Quantum Factoring
(and why you should try to factor P^2Q)

Seyoon Ragavan (MIT)

Based on joint work with Gregory D. Kahanamoku-Meyer*, Vinod Vaikuntanathan*, and Katherine Van Kirk†
*MIT, †Harvard



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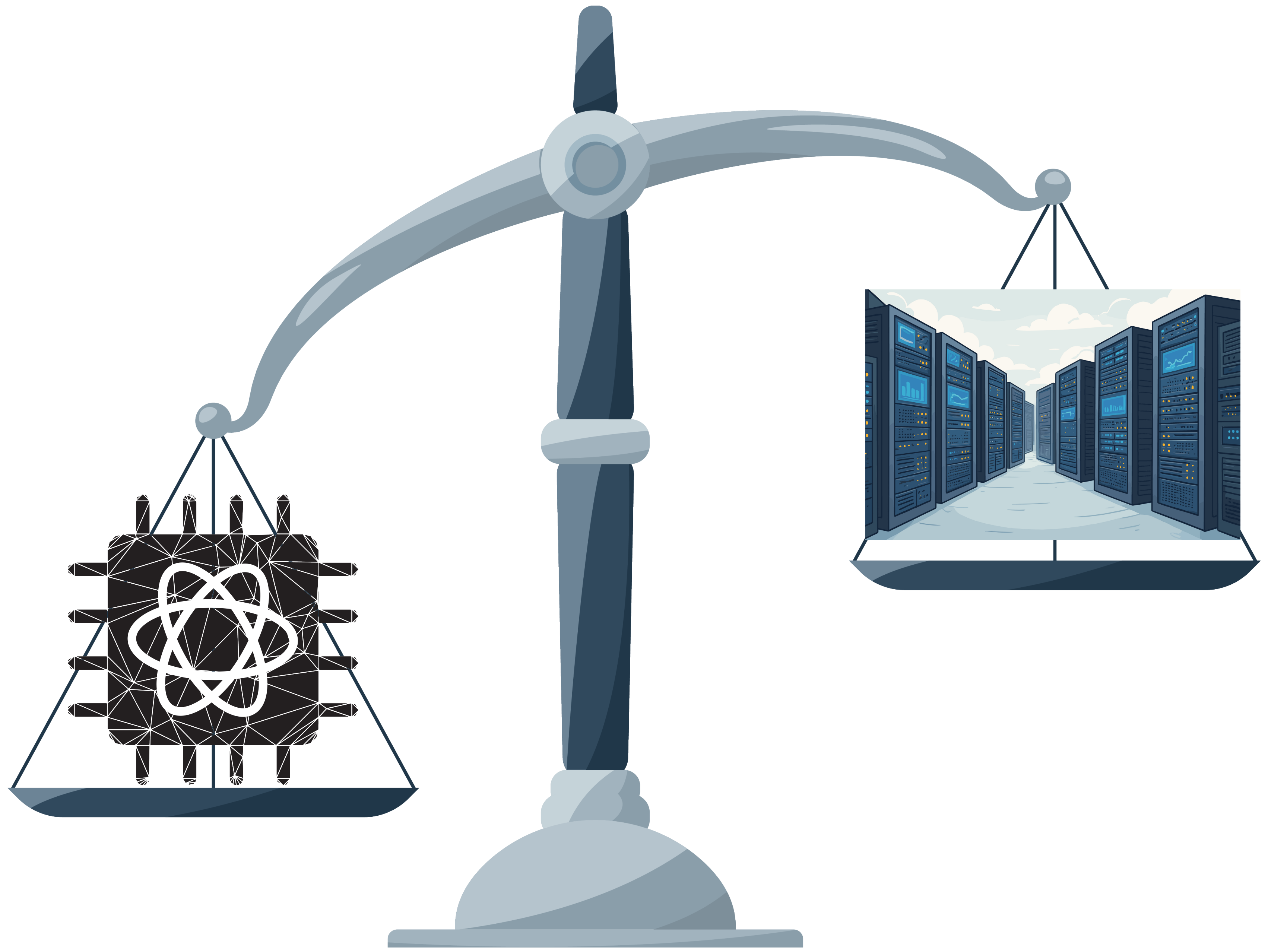

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Given a large integer N, find its prime factorisation in $\mathsf{poly}(\log N)$ time.

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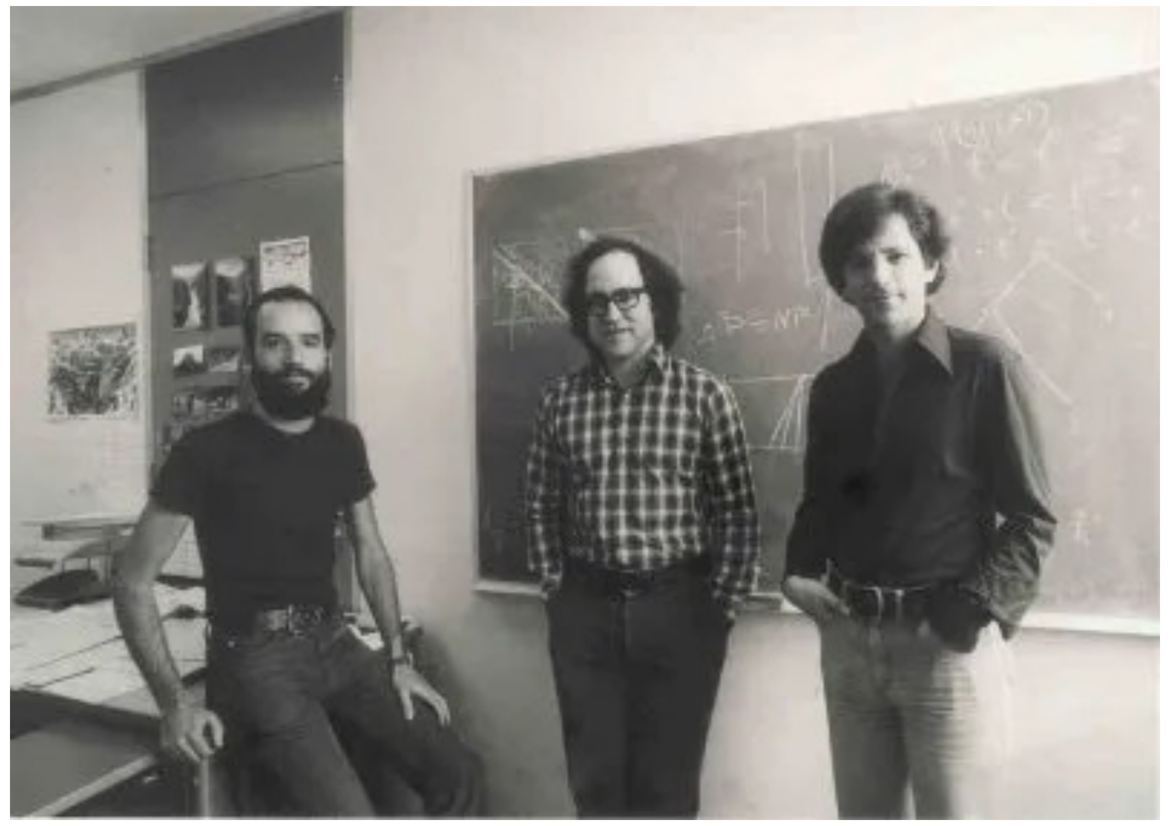

