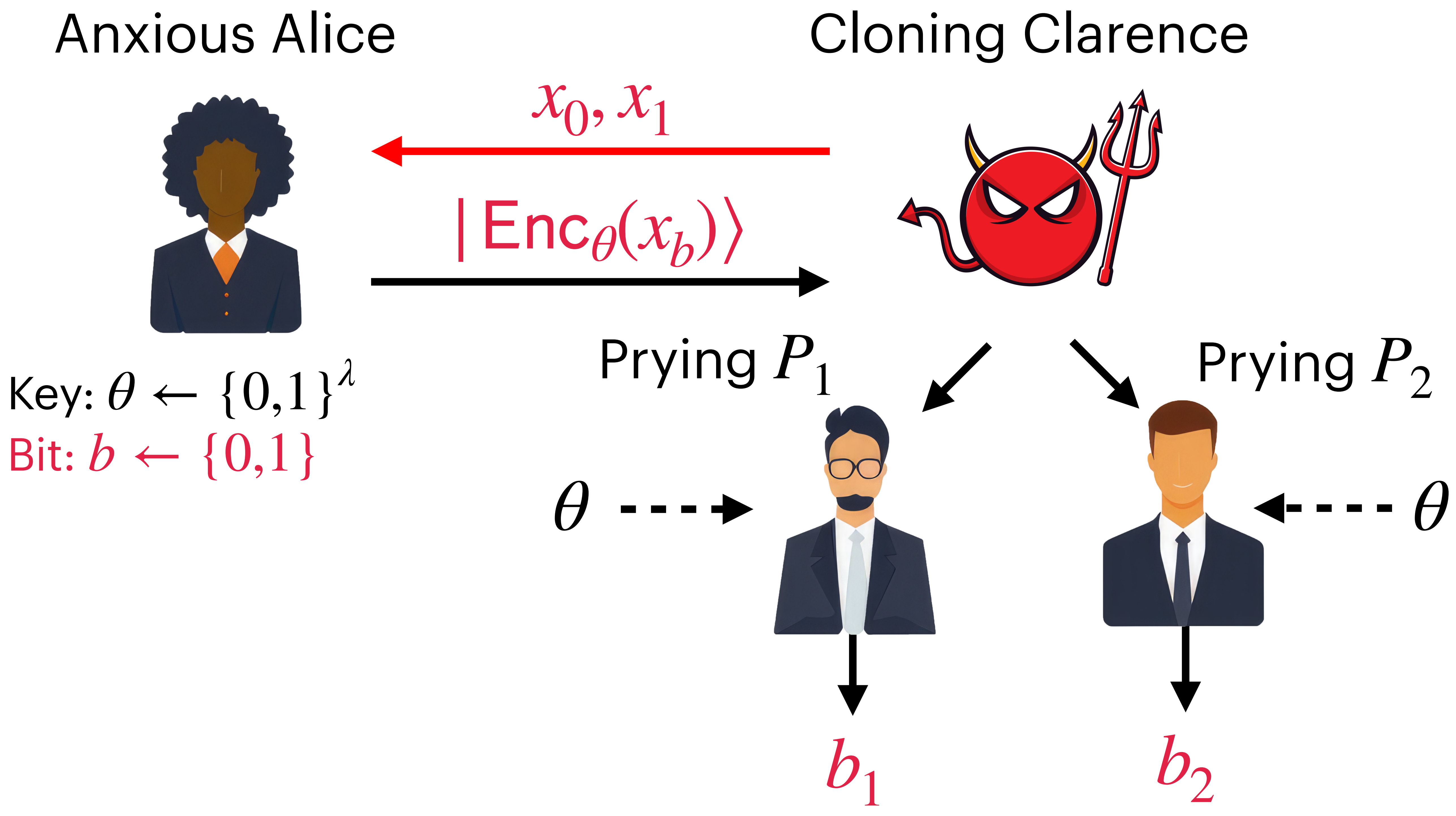

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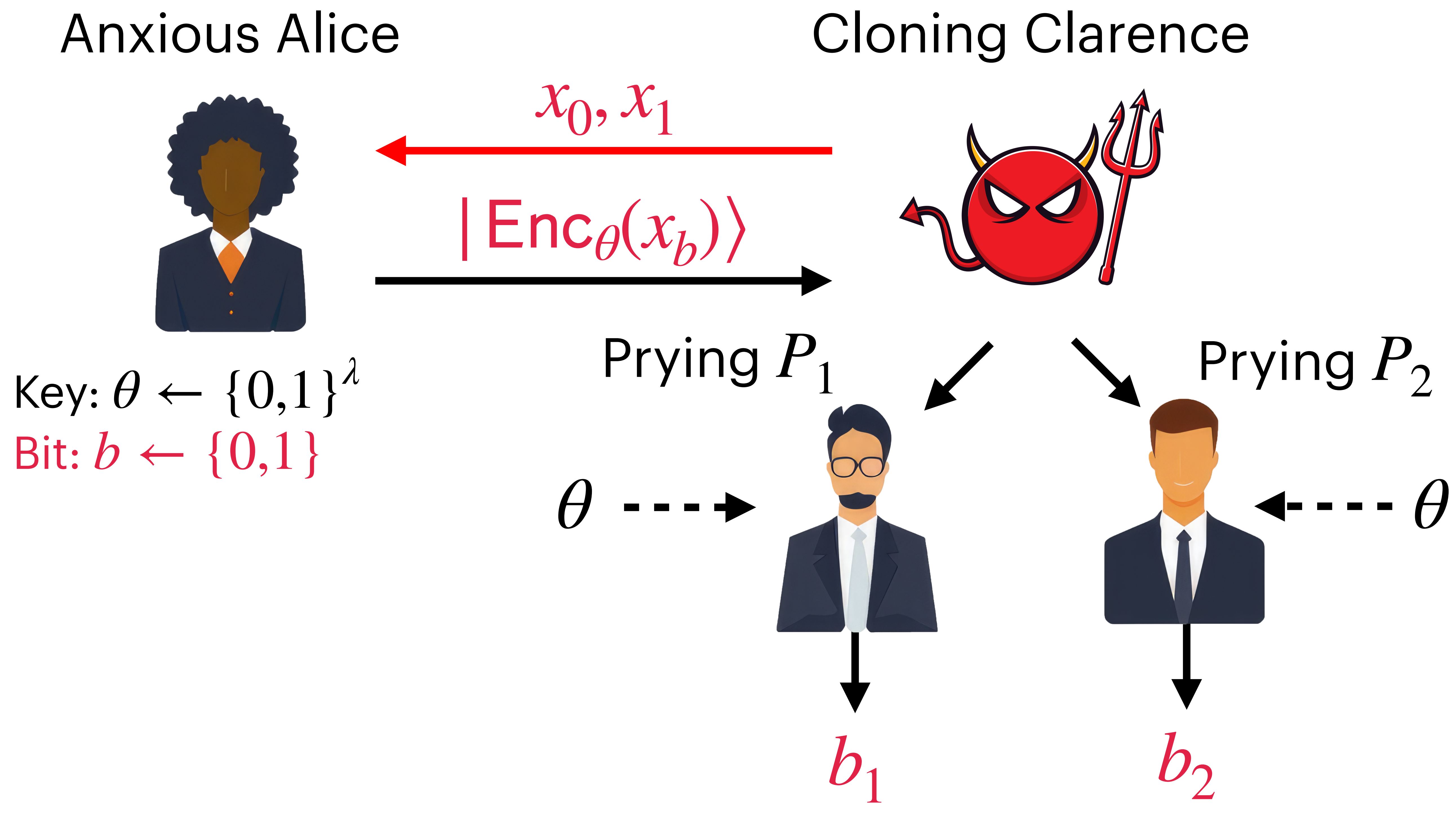

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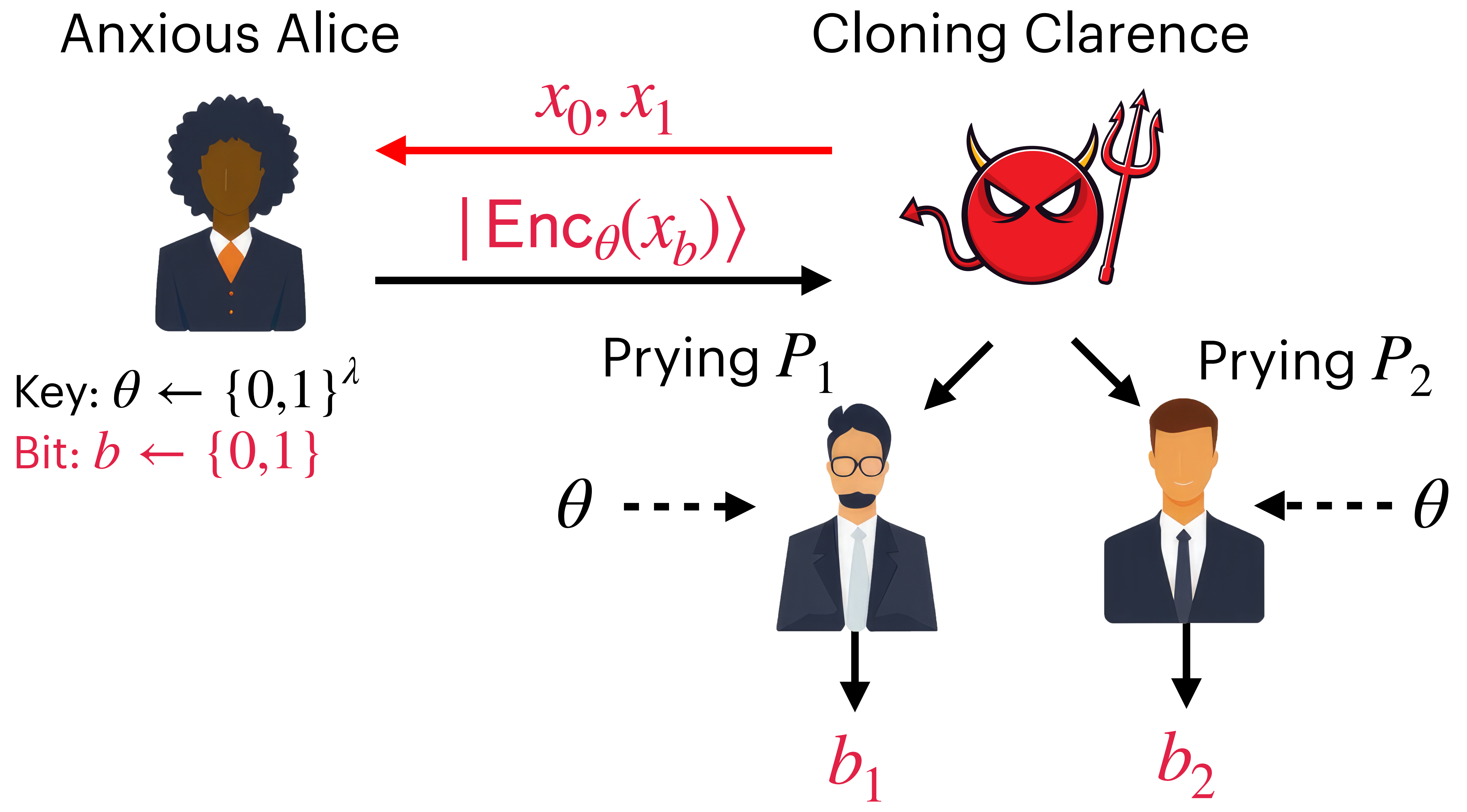

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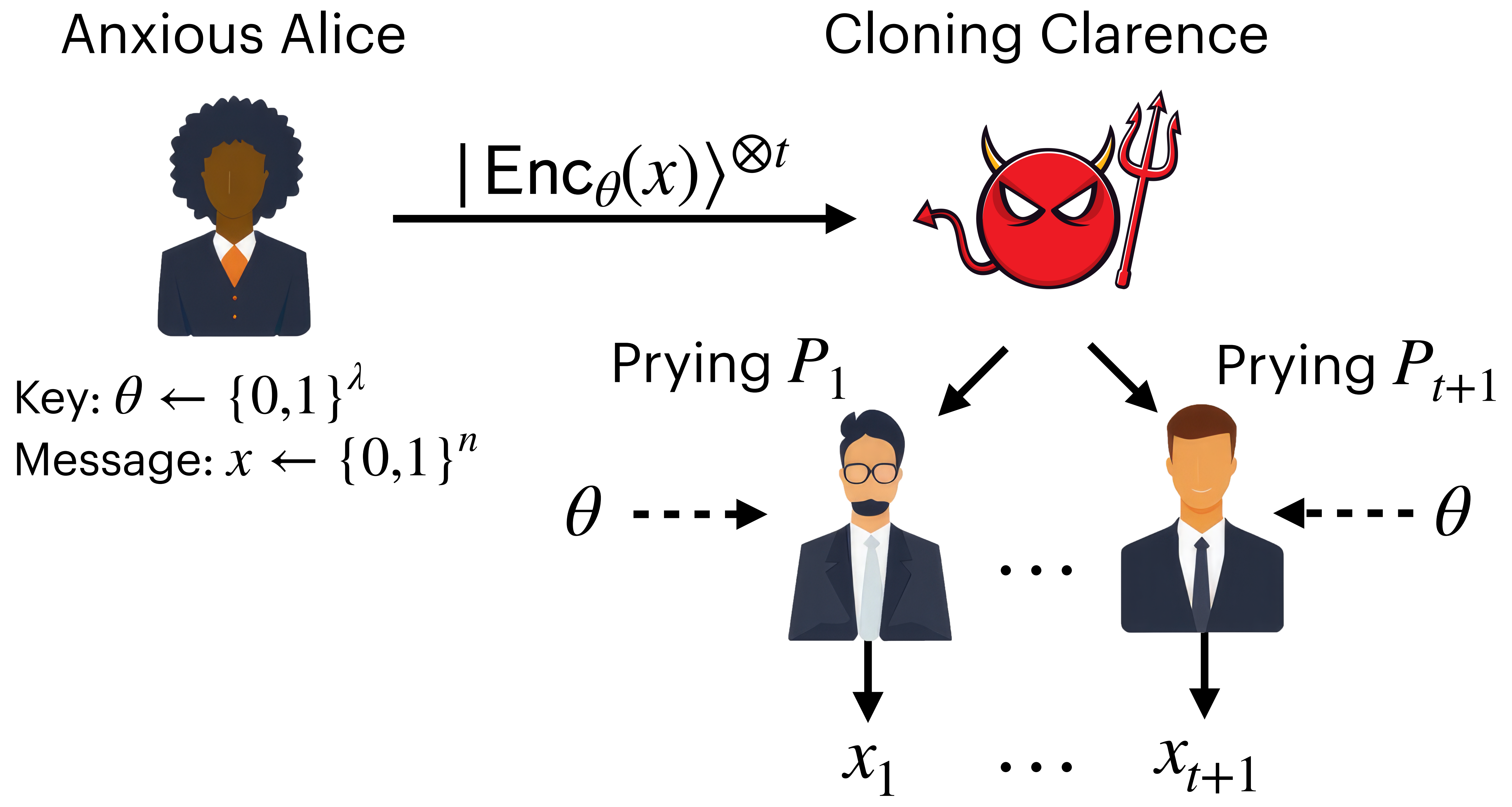
Actually proving IND-security is a notoriously difficult open problem!

