Parallel Spooky Pebbling Makes Regev Factoring More Practical

[arXiv:2510.08432]

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Thanks Greg and Katherine for many of these slides!


PEBBLERS
ENDGAME

Quantum factoring algorithms

Given an n-bit integer N, compute its prime factorisation in poly(n) time.

[1] $O(\cdot)$ hides constant factors. $\widetilde{O}(\cdot)$ hides poly($\log n$) factors.

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- Key idea: run quantum period finding on $g(x) = a^x \bmod N$

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Bottleneck: computing $g(x)$

Task: compute $|x\rangle|0\rangle \mapsto |x\rangle|g(x)\rangle = |x\rangle|a^x \bmod N\rangle$

Shor

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  - Gates: $O(n^3)$
  - Qubits: 1 for x, n for $a^x \bmod N$, and $n + O(1)$ ancilla qubits

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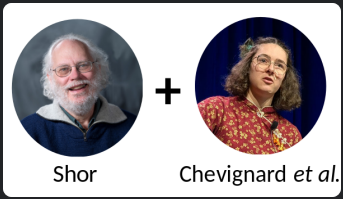

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- Gidney: concrete improvements to Chevignard et al.

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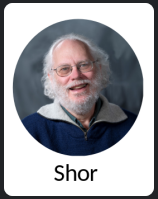
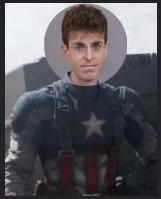


Core of Shor's algorithm

$g(x) = a^x \bmod N$, where
$x \leq 2^{O(n)}$.



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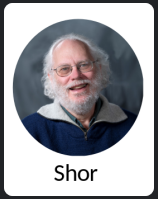
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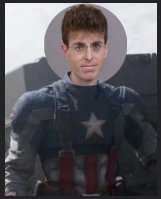

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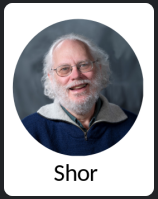
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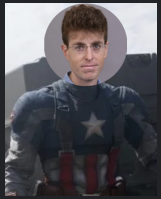

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Roadmap

- Previous approaches to Regev
  - Original algorithm: fewer gates, but irreversible so more qubits
  - Making things reversible but concretely inefficient: Fibonacci arithmetic

- Pebbling!
  - Pebble games: accepting irreversibility
  - Previous work: adding parallelism *or* spookiness

- Our work: parallelism and spookiness join forces
  - Example
  - Pebbling results
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Intuition: how to compute a^x mod N when x is a power of two?

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General case: multiply in extra factors of a along the way...

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General case: multiply in extra factors of a along the way...
Gate count: $O(\log x)$ multiplications of n-bit integers $\Rightarrow \widetilde{O}(n \log x)$

Regev's gate savings

Recall: computing $a^x \bmod N$ takes $\widetilde{O}(n \log x)$ gates

Shor

- Goal: compute $a^x \bmod N$ for $x \leq 2^{O(n)}$

Regev

- Goal: compute $\prod_{i=1}^{\sqrt{n}} a_i^{x_i} \bmod N$ for $x_i \leq 2^{O(\sqrt{n})}$

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  - Gates to naively compute all $\sqrt{n}$ terms: $\sqrt{n} \cdot \widetilde{O}(n^{3/2}) = \widetilde{O}(n^2)$ 😢
- Regev magic: if the a_i's are all small, can essentially just pay the cost of computing one term! 😃

Regev's qubit problem

How to compute $|x\rangle|0\rangle \rightarrow |x\rangle|a^x \bmod N\rangle$ quantumly?

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- **Issue:** squaring mod N is not reversible!
- Regev's original approach: square out of place to get $|a\rangle|a^2\rangle|a^4\rangle \ldots |a^x\rangle \Rightarrow O(n \log x)$ qubits 😢

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What if we restructure our computation so each step is reversible?