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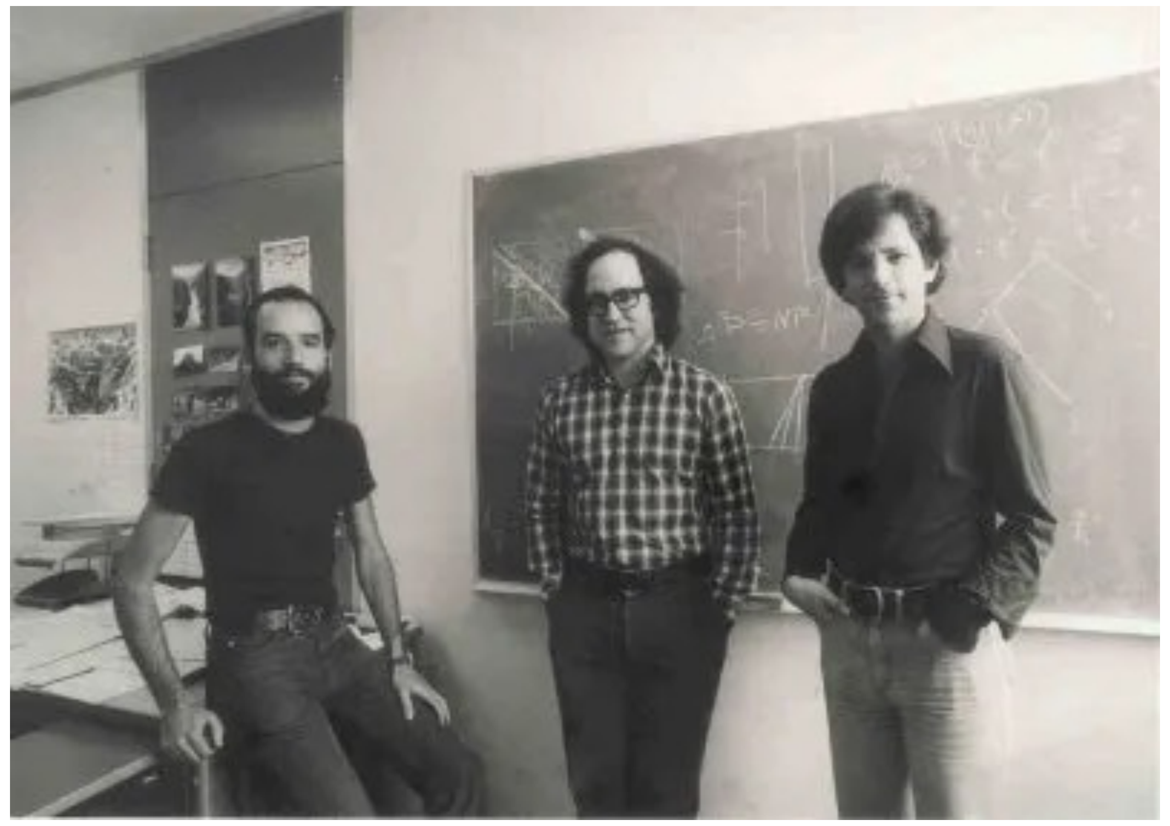

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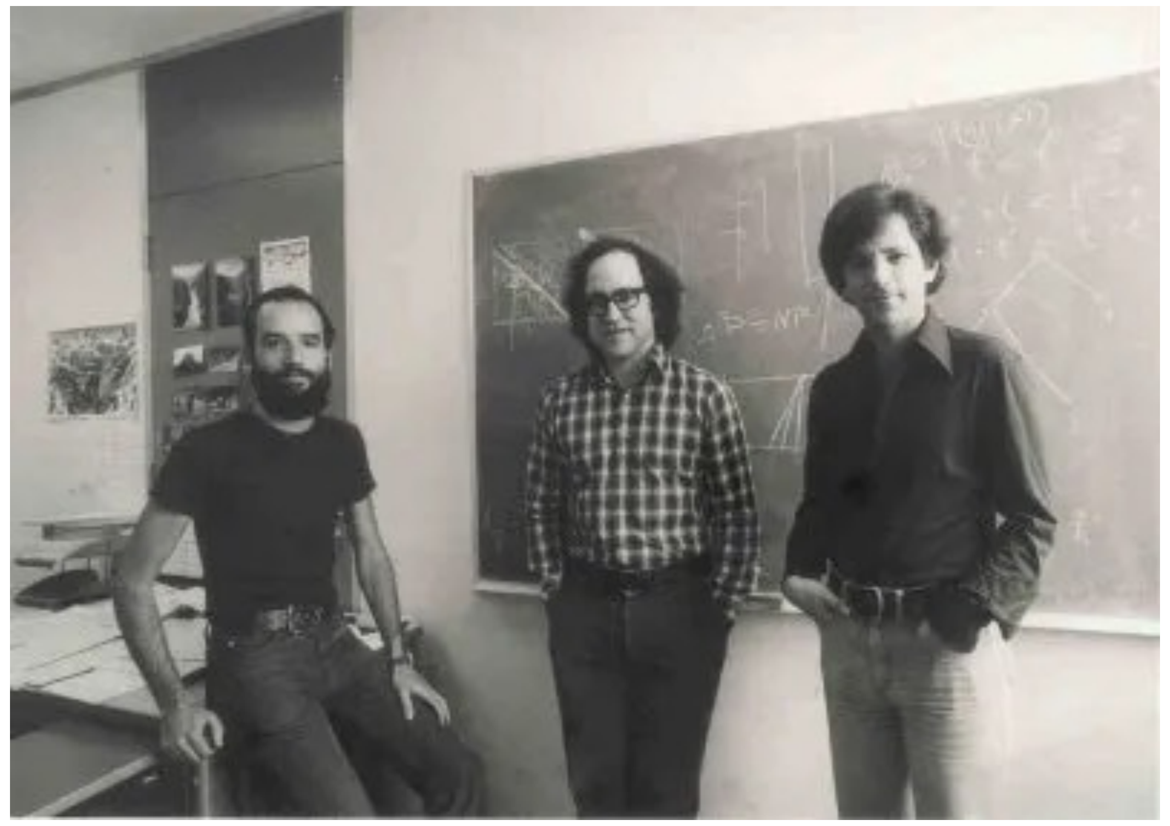
- $c > 0$ and $\alpha \in \left[0, 1\right]$
  - $\alpha = 0$: $(\log N)^c \to$ the dream
  - $\alpha = 1$: $N^c \to$ brute force