(Need Goldreich-Levin-style argument where P_1, P_2 both get the <u>same</u> challenge r)

This Talk: Multi-Copy Search Security



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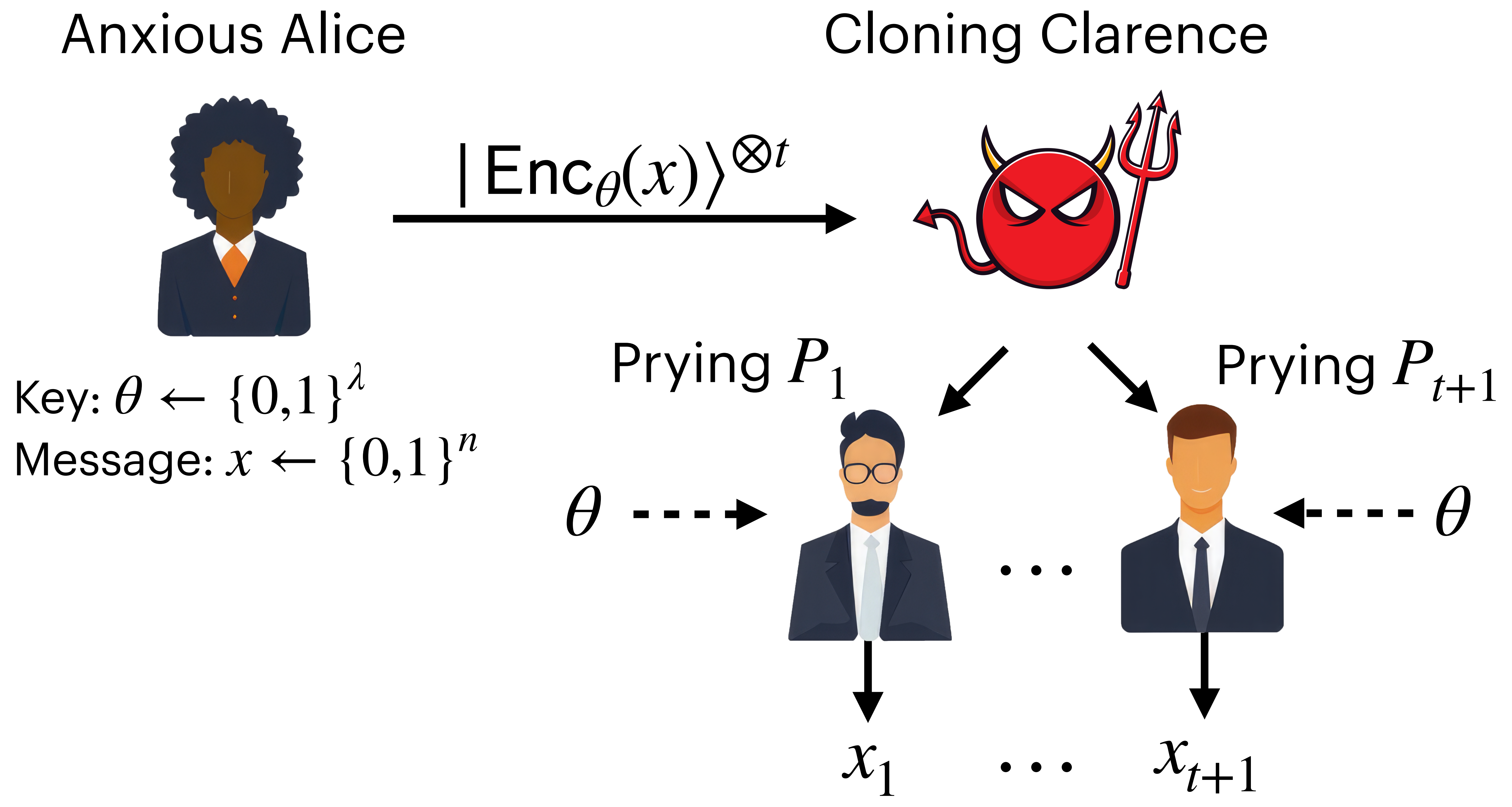

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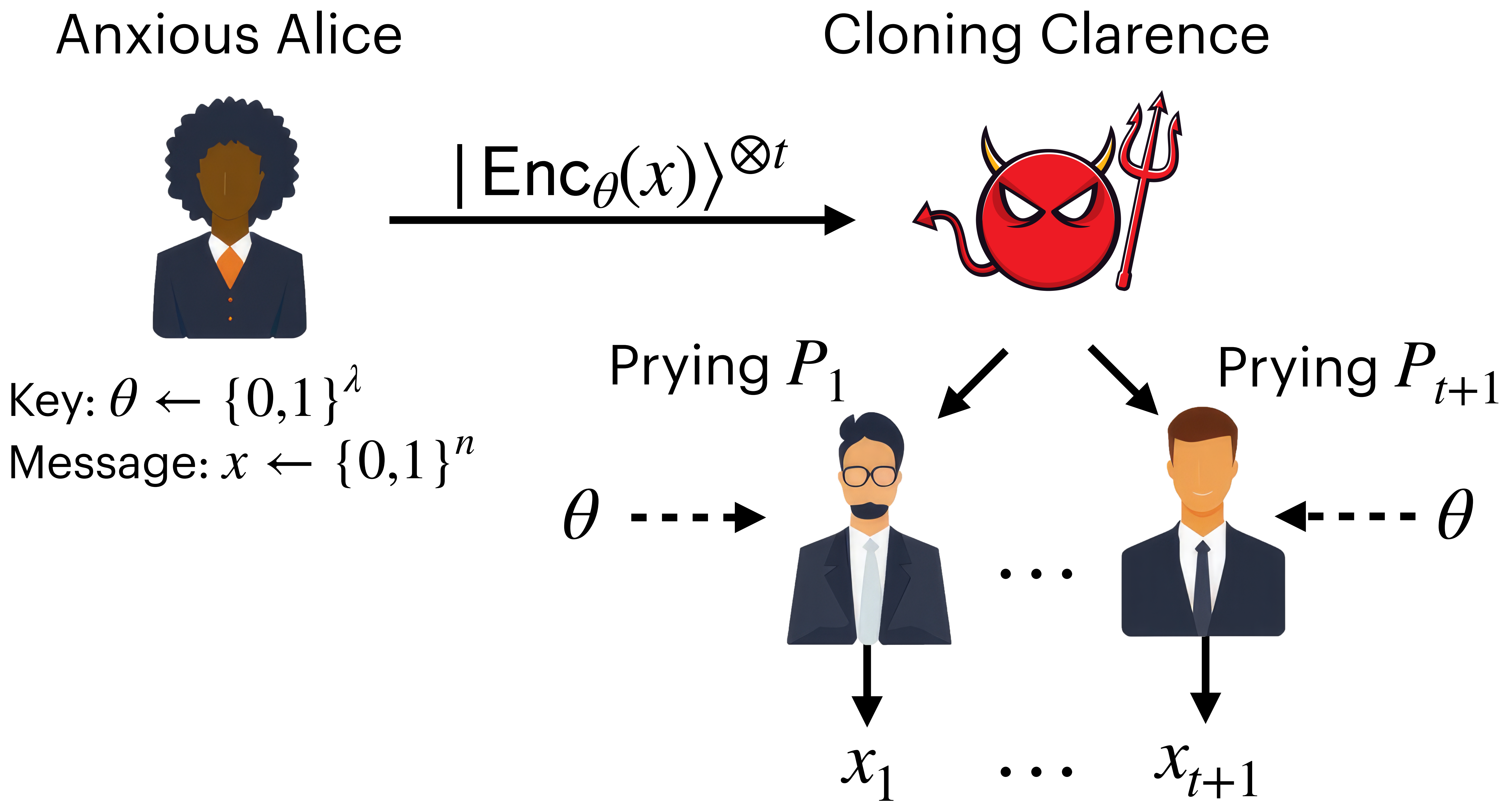

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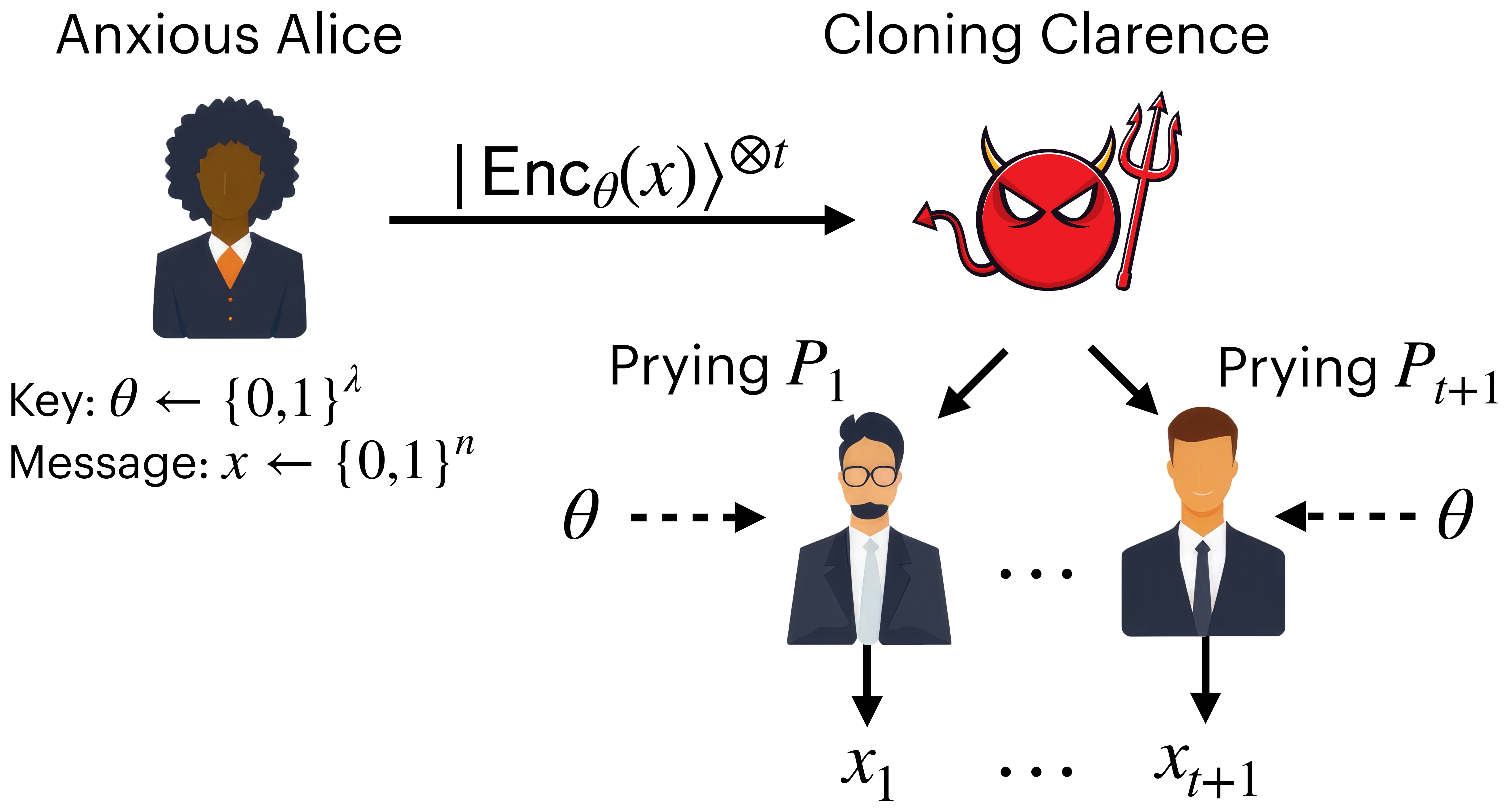

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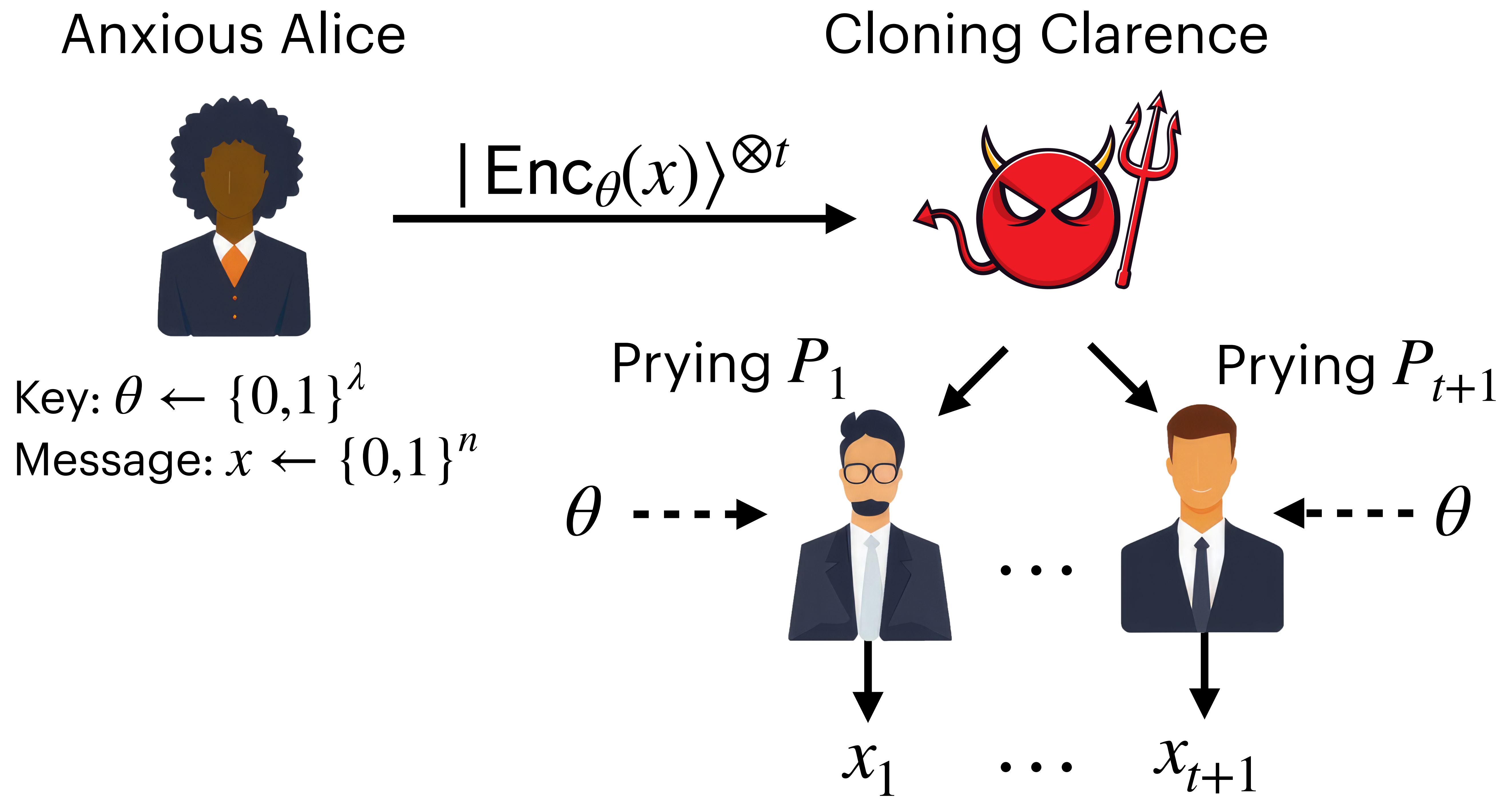

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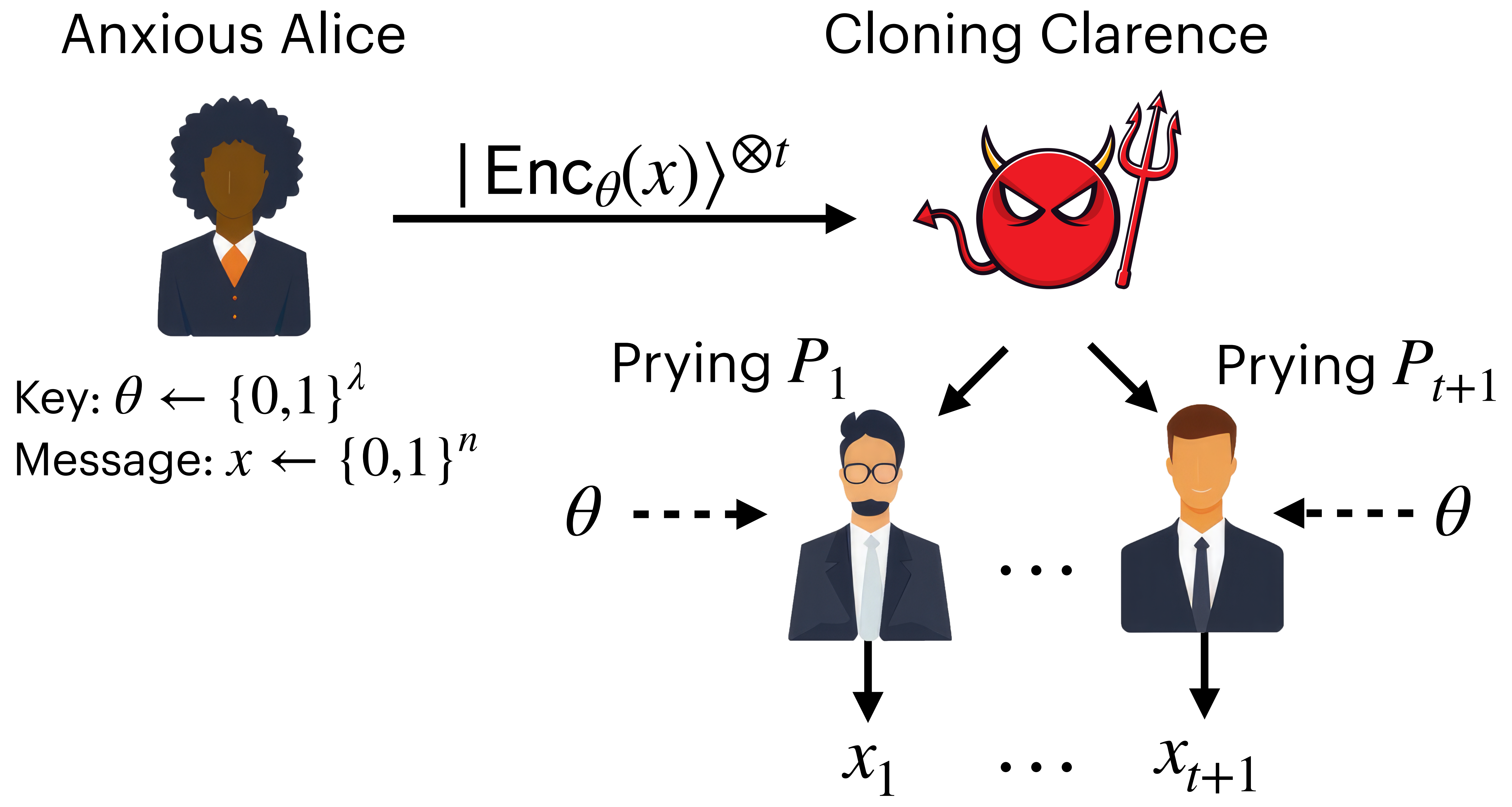