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To compute $a^x \bmod N = \prod_i \left(a^{2^i}\right)^{x_i} \bmod N$:

- **classically** precompute $a, a^2, a^4, a^8, \cdots \pmod N$
- Multiply together for all i where bit $x_i = 1$

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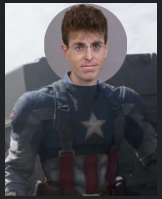

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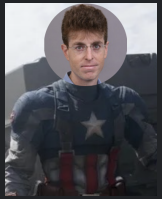

Regev's factoring speedup comes from multiplied values being **small**

😢 $a_j^{2^i}$ mod N in general *not* small

- Applying Shor's rearrangement would **kill Regev's improvement** in gate count

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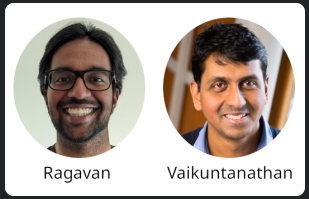

Let's do "repeated squaring," but with **Fibonacci number** exponents instead of powers of 2

😃 "Squaring" is **reversible** in this representation!!

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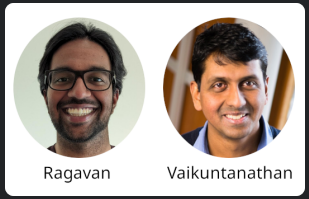

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😢 Large constant factors → Shor still wins in practice

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Is it possible to do better?

Pebble games for reversible computation

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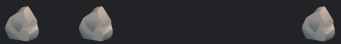

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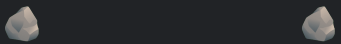

Space usage (max # pebbles): $O(k)$ registers
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Using less space: a recursive strategy

1. Pebble from start to $k/2$

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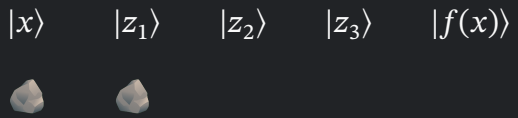

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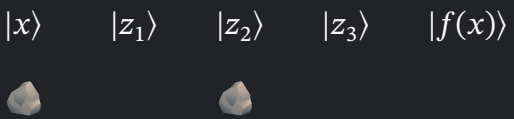

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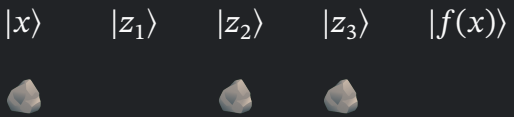

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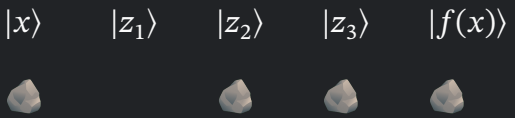

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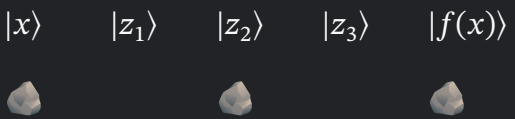

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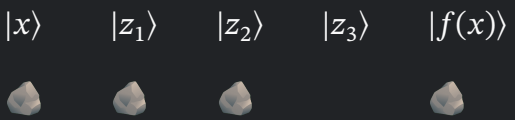

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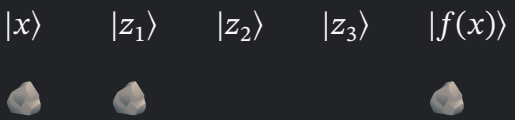

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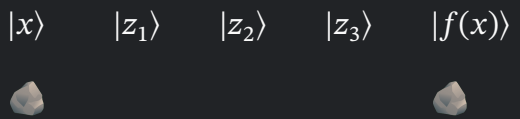

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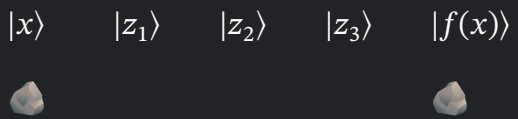

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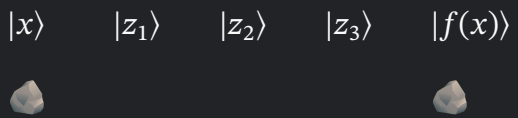

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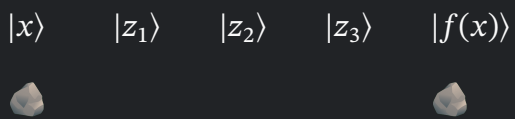