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  - Uses class groups of binary quadratic forms

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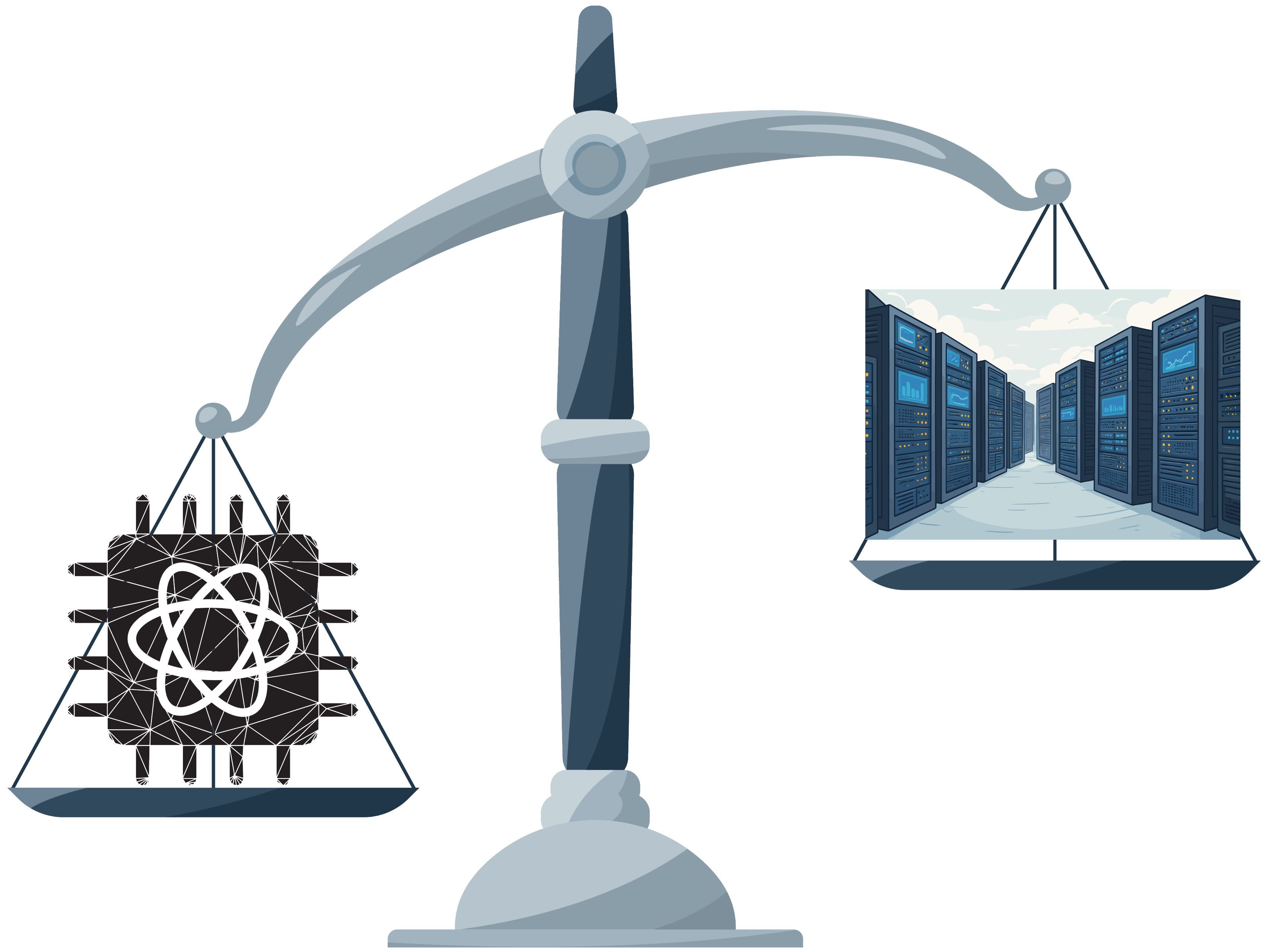