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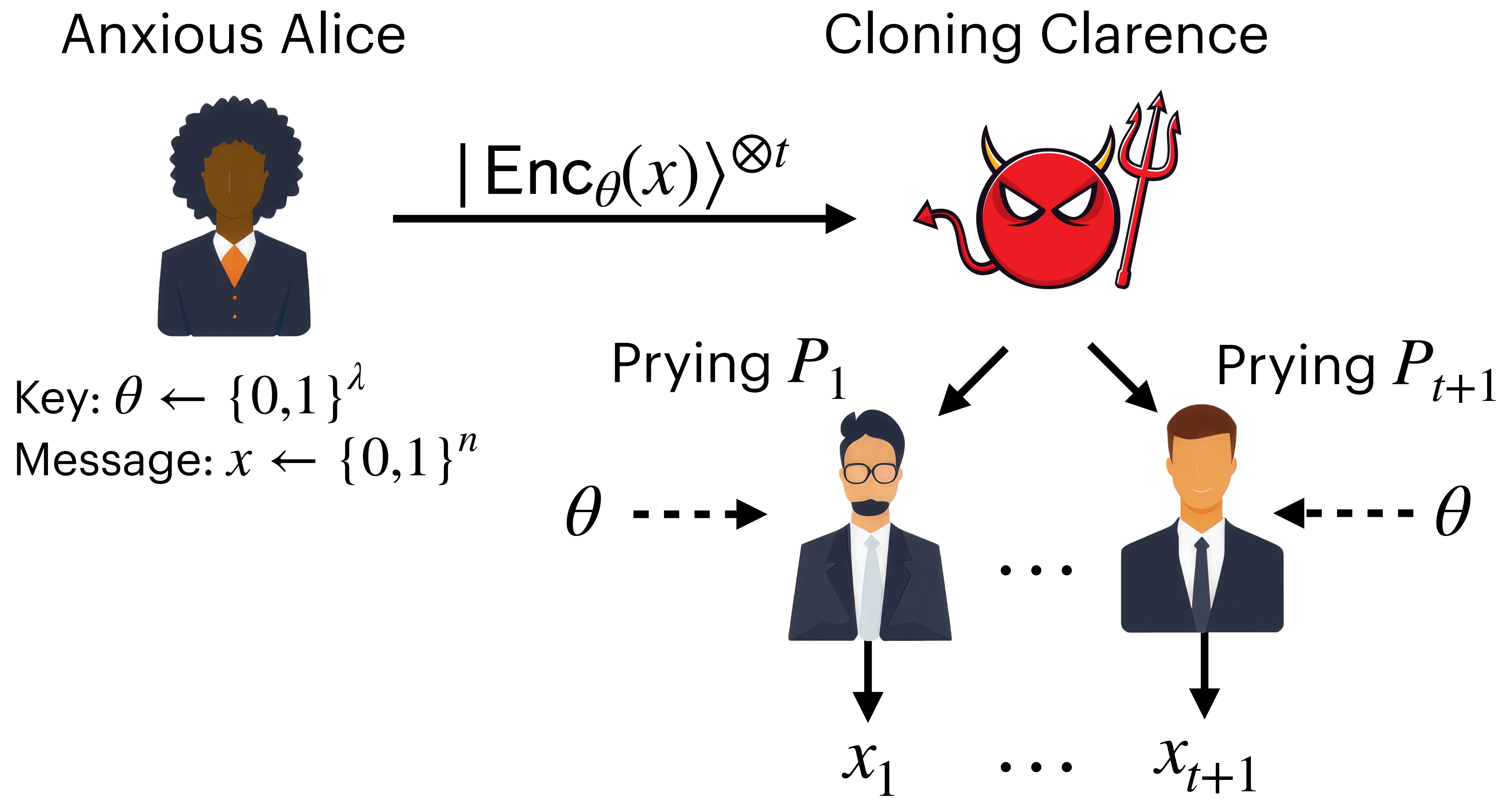

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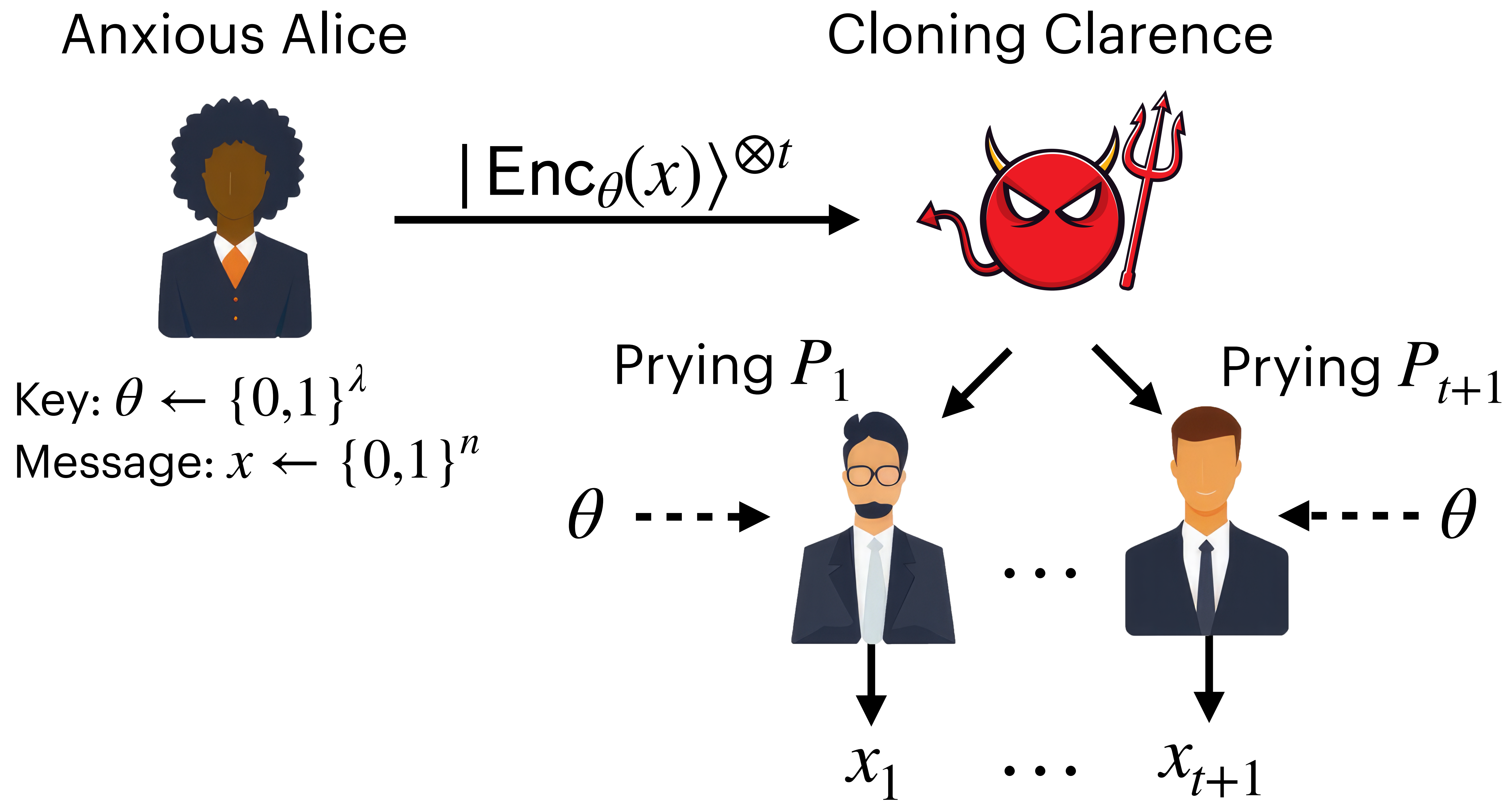

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Ideal goal: a construction which doesn't admit any strategies achieving better than 2^{-n}

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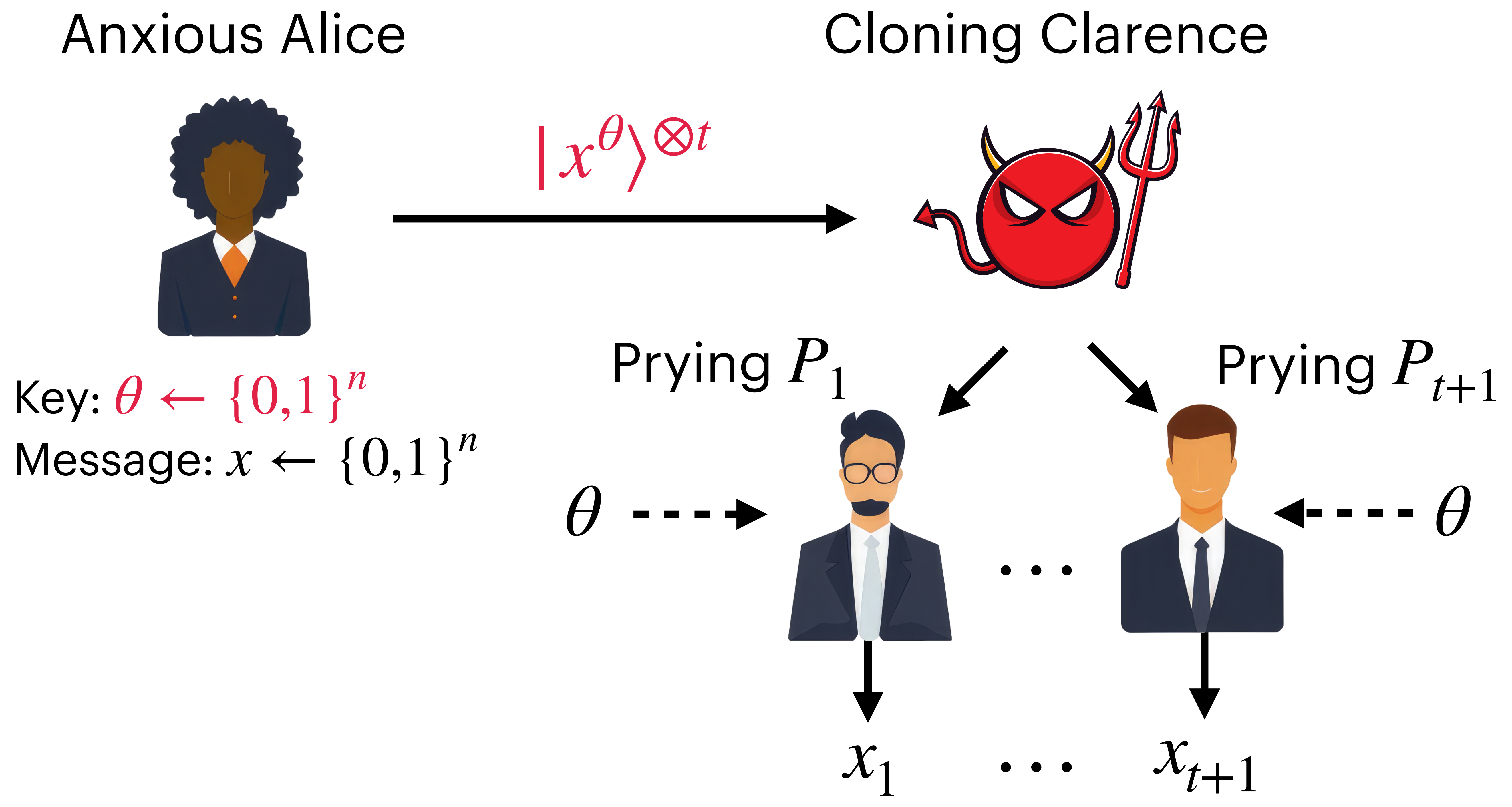

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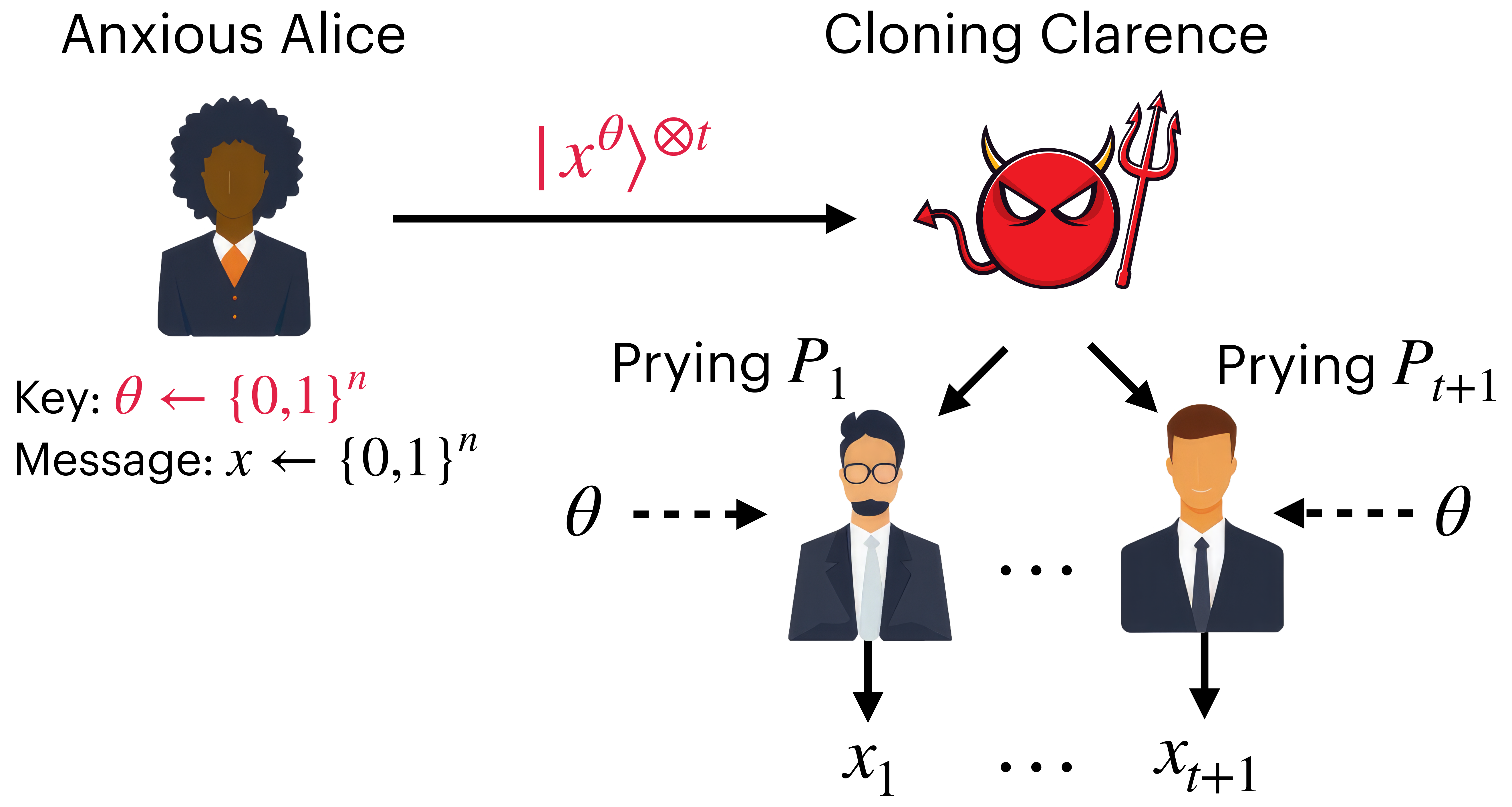

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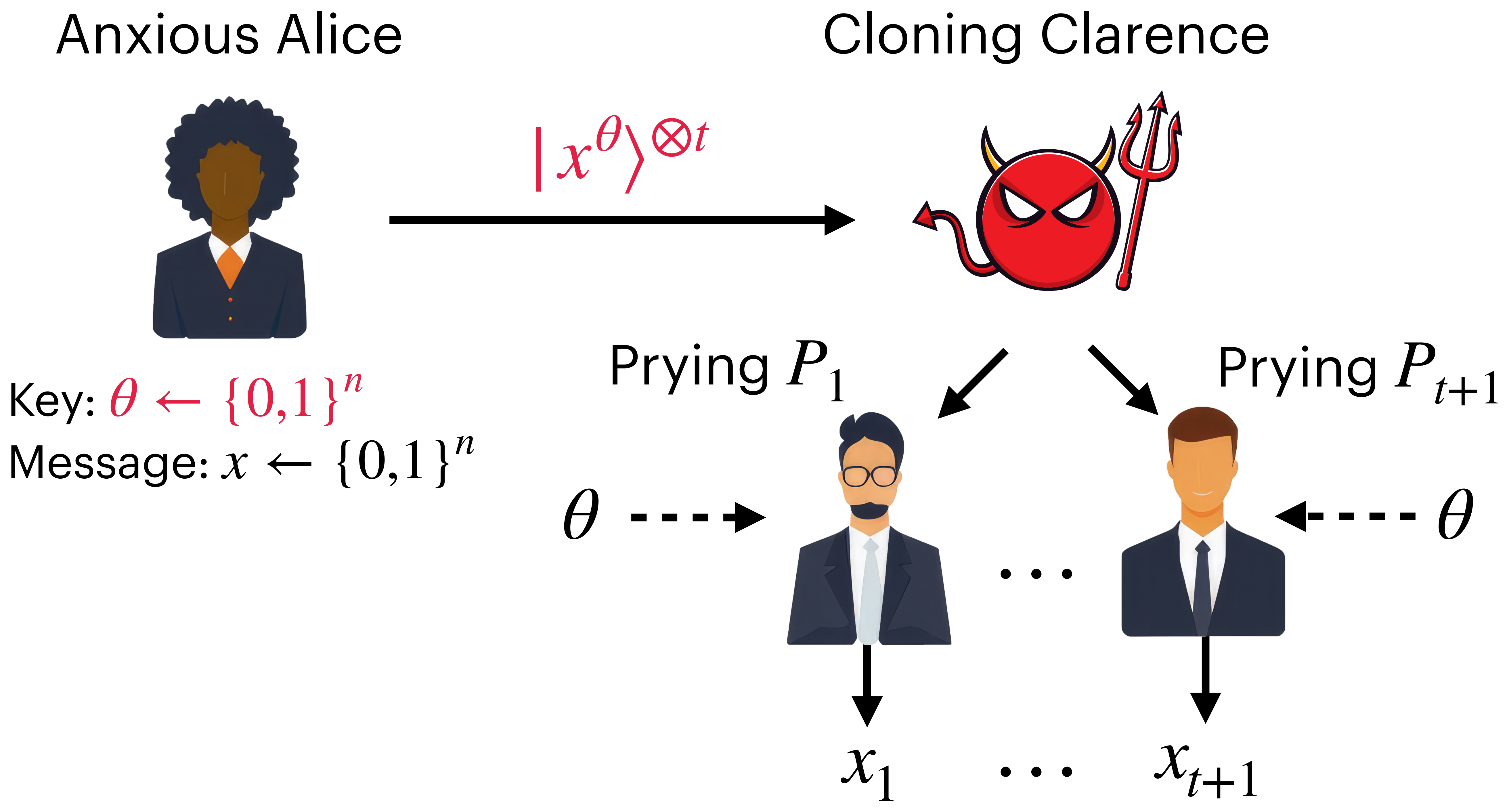