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Generalizing this strategy: $O(2^{1/\epsilon} \log k)$ pebbles, pebbling time $O(k^{1+\epsilon})$ 😐

Roadmap

- Previous approaches to Regev
  - Original algorithm: fewer gates, but irreversible so more qubits
  - Making things reversible but concretely inefficient: Fibonacci arithmetic

- Pebbling!
  - Pebble games: accepting irreversibility
  - Previous work: adding parallelism *or* spookiness

- Our work: parallelism and spookiness join forces
  - Example
  - Pebbling results
  - Factoring results

Resource 1: parallelism

Setting:

- Pebble operations in parallel are allowed as long as they act on disjoint pebbles

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Example: space usage $O(4^{\sqrt{\log k}})$ pebbles, $O(k)$ pebbling depth

Depth is optimal up to a constant factor!

Resource 2: quantum spookiness

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We've turned $|z_i\rangle$ into a ghost!

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Rules:

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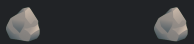

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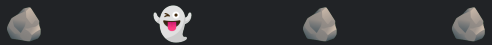

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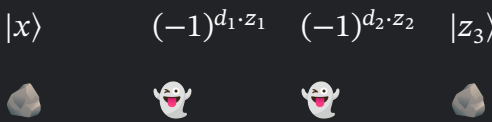

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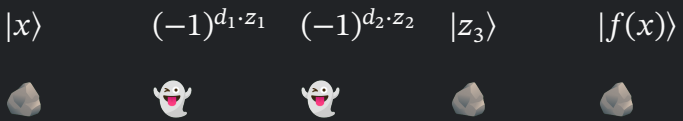

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We've been tempted too strongly by the power of dark magic...

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$|f(x)\rangle$

Space: $O(\log k)$ pebbles 😃

Spooky pebble games

1. Blast straight to $k/2$, leaving ghosts
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Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $T(k) = O(k) + 2T(k/2)$

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Why is spookiness so powerful?

Without spookiness

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With spookiness

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- **What spookiness does NOT enable:** moving pebbles faster, without leaving a mess
- **What spookiness does enable:** if we're okay with making a mess (to be cleaned up later), can move pebbles faster

Our work: parallelism and spookiness join forces

Roadmap

- Previous approaches to Regev
  - Original algorithm: fewer gates, but irreversible so more qubits
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Can we do better using spookiness and parallelism in tandem?

Example: optimal-depth parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

Step number: 0

Max. pebble count: 1

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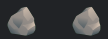
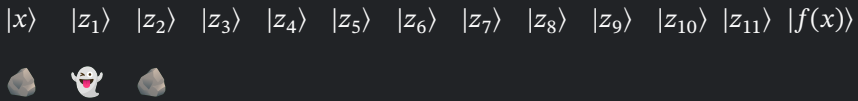


Step number: 1

Max. pebble count: 2

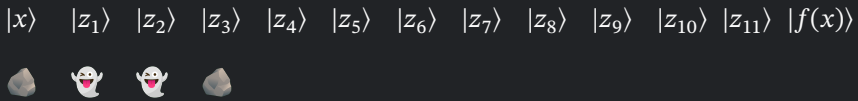
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 2

Max. pebble count: 2

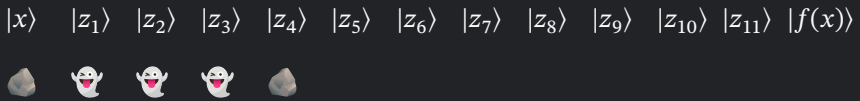
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 3

Max. pebble count: 2

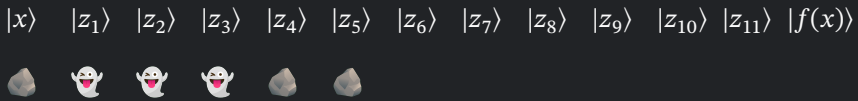
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 4