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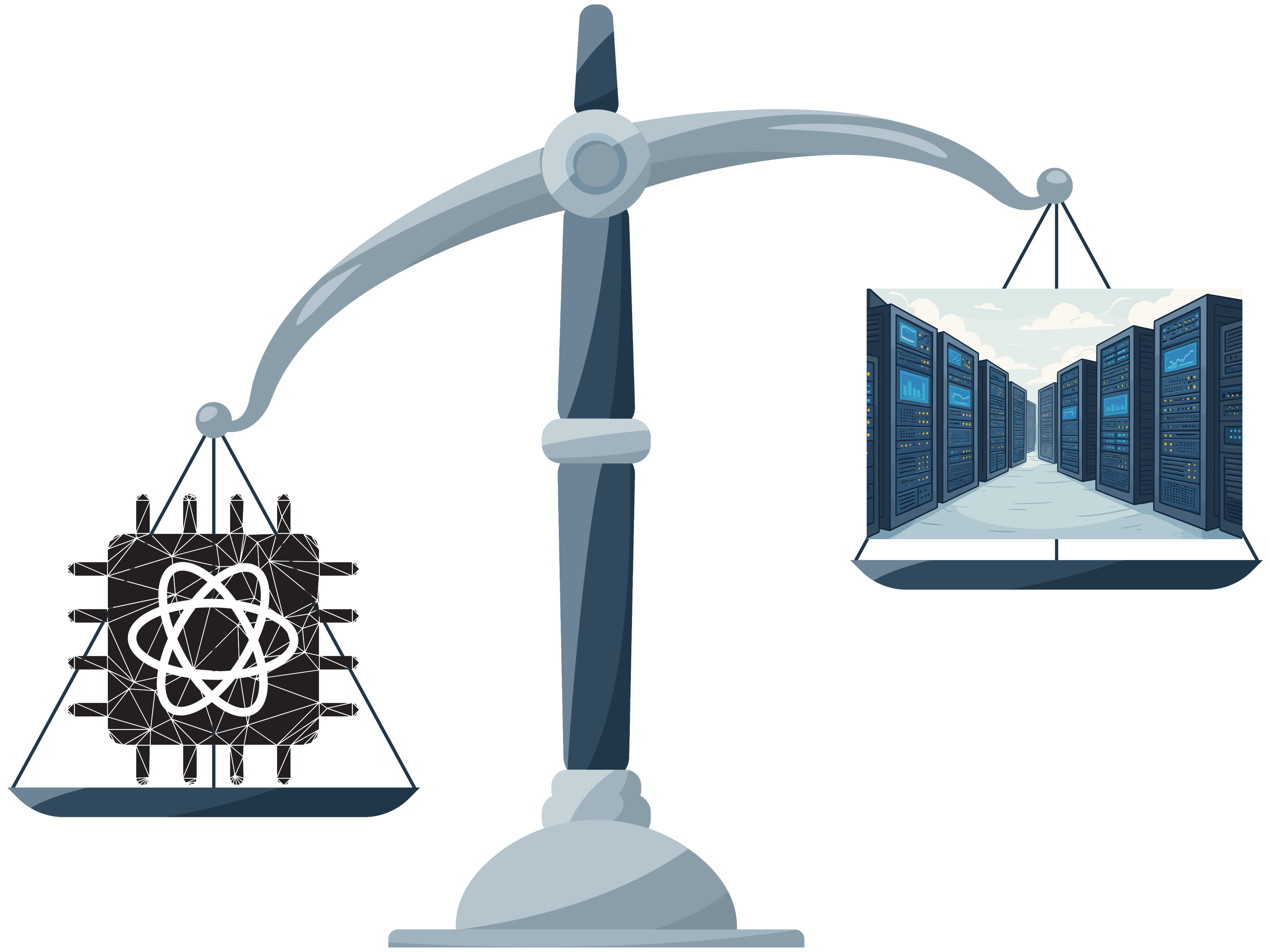

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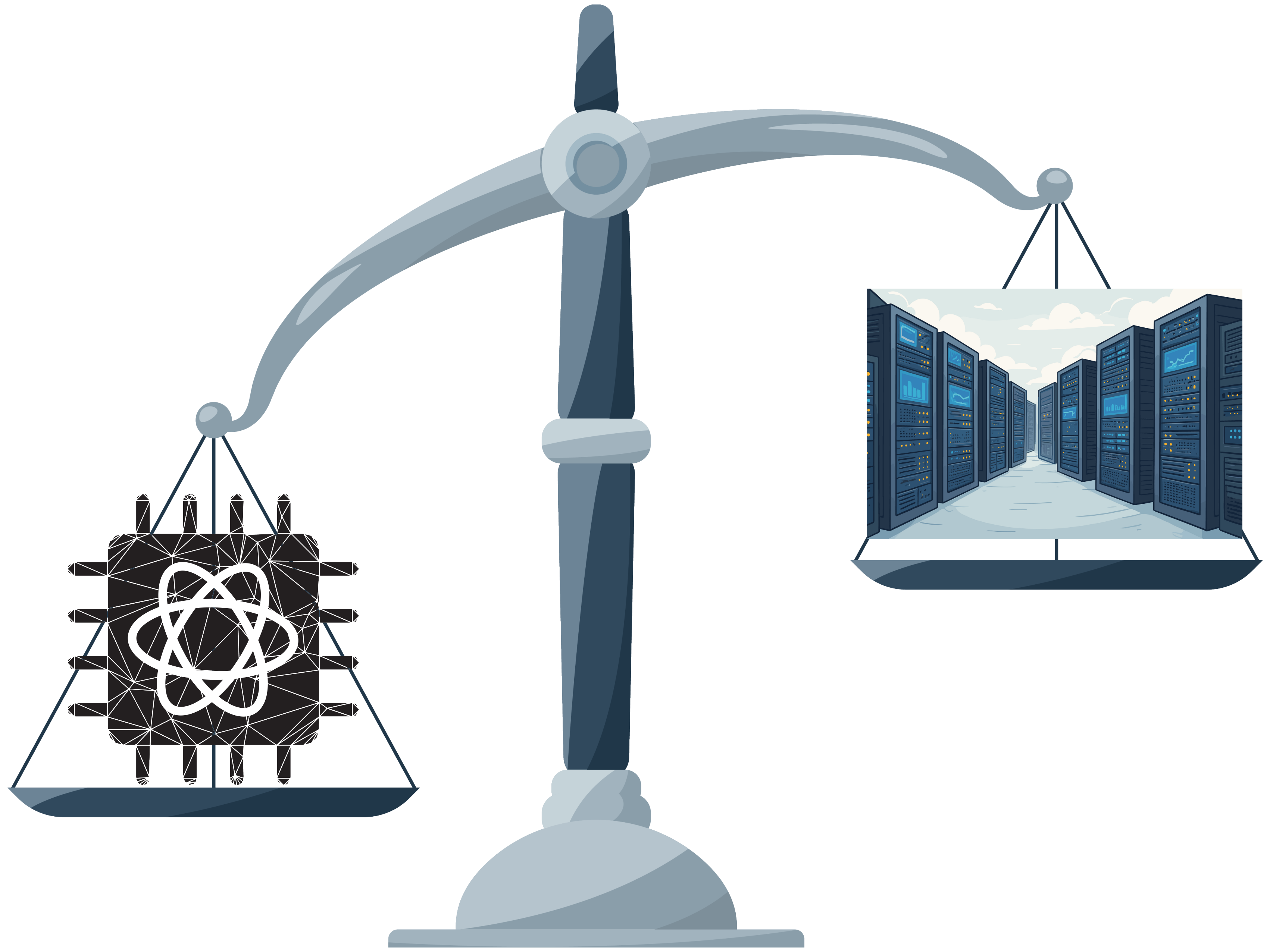