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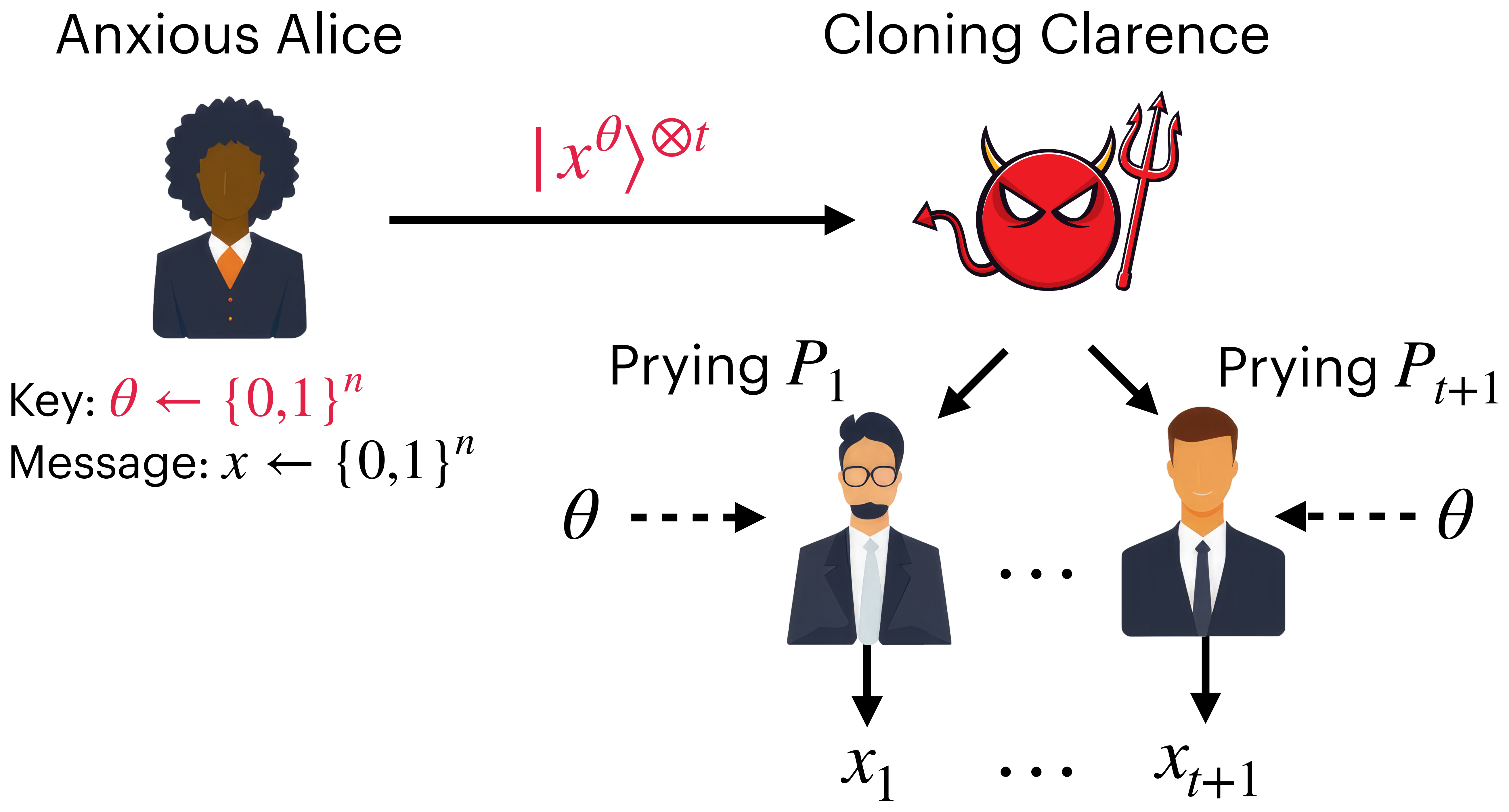

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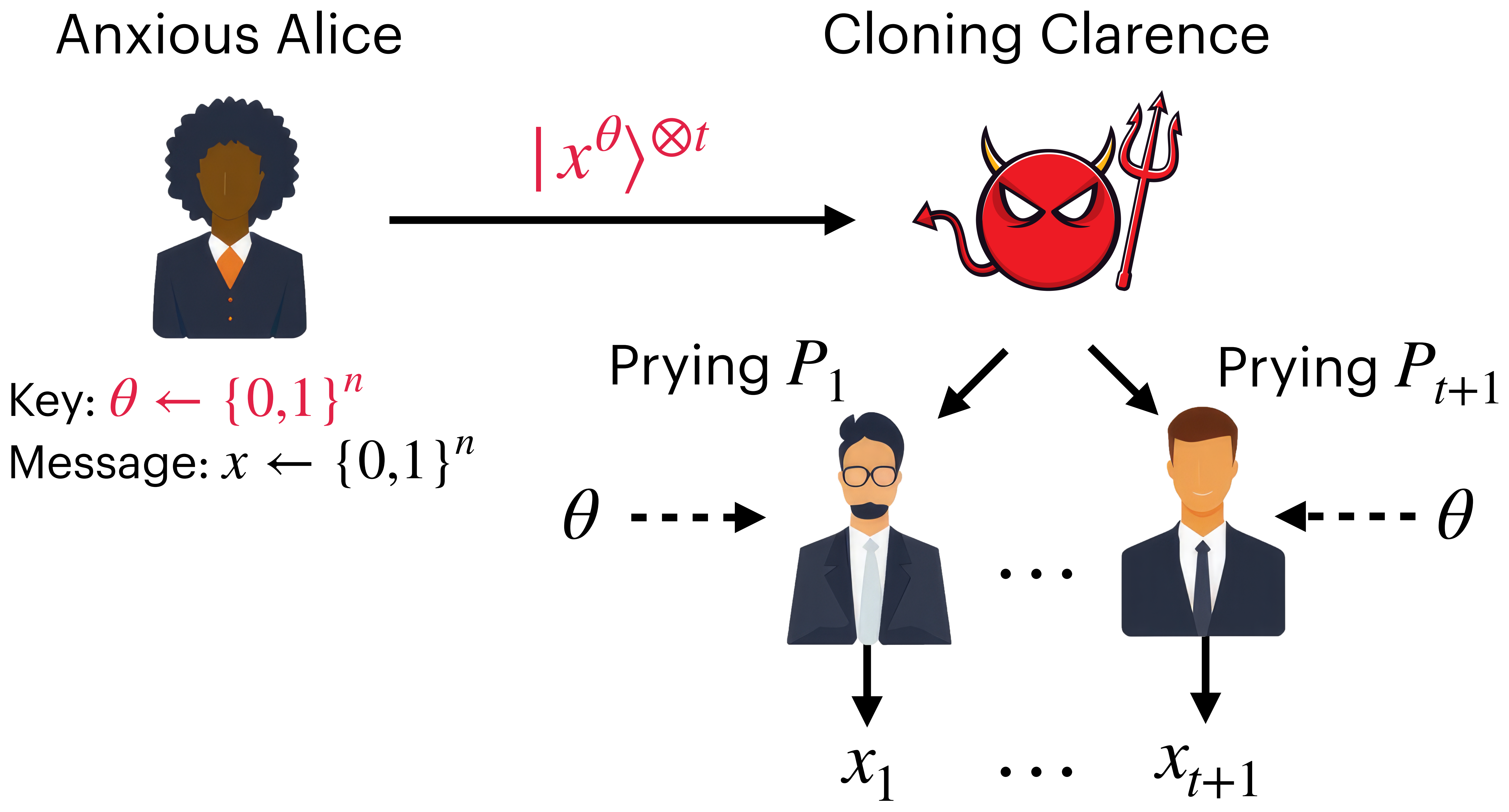

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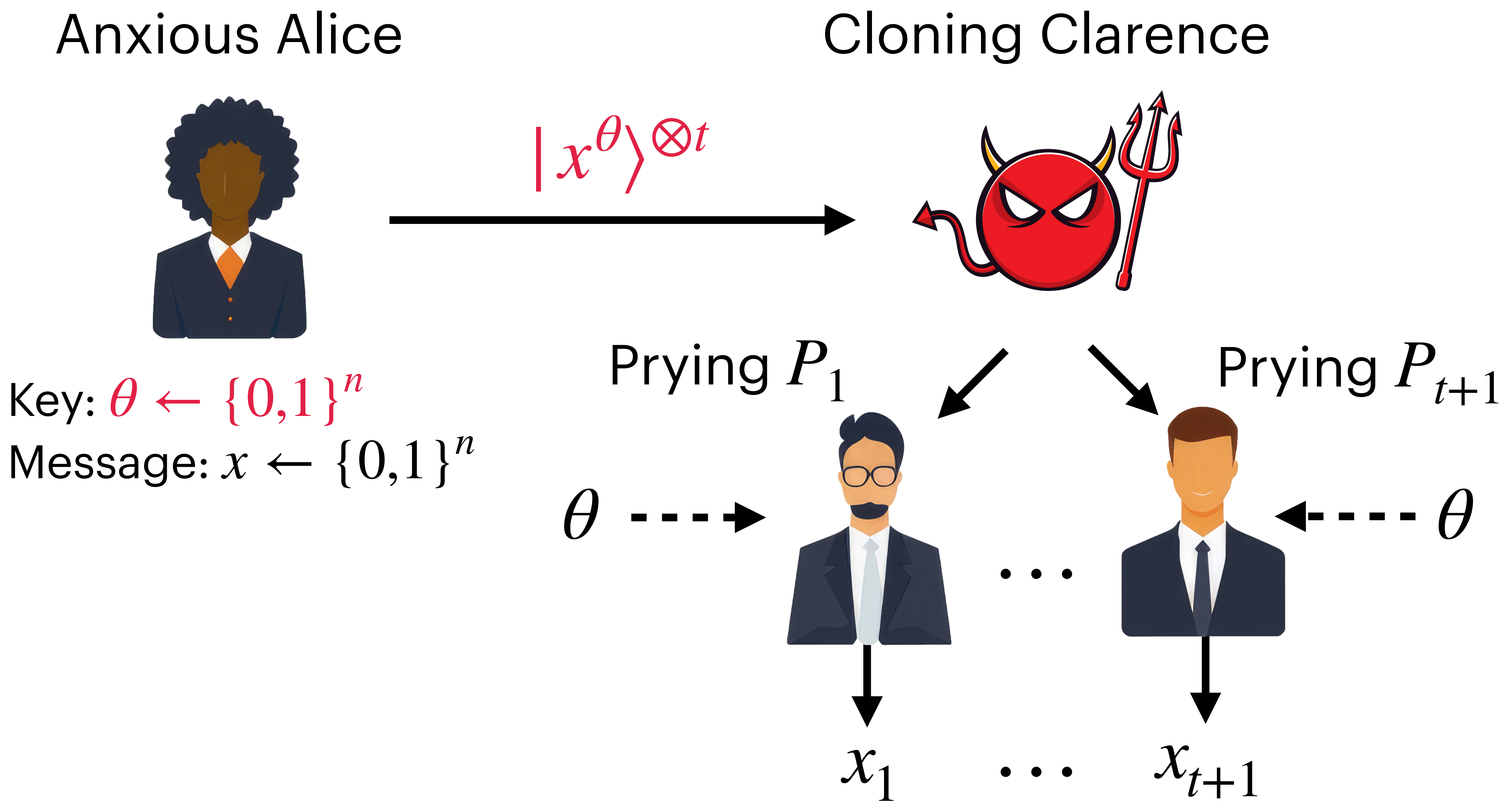

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  - All players can mix-and-match these results to recover x after receiving θ

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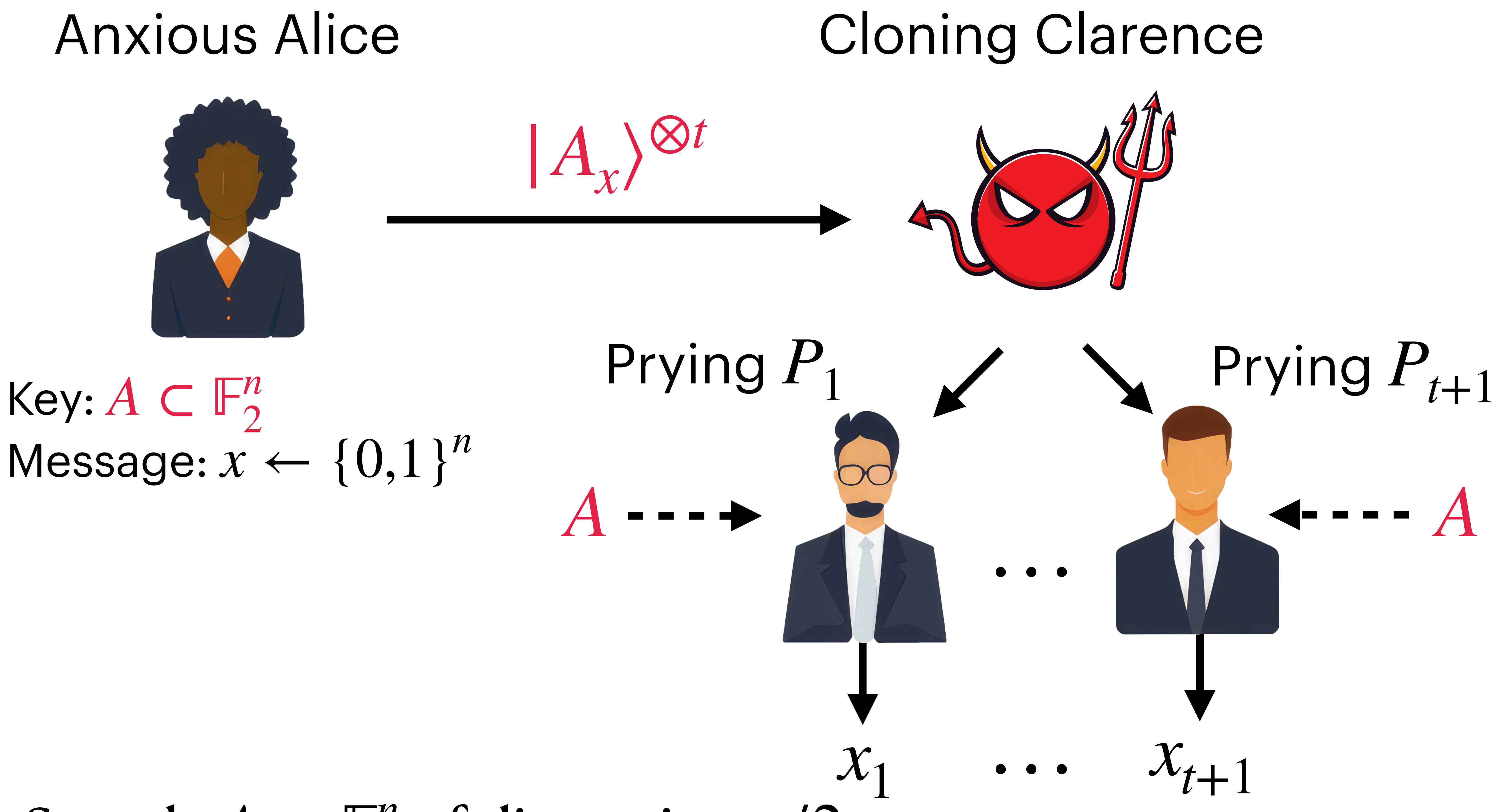
Previous Candidate 2: Subspace Coset States