Max. pebble count: 2

Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 5

Max. pebble count: 3

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 6

Max. pebble count: 3

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 7

Max. pebble count: 3

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 8

Max. pebble count: 4

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 9

Max. pebble count: 4

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 10

Max. pebble count: 5

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 11

Max. pebble count: 6

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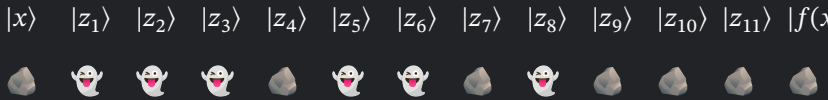


Step number: 12

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 13

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 14

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 15

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 16

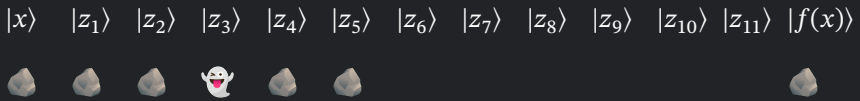
Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$

Step number: 17

Max. pebble count: 7

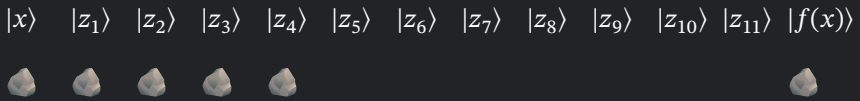
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 18

Max. pebble count: 7

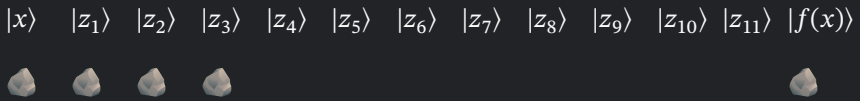
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 19

Max. pebble count: 7

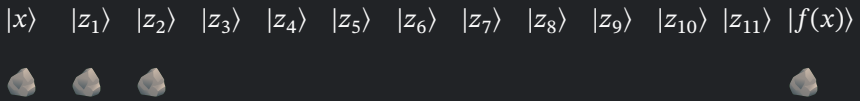
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 20

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 21

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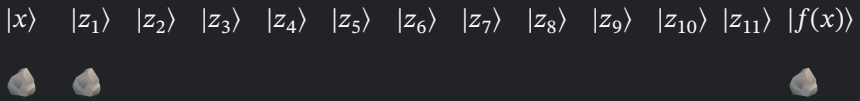
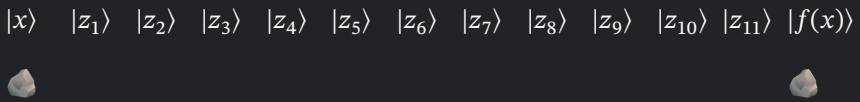
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 22

Max. pebble count: 7

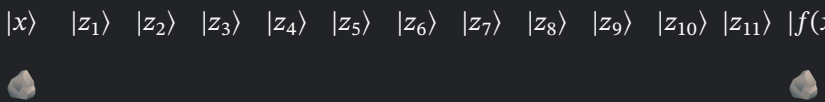
Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: 23

Max. pebble count: 7

Example: optimal-depth parallel spooky pebble game for $k = 12$



Step number: $23 = 2k - 1$, which is optimal 🤩

Max. pebble count: 7

Roadmap

- Previous approaches to Regev
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Explicit construction

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Automated search

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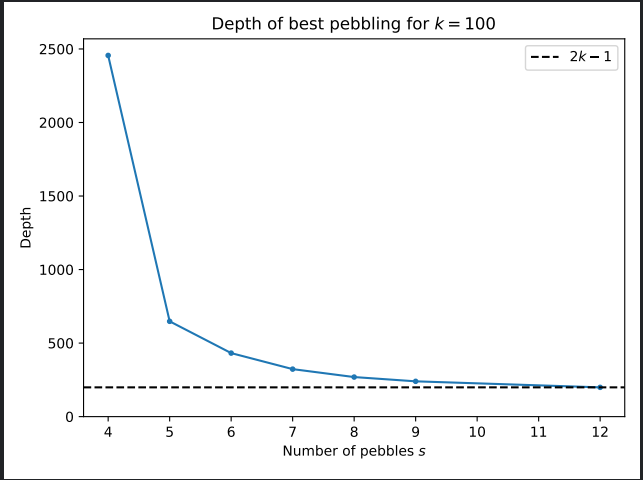
- Finds lowest-depth solution for *any* fixed number of pebbles s and length k

Numerical results



Pebbling summary

To pebble to length k:

Algorithm	Parallelism?	Spookiness?	Space	Depth
Bennett ’89	N	N	$O(2^{1/\epsilon} \cdot \log k)$	$O(k^{1+\epsilon})$

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Recap

Hope seemed lost...