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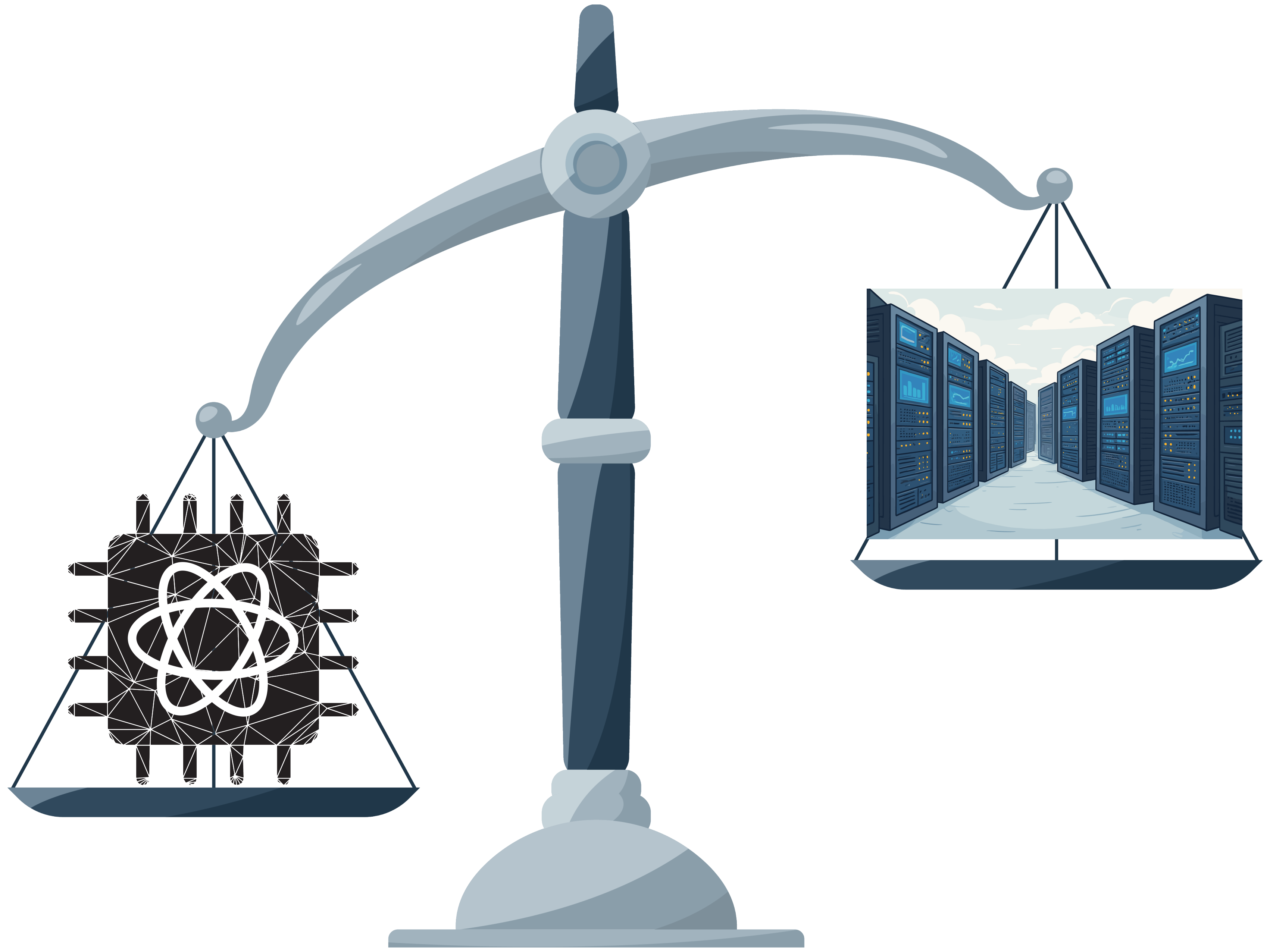

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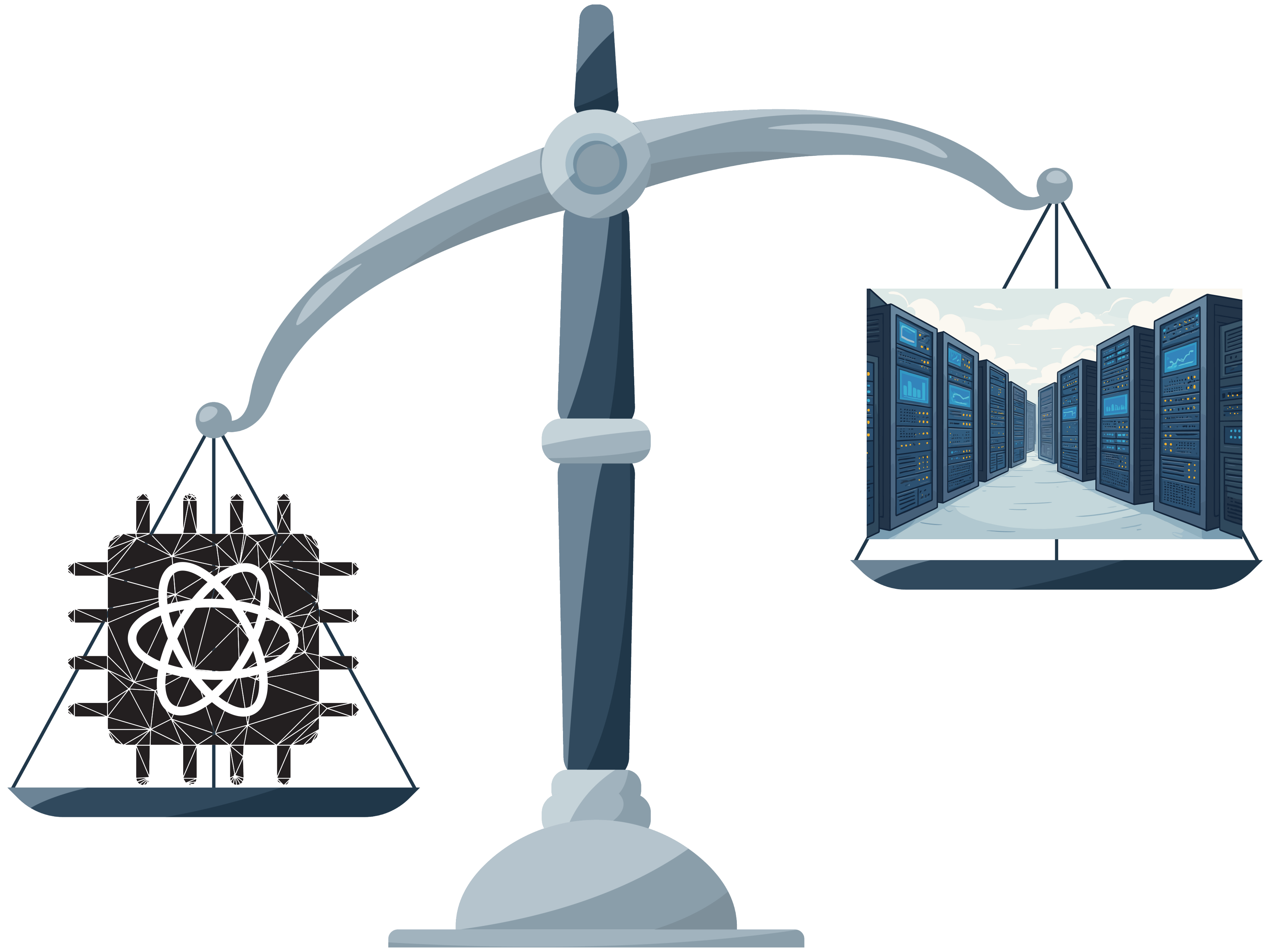