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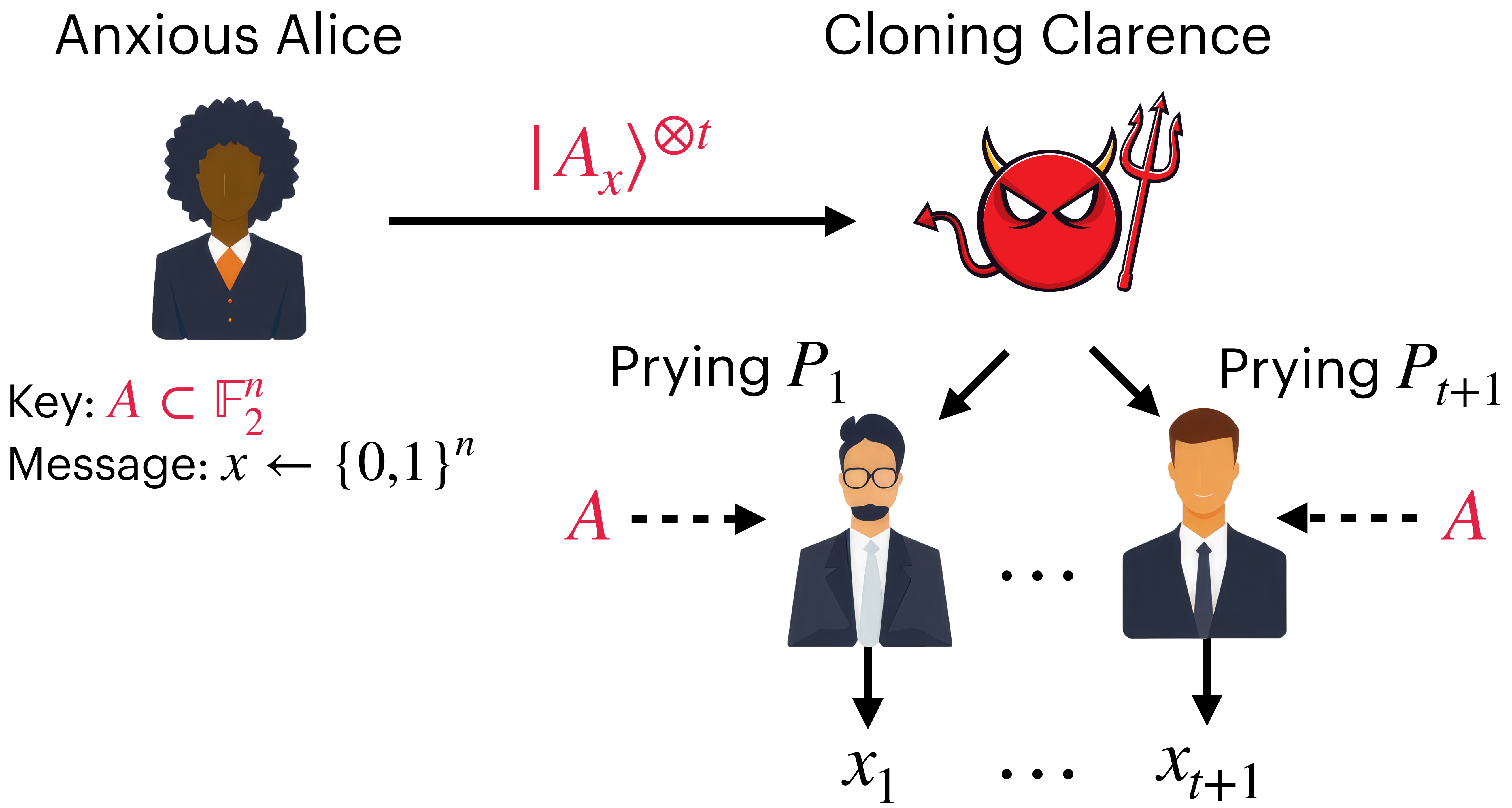

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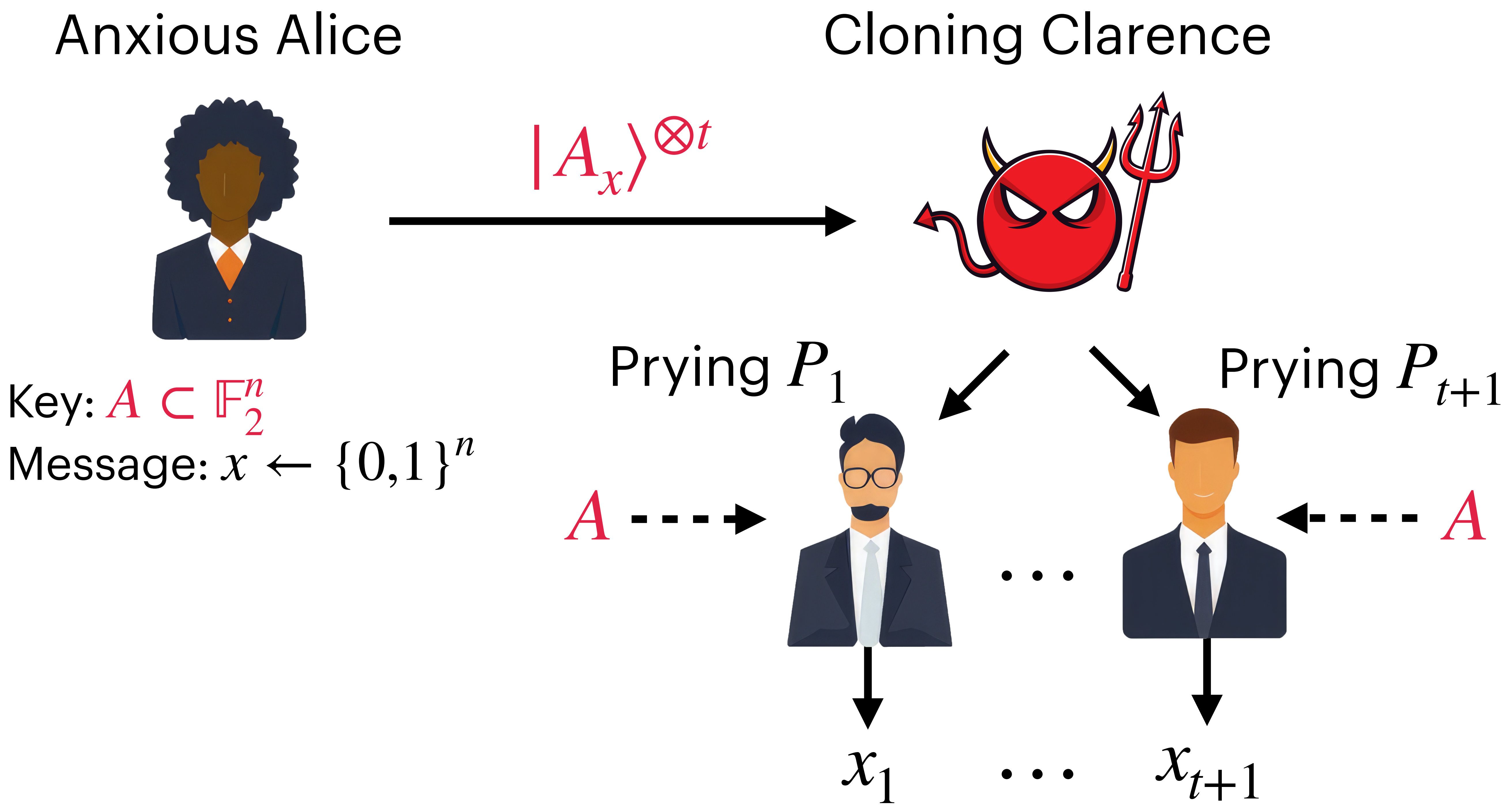

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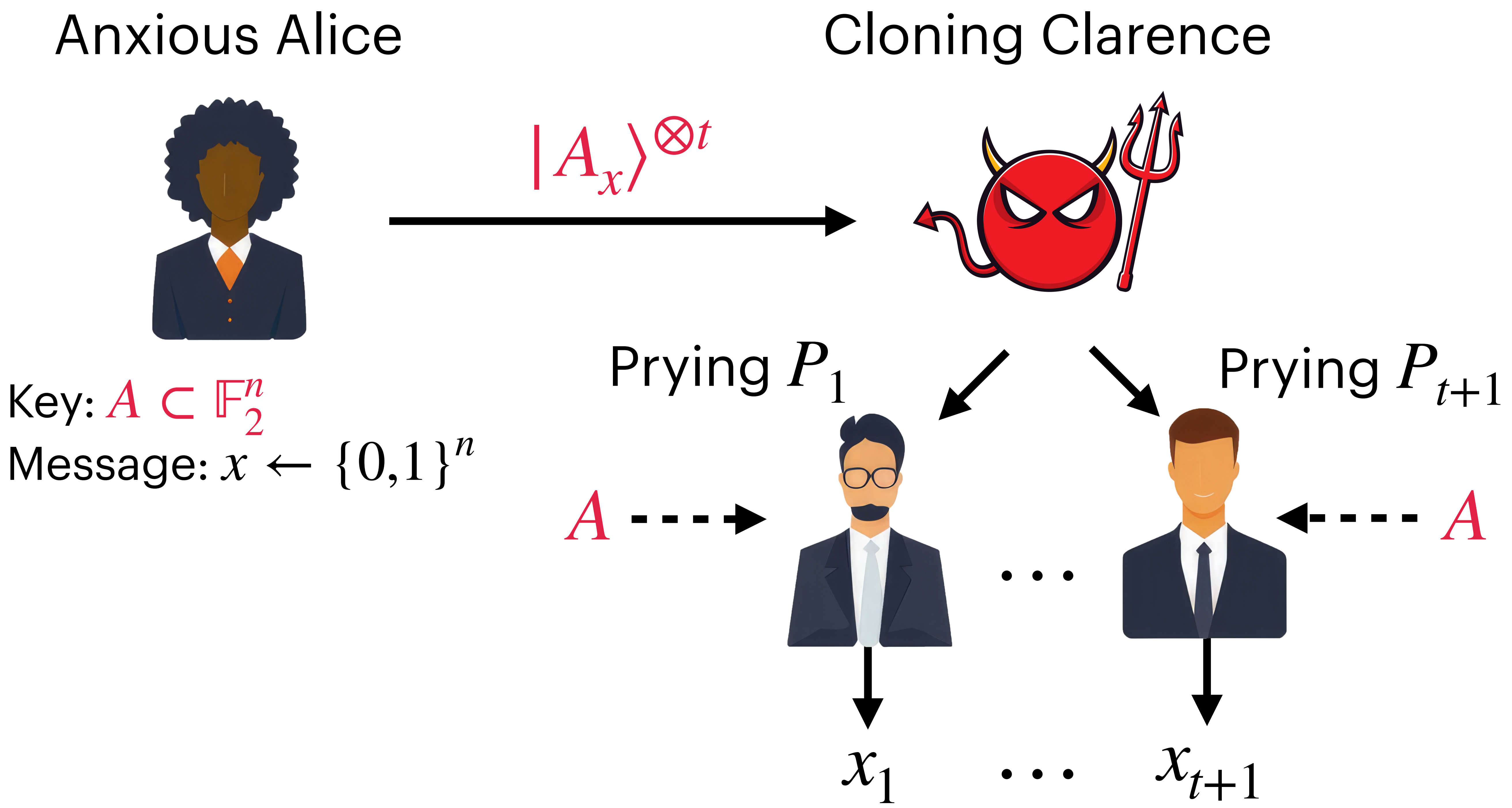

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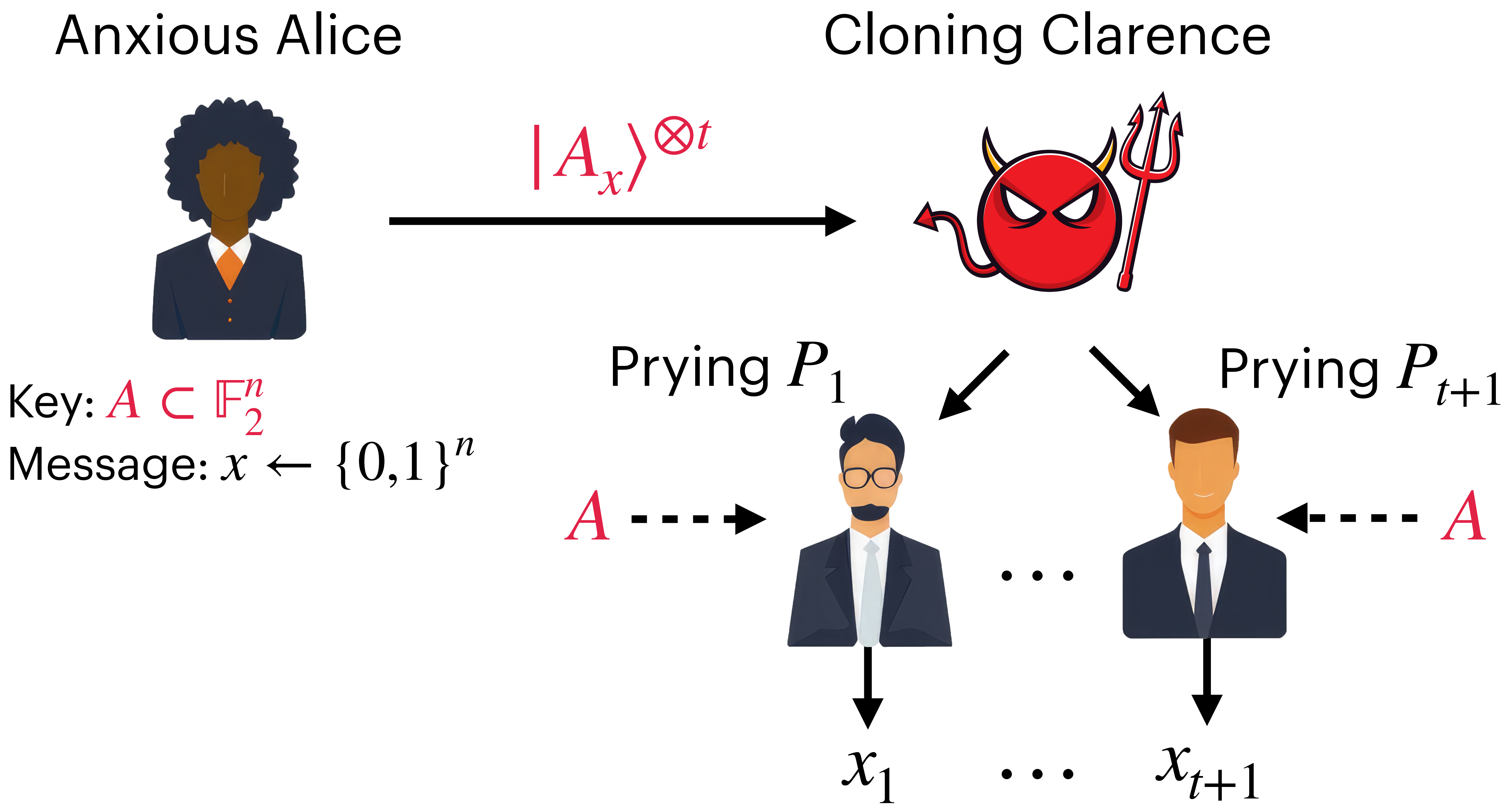

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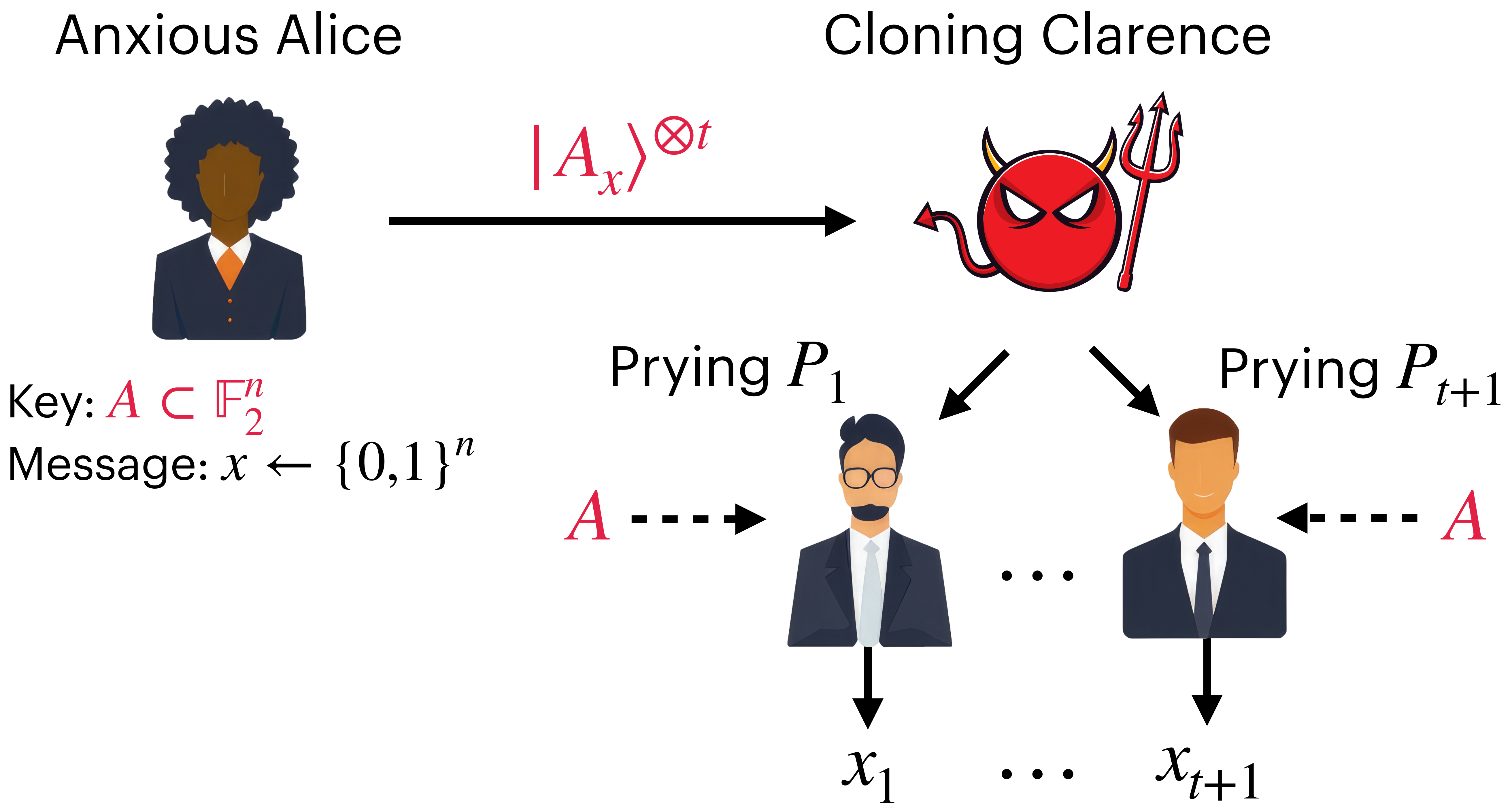