Recap

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Recap

But then Bennett broke through the reversibility barrier with pebbling!

Recap

And Blocki, Holman, and Lee parallelized this!

Recap

Using spookiness, Gidney, Kornerup, Sadun, and Soloveichik brought even more hope

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Recap

And a ragtag crew of friends came along to put it all together 😃

Recap

Pebblers! Assemble.

Recap

Pebblers! ~~Assemble~~ Estimate concrete costs for RSA-2048 and RSA-4096.

Factoring results

For factoring 4096-bit RSA:

- all metrics counted in n-bit multiplications and evaluated per shot
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Our results (fewer pebbles)	561	600	$8n$

Note: this is shamelessly focusing on **per-shot depth/gates**, our best metrics...

Takeaways

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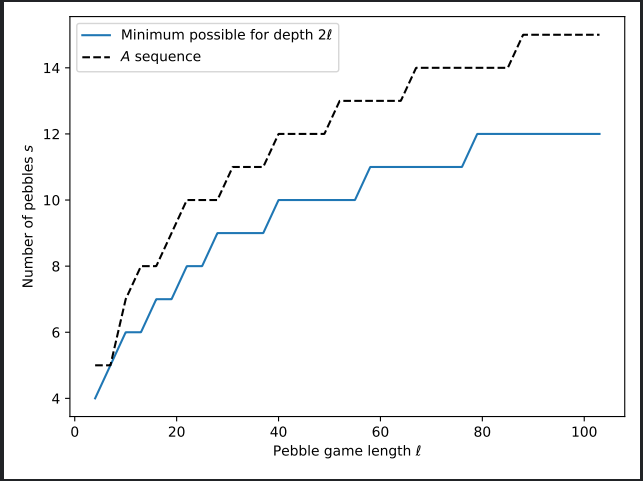
- "This paper creates **slack**" for further Regev modifications in the future
- Efficiently evaluating sequential algorithms is **widely applicable** beyond factoring
- **Generalization:** parallel spooky pebbling on arbitrary graphs

Thank you!

arXiv:2510.08432

Backup

Numerical results



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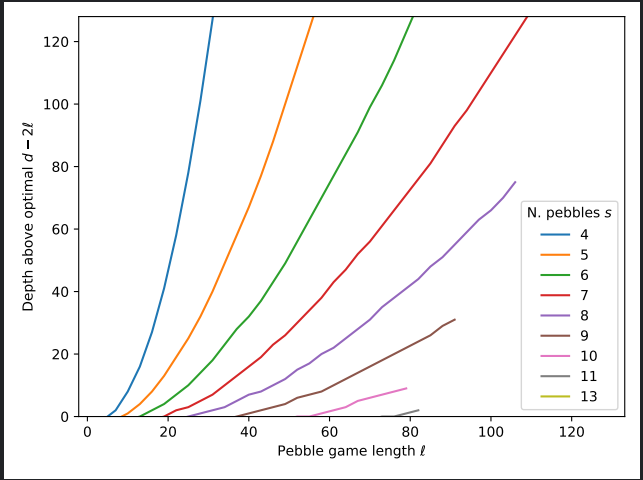

Making Shor reversible

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$|1\rangle$

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Break up x into its individual bits x_i:

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a^x \bmod N &= a^{\sum_i 2^i x_i} \bmod N \\
&= \prod_i a^{2^i x_i} \bmod N \\
&= \prod_i \left(a^{2^i}\right)^{x_i} \bmod N \\
&= \prod_i c_i^{x_i} \bmod N
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Each iteration is a controlled multiplication by classical c_i—which is reversible!