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- **Anything better?**


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ALGORITHMS FOR FACTORING P^2Q
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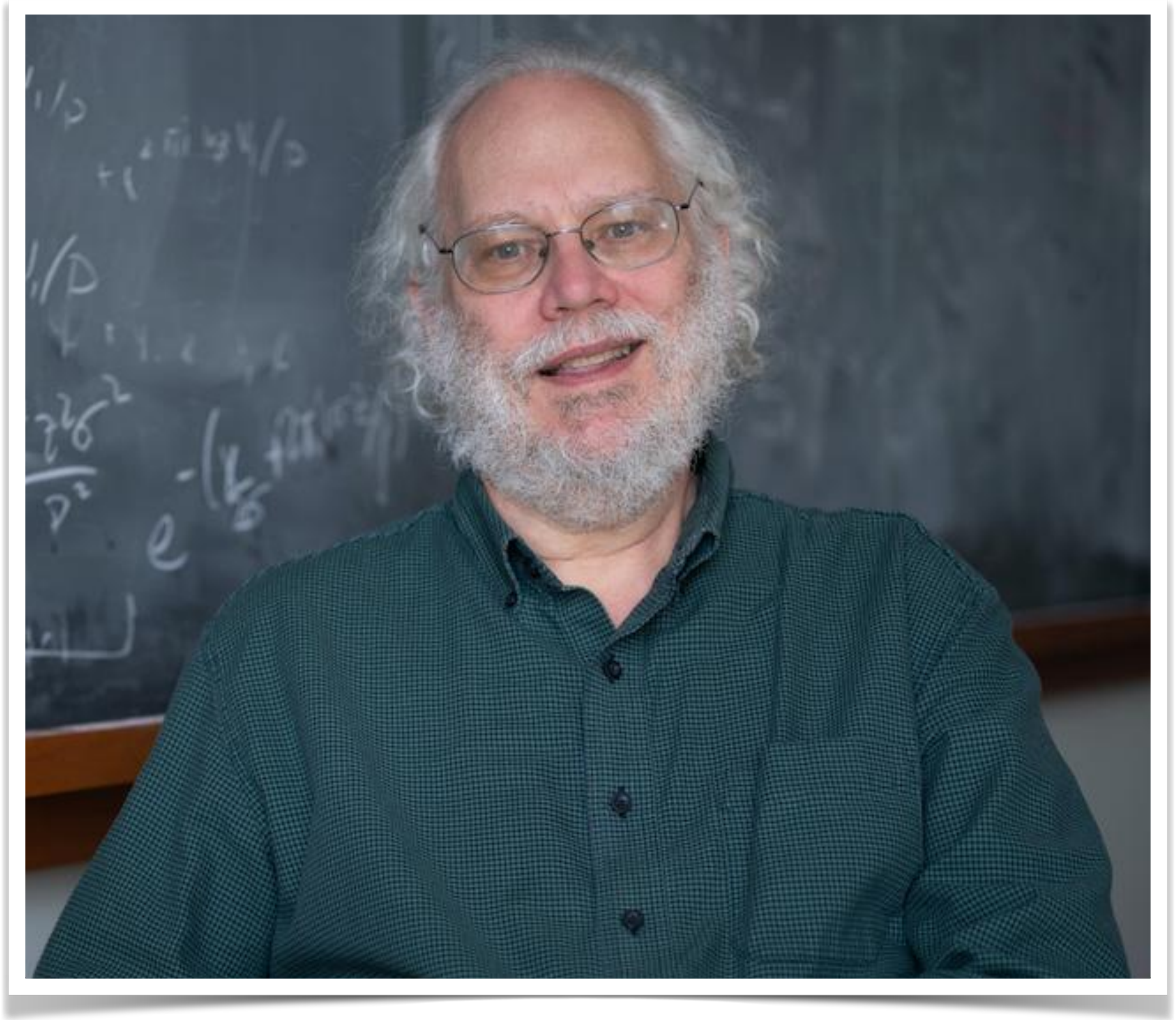
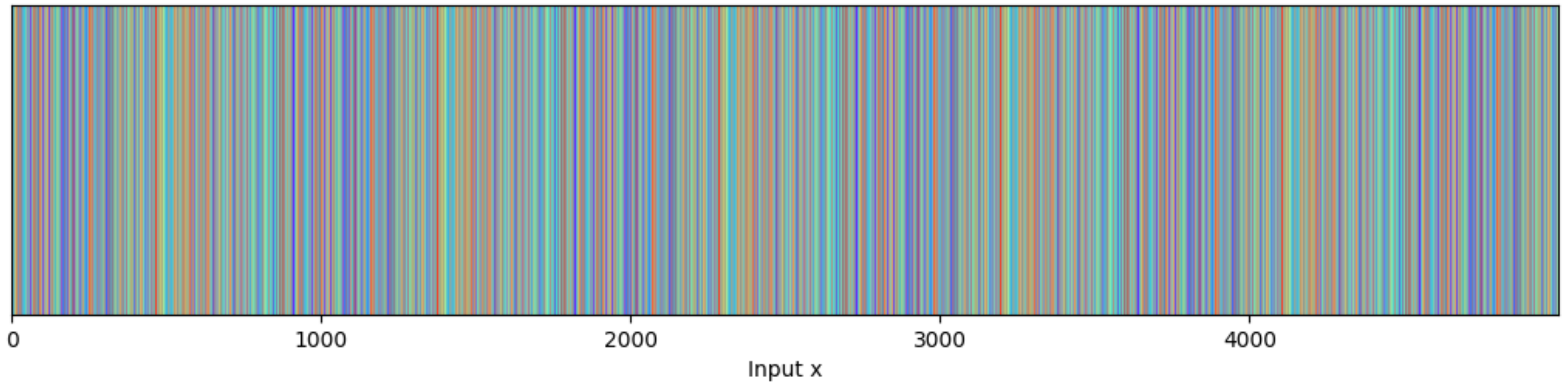
Shor's Algorithm: A Sketch