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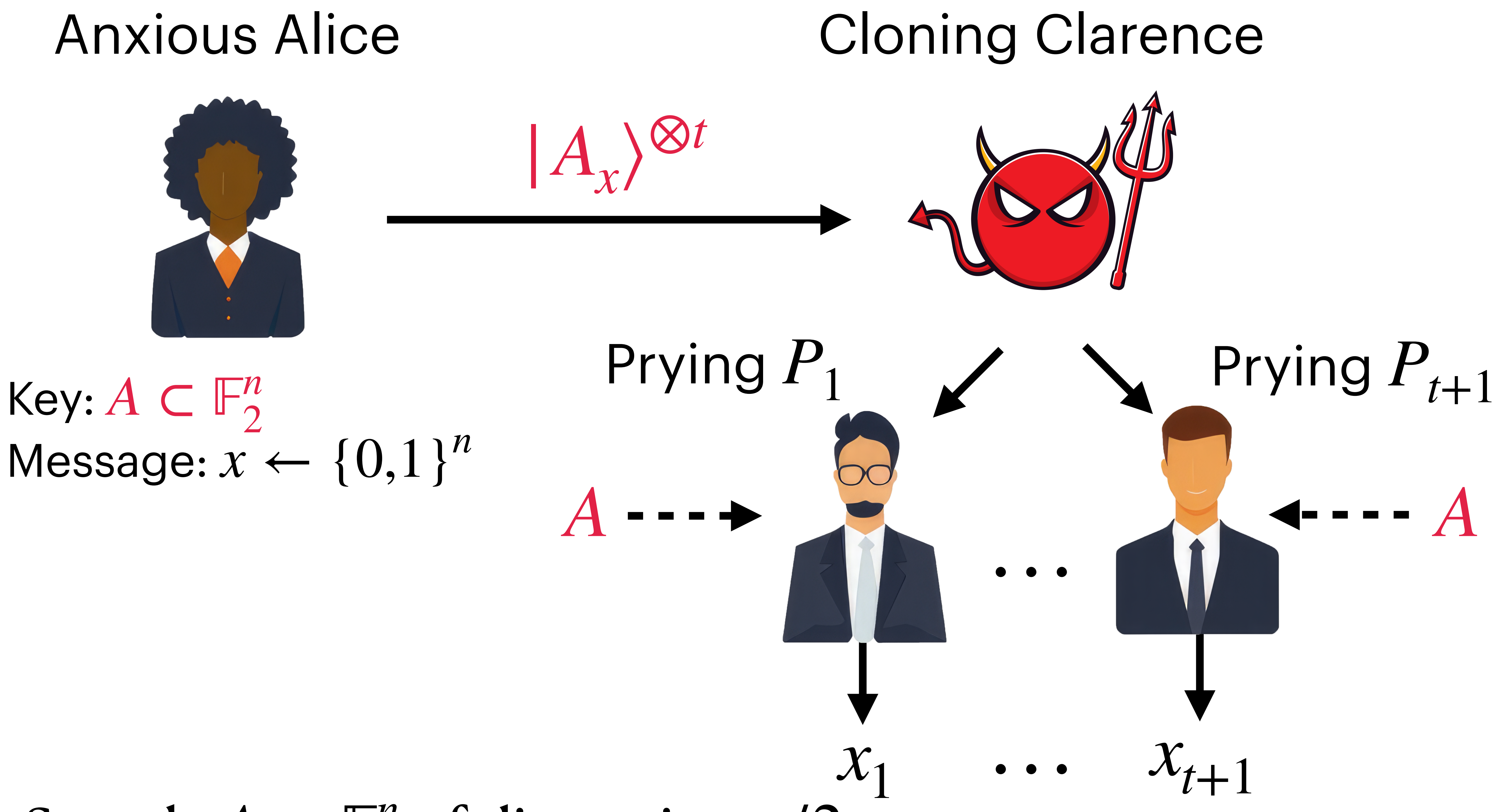

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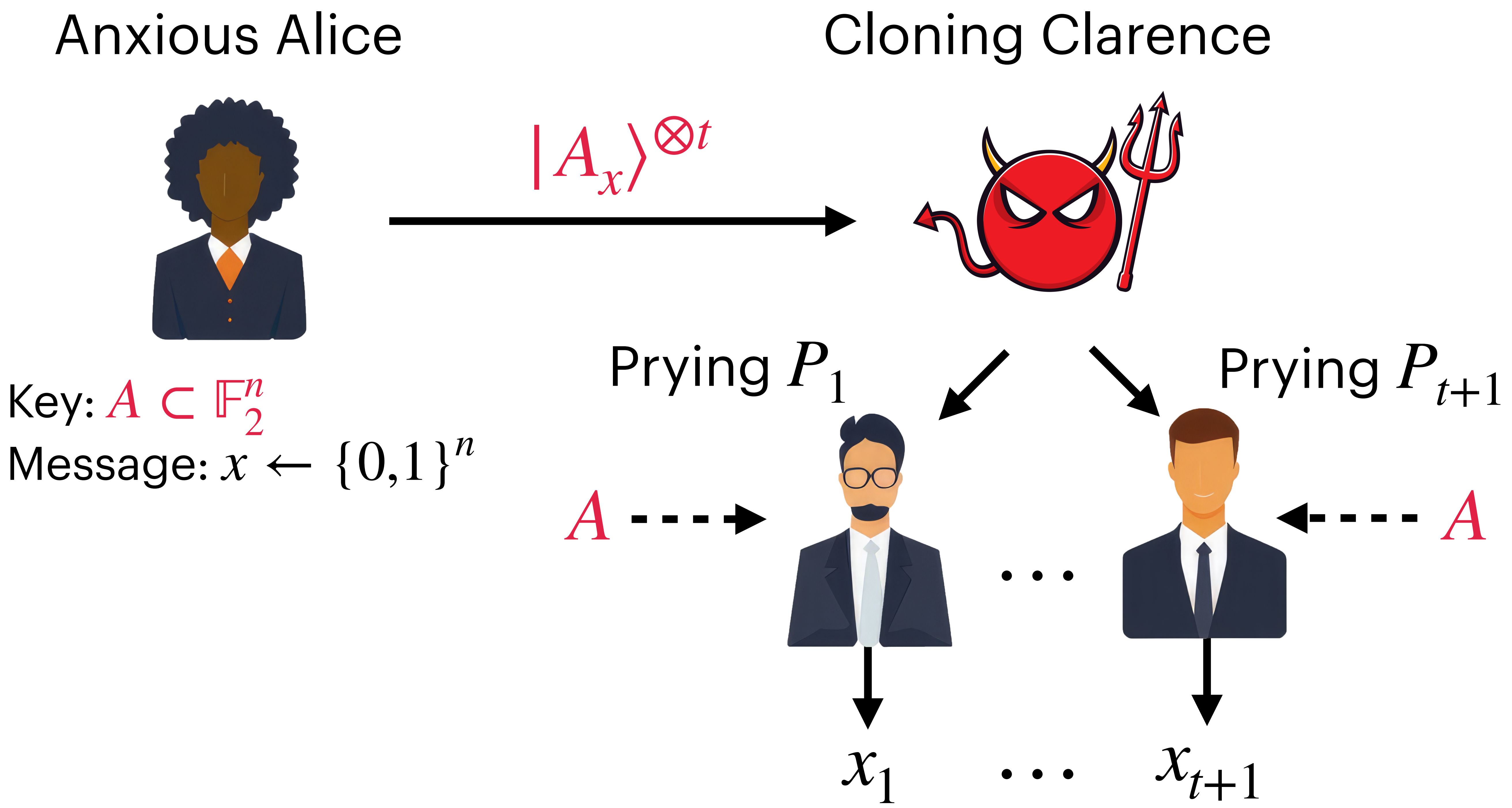

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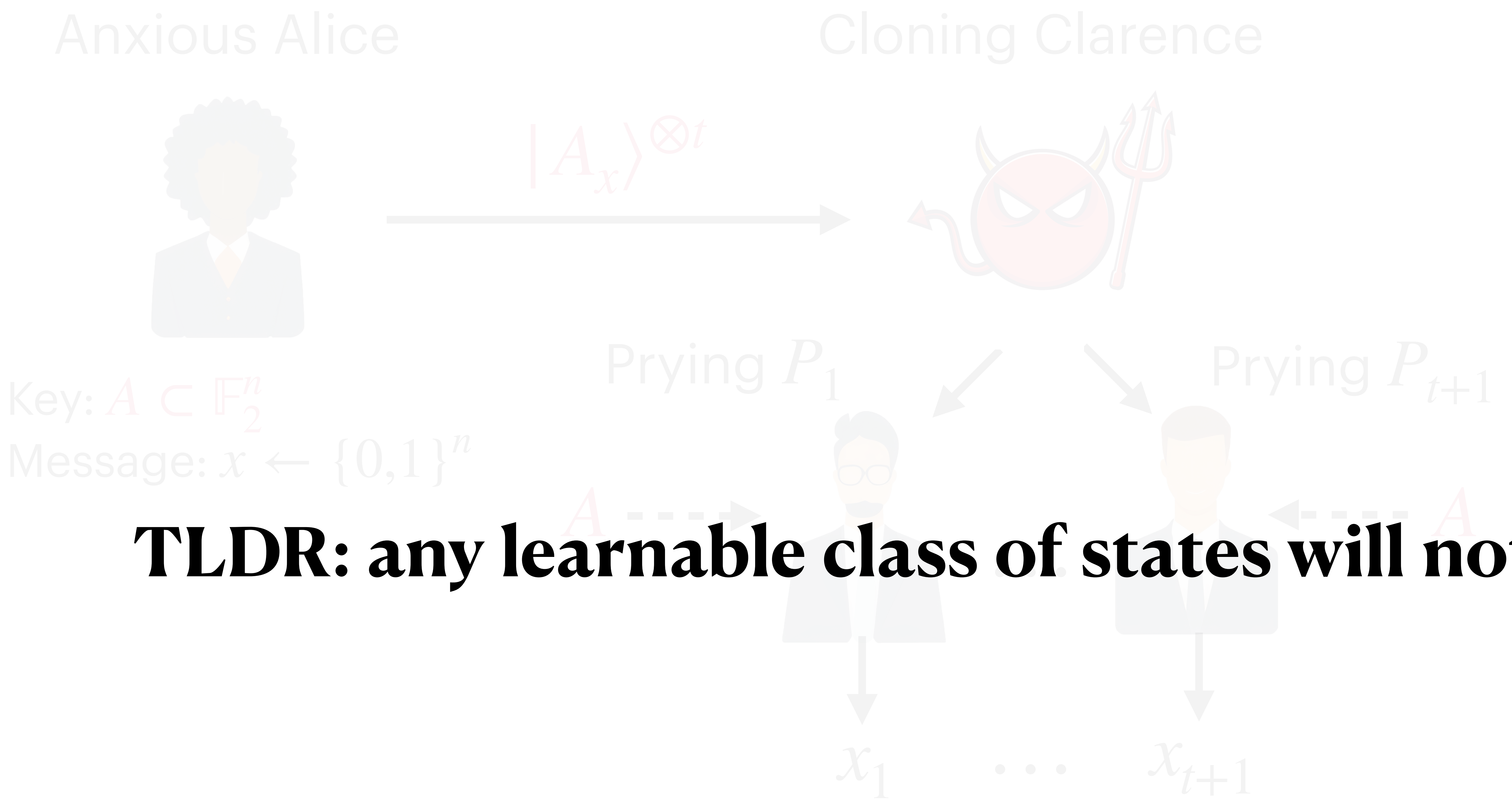

TLDR: any learnable class of states will not suffice for multi-copy security!

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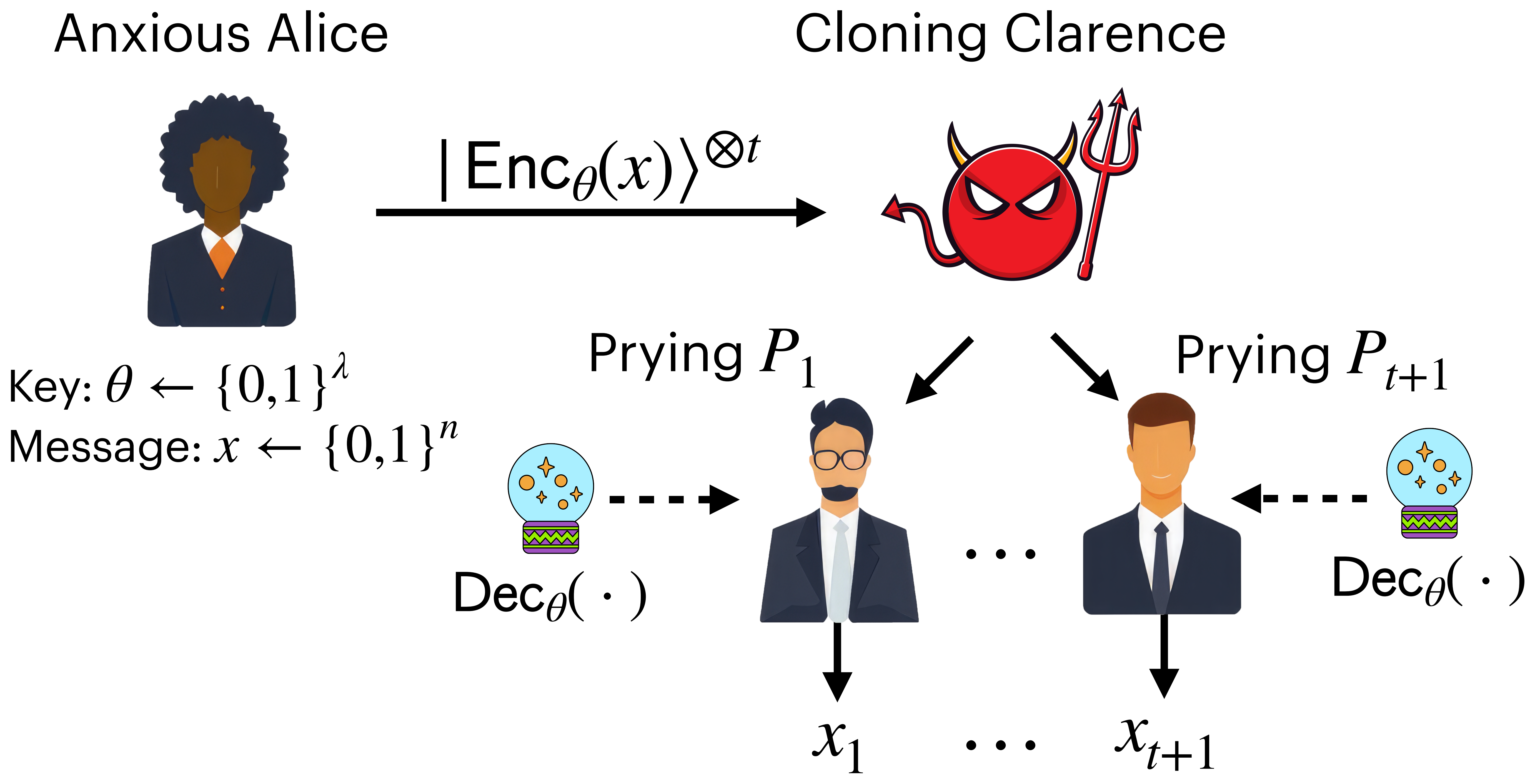
Our Model: One Oracle Query