Preliminary: Quantum Period Finding

- Periodic function $f : \mathbb{Z} \to \mathbb{Z}$ with unknown period T
  - $x \equiv y \pmod{T} \Rightarrow f(x) = f(y)$



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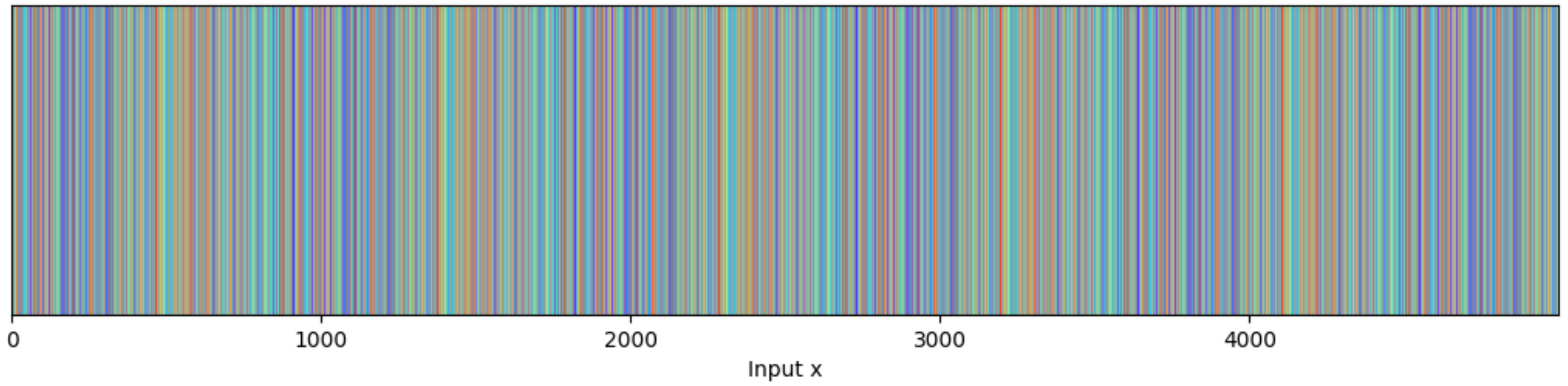

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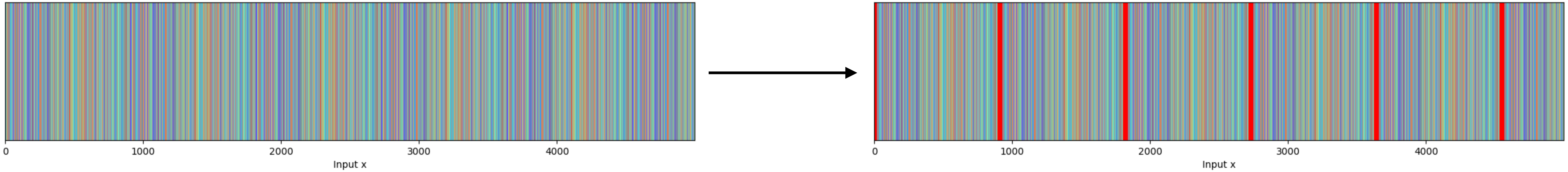
  - $x \equiv y \pmod{T} \Rightarrow f(x) = f(y)$
  - T is exponentially large
- Informal theorem statement: can quantumly recover a uniformly random multiple of $1/T$ (and hence T itself) using essentially only the time/space needed to compute $f(x)$ for $|x| \le \mathrm{poly}(T)$



Shor overview: finding square roots of 1

- Goal: find $z \not\equiv \pm 1 \bmod N$ such that $z^2 \equiv 1 \bmod N$
  - N divides $z^2 - 1 = (z-1)(z+1)$ but not either factor individually
  - Hence $\gcd(z-1, N)$ is a nontrivial divisor of N

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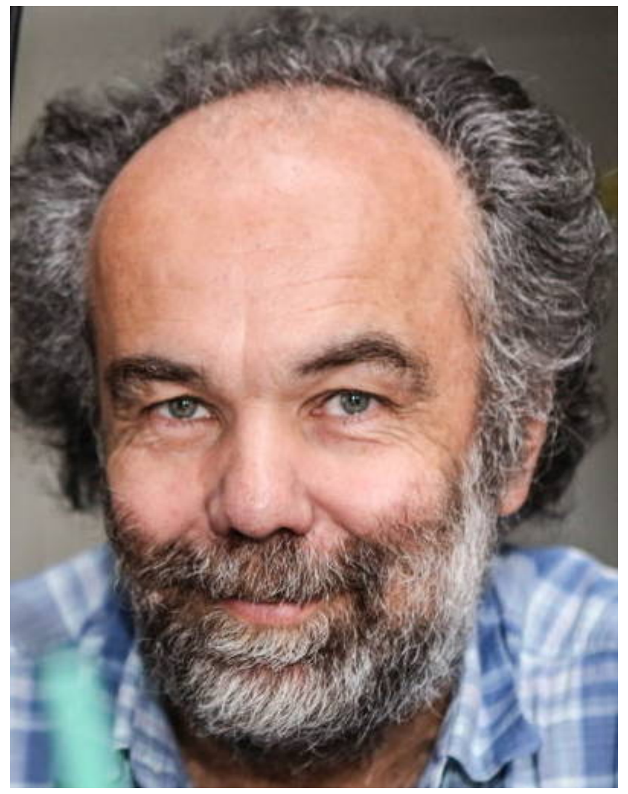
  - *"Luck" is with respect to the randomly chosen base*