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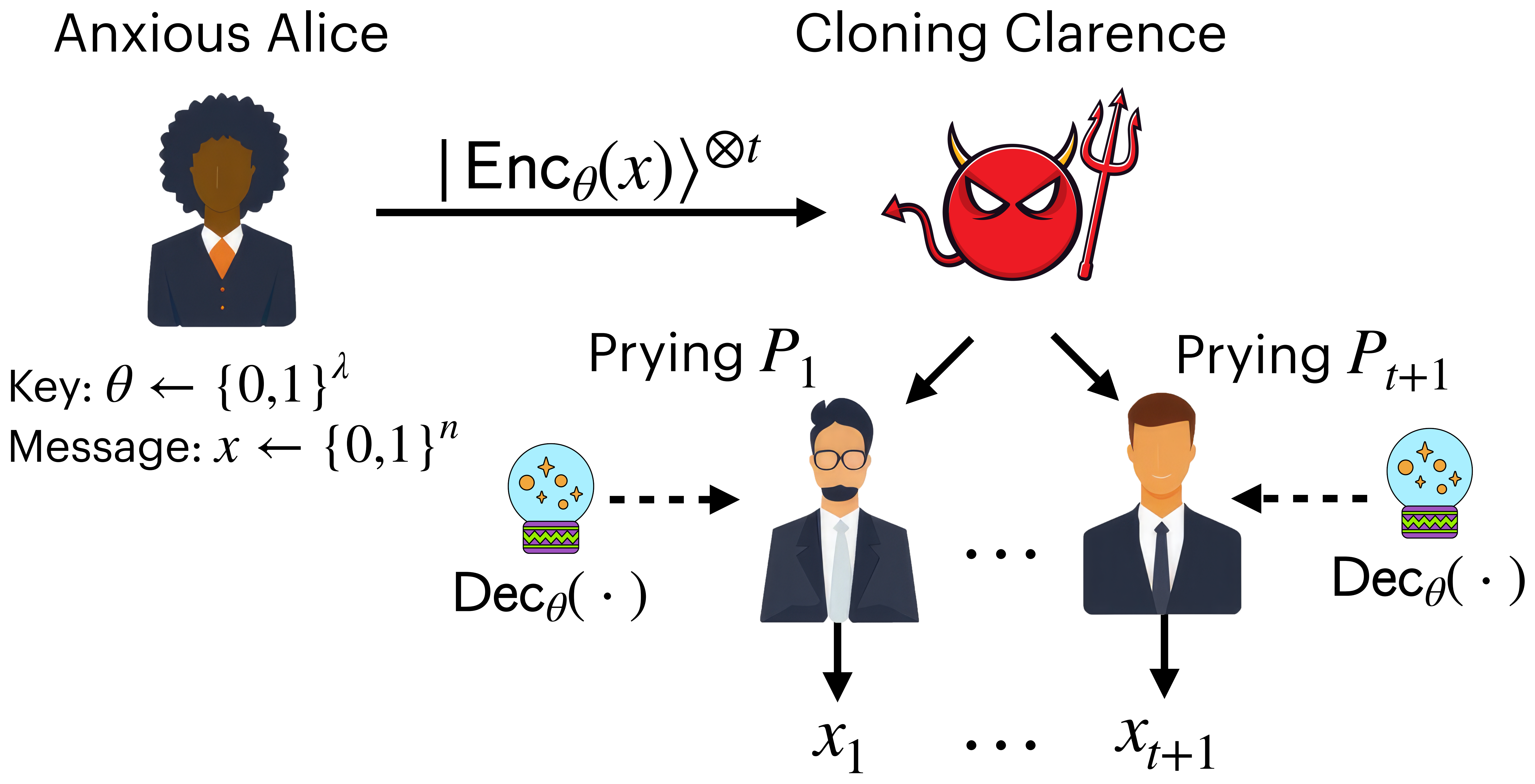

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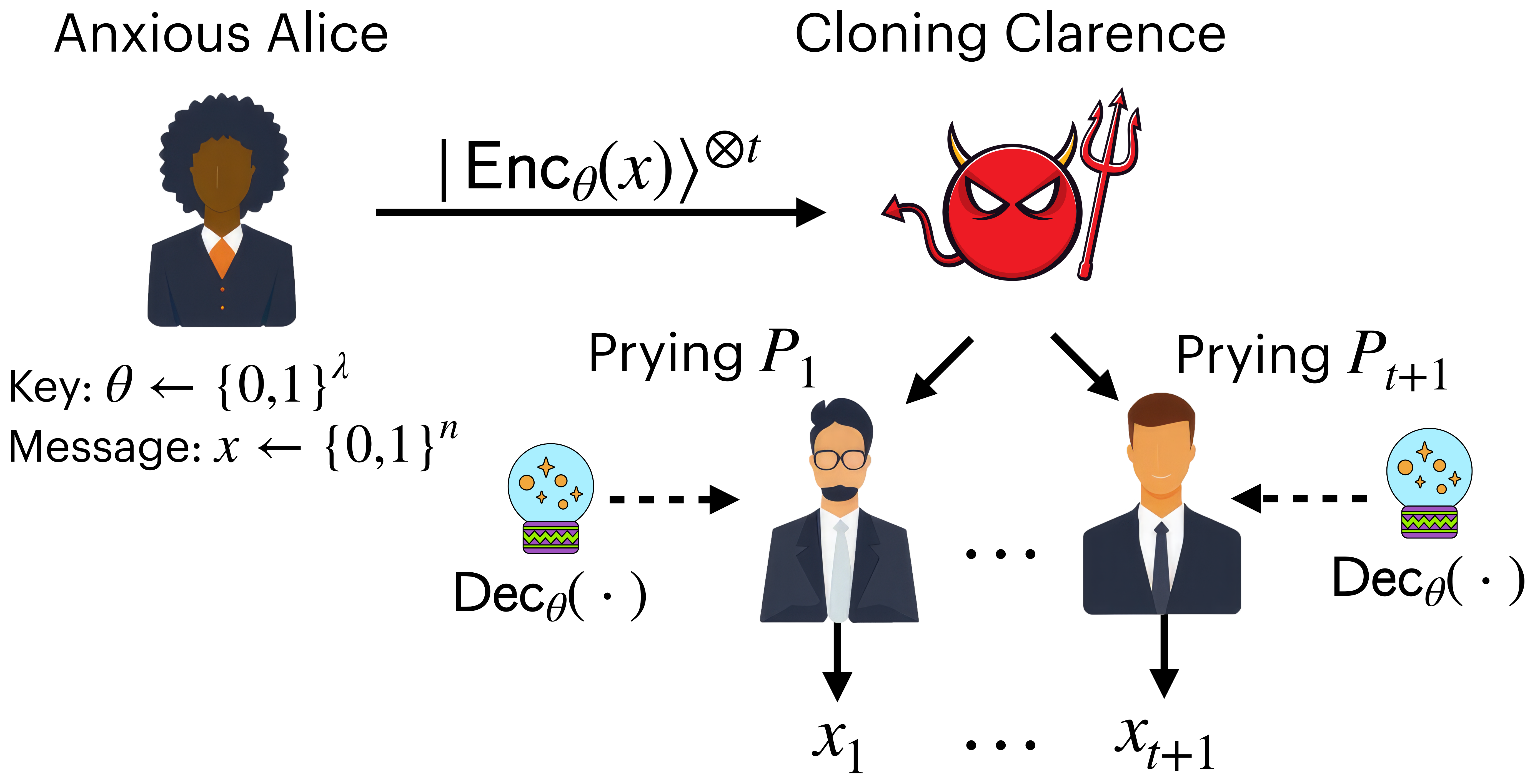

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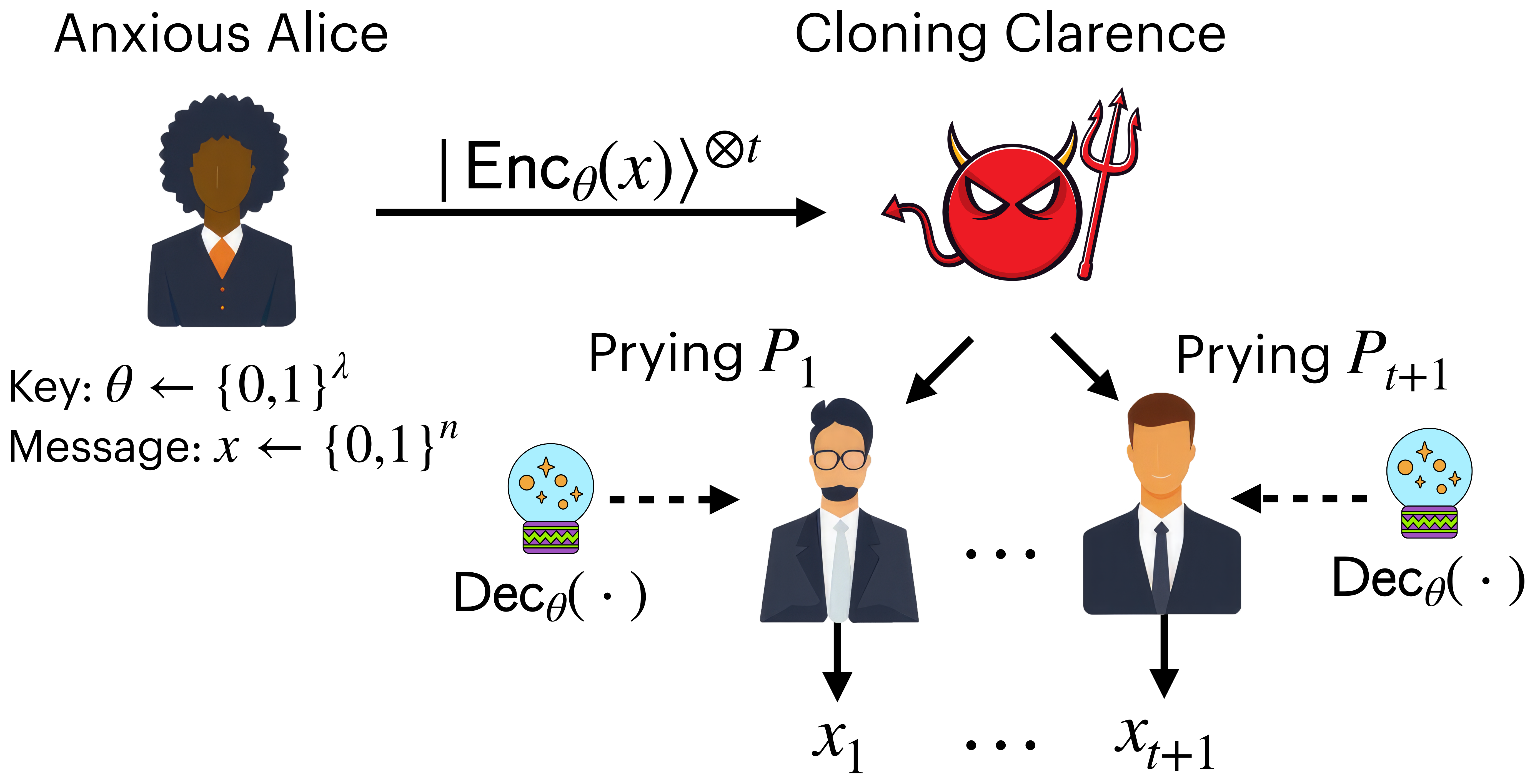

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- Application of our $t = 1$ result to black hole physics (Hawking radiation decoding)
  - Relies on the above properties

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Our Construction and Analysis

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  - Implies pseudorandom states are multi-copy unclonable → a natural candidate for unclonable encryption!

Pseudorandom States: Construction

Ji-Liu-Song '18, Brakerski-Shmueli '19

- Theorem: if $\left\{f_k : \{0,1\}^n \rightarrow \{0,1\} : k \in \{0,1\}^\lambda\right\}$ is a PRF, then

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- Alternate way of writing this: $\frac{1}{2^{n/2}} \mathsf{U}_{f_k} \sum_{z=0}^{2^n-1} |z\rangle = \mathsf{U}_{f_k} \mathsf{H}^{\otimes n} |0^n\rangle$, where

$$\mathsf{U}_f := \sum_{z=0}^{2^n-1} (-1)^{f(z)} |z\rangle\langle z| \text{ is a phase oracle for } f$$

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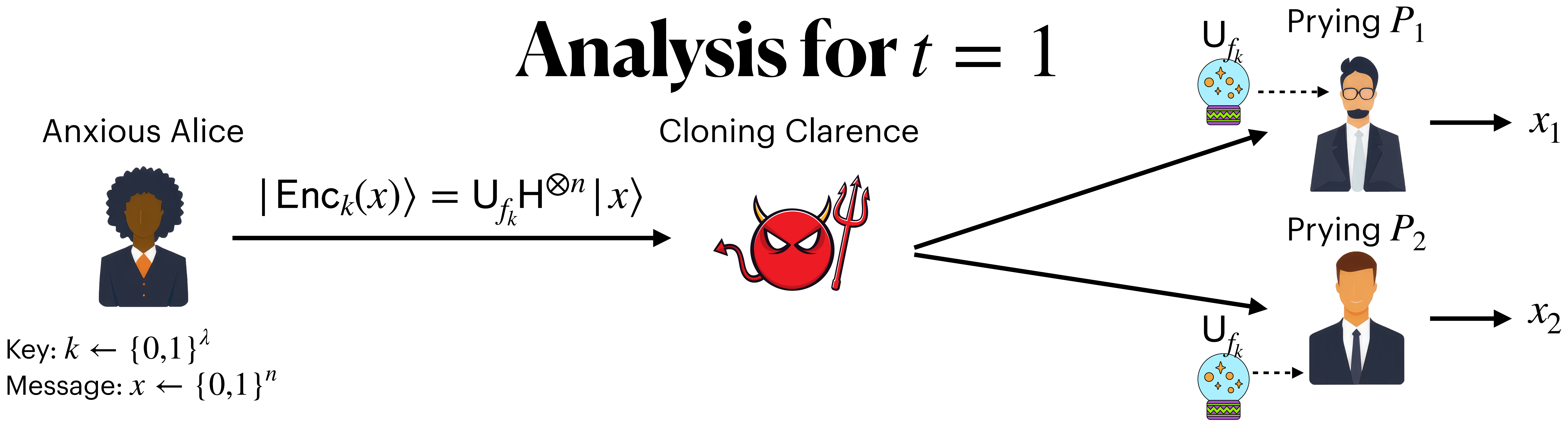
- Decryption simply applies $\mathsf{H}^{\otimes n}\mathsf{U}_{f_k}$ and measures in the standard basis
- Werner '98: these states are not multi-copy clonable by a computationally bounded adversary
- Our goal is to prove a stronger theorem ("useful no-cloning"): Cloning Clarence cannot construct *any* $t+1$ states that reveal x given k
  - This is why we go to an oracle model

Analysis for $t = 1$



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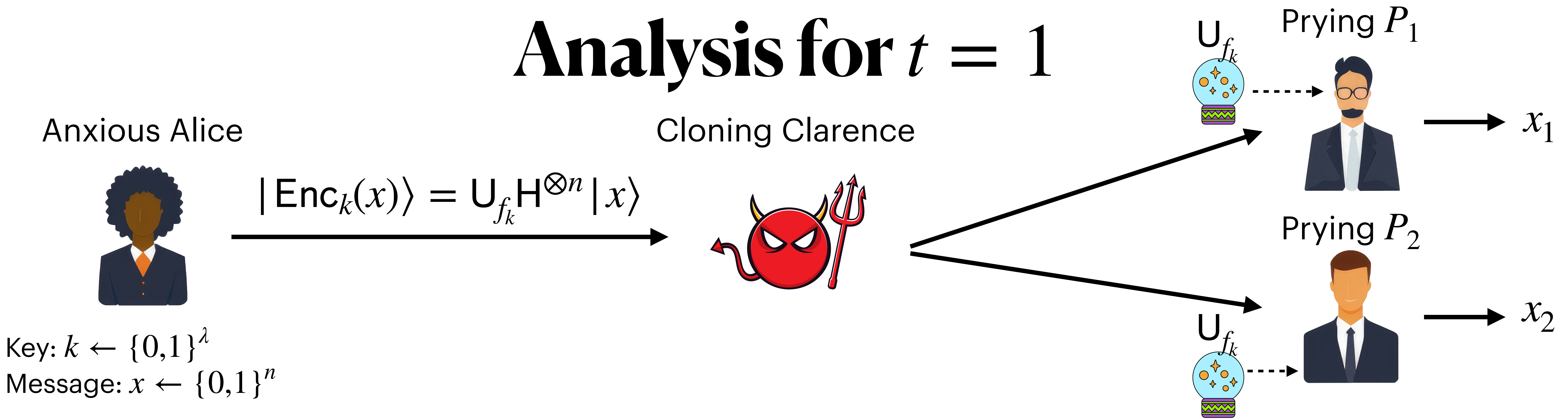

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- This is the only step where we will use the fact that the adversaries are computationally bounded

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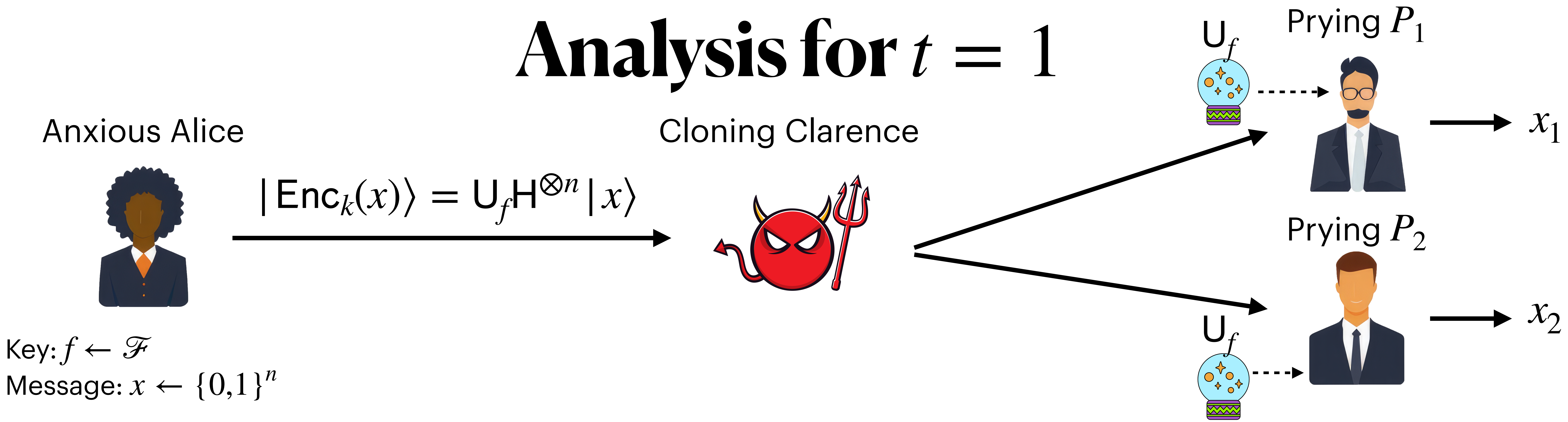

- Step 2: switch to an entanglement-based formulation where Alice and Clarence begin the game sharing entangled qubits

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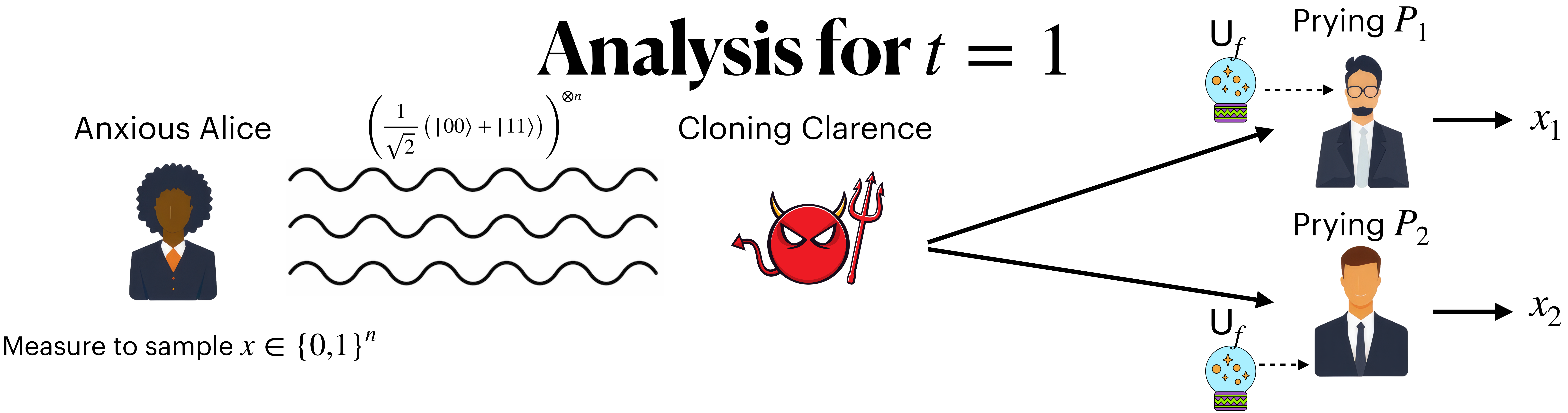

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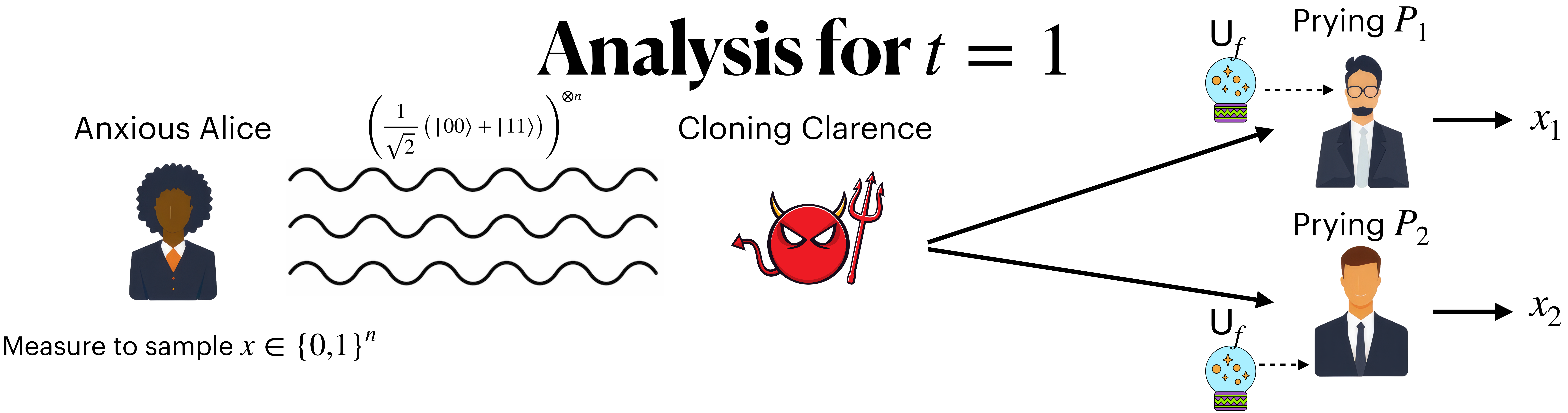

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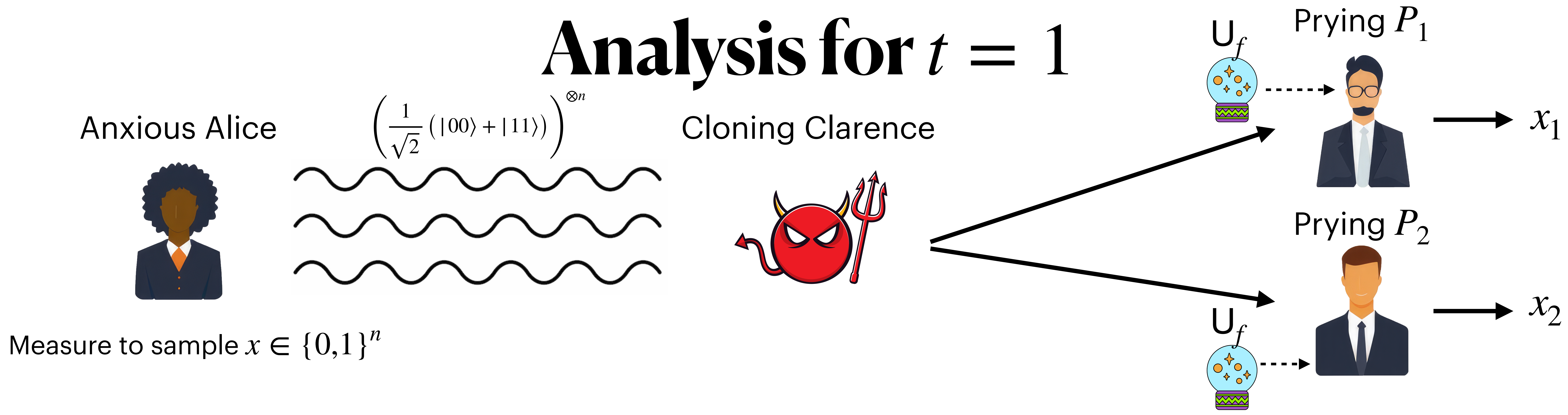

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3. P_1, P_2 make one query to U_f and output their guesses. Win if $x_1 = x_2 = x$.

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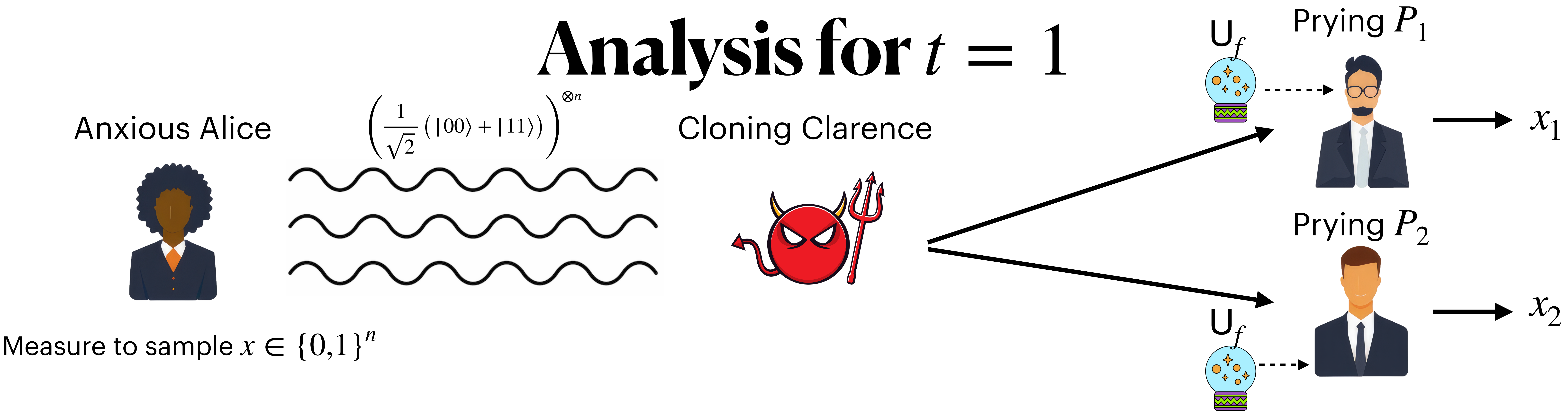

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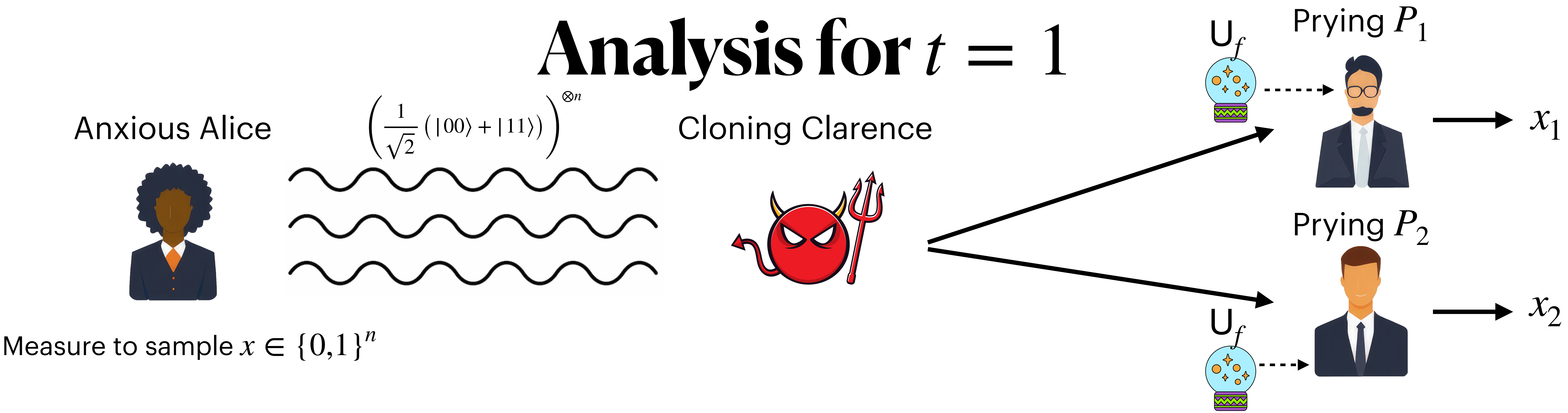

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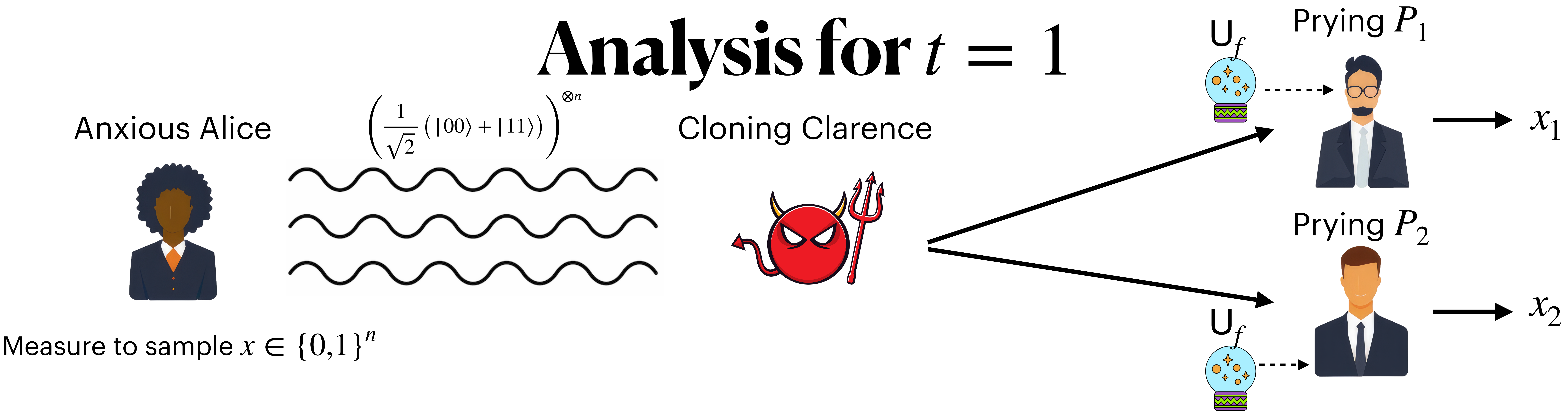

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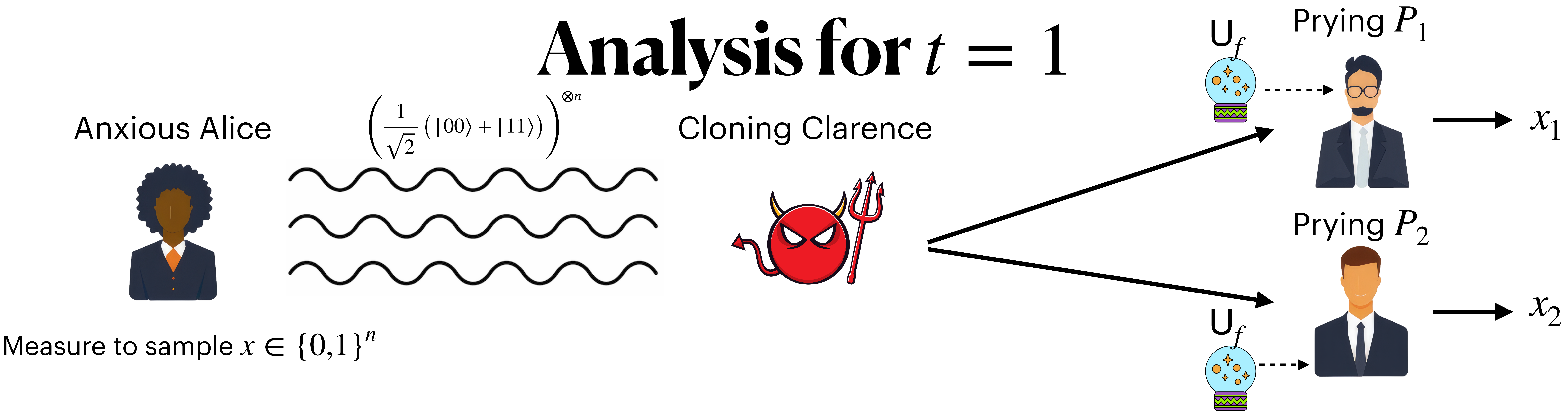

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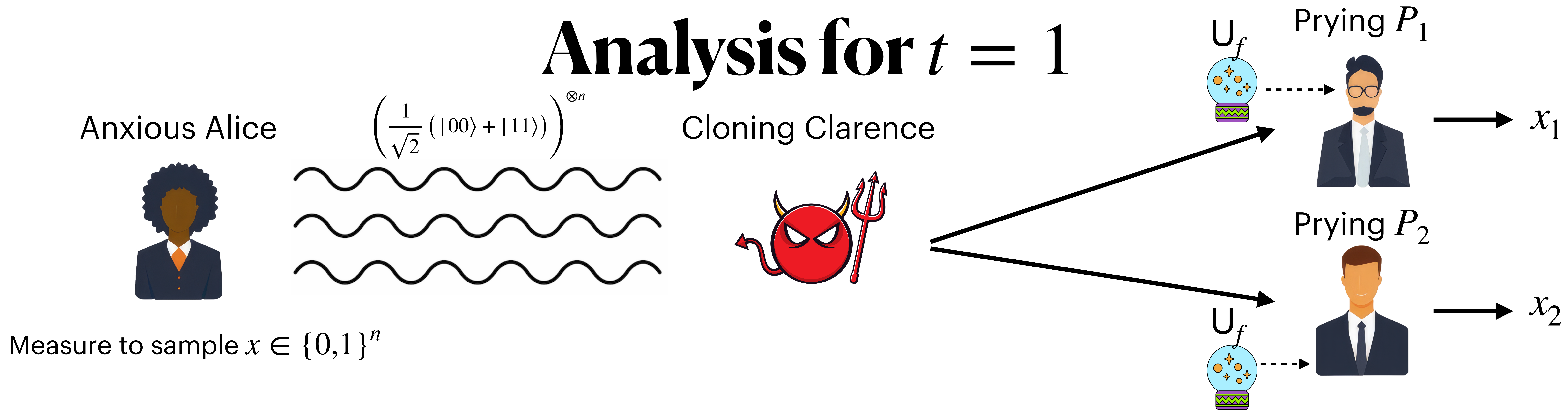

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- Our approach: analyse using binary types (AGQY22) and a novel generalisation called subtypes

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- Then

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- IND security in the plain model with classical decryption keys and distinguishing advantage $\mathsf{negl}(\lambda)$?
  - Open even for the single-copy case!