Factoring P^2Q with LPDS12: A Sketch

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- Legendre symbol essentially indicates whether this is the case:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a nonzero quadratic residue modulo } p\text{; and} \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p\text{; and} \\ 0, \text{ if } a \text{ divisible by } p\,. \end{cases}$$

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- Theorem (from Euclid to Schönhage 1971): can compute $\left(\frac{a}{N}\right)$ efficiently without knowing the factorisation of N — in fact, in time $\tilde{O}(\log N)$

Factoring from Jacobi Symbol Periodicity

- For RSA integers ($N = PQ$): product of two periodic functions with smaller periods but itself only has period N

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- What about $N = P^2Q$?

$$\left(\frac{a}{N}\right) = \left(\frac{a}{P}\right)^2\left(\frac{a}{Q}\right) = \left(\frac{a}{Q}\right)$$, which is periodic* with period Q!

* modulo minor technical caveats; could have $\left(\frac{a}{P}\right) = 0$ for a tiny fraction of inputs a

Quantumly Factoring $N = P^2Q$

Li, Peng, Du, Suter (2012)

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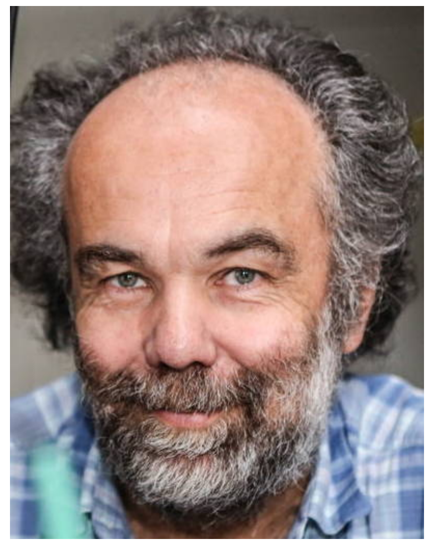

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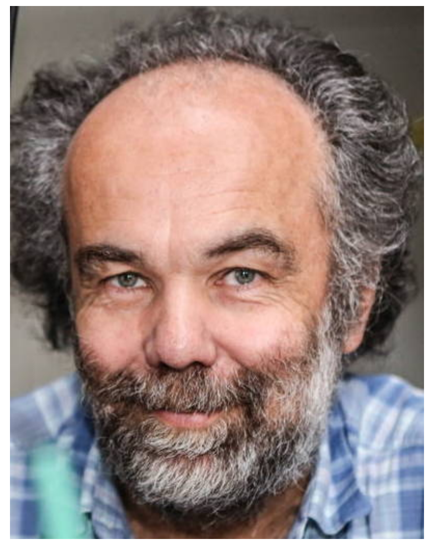

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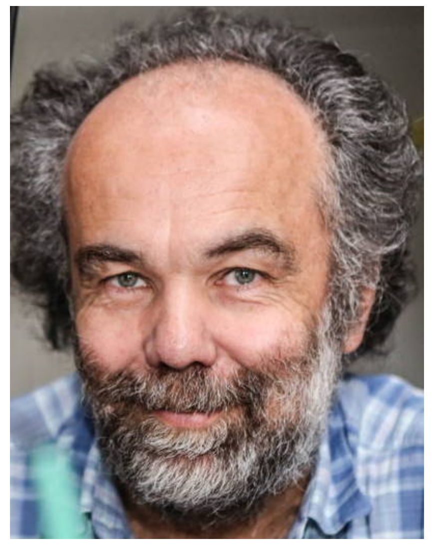

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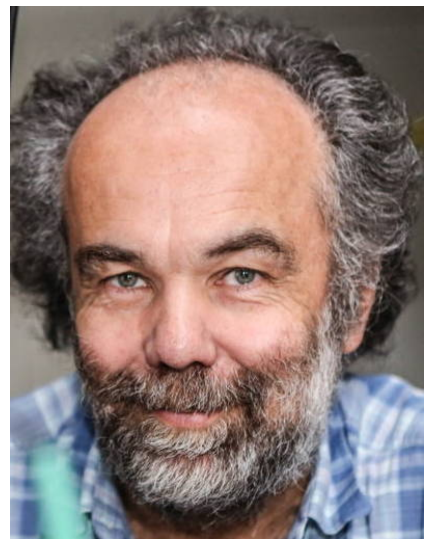
  - K**R**VV'25: the space can be brought down to $\tilde{O}(\log Q)$



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- The time needed to quantumly factor is $\tilde{O}\left((\log N)^{3/2}\right)$ (Regev)…
  - … unless N has a square factor, in which case we just need $\tilde{O}(\log N)$ time (Jacobi)

Thank you for your attention! Questions?

Bonus Slides

Algorithms Computing the Jacobi Symbol

ft. Euclid, 2000 years ago

<u>Jacobi Properties</u>

- Periodicity: $\left(\frac{a}{b}\right) = \left(\frac{a \bmod b}{b}\right)$
- Reciprocity:* $\left(\frac{a}{b}\right) = (-1)^{f(a,b)}\left(\frac{b}{a}\right)$

  for a very simple f

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$$f(a,b) = \begin{cases} 0, \text{ if } a \equiv 1 \pmod 4 \text{ or } b \equiv 1 \pmod 4 \\ 1, \text{ if } a \equiv b \equiv 3 \pmod 4 \end{cases}$$

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Greatest Common Divisor (GCD) Properties

- Periodicity:
  $\gcd(a, b) = \gcd(a \bmod b, b)$
- Reciprocity: $\gcd(a, b) = \gcd(b, a)$

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Algorithms Computing the Jacobi Symbol

ft. Euclid, 2000 years ago

<u>Jacobi Properties</u>

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  for a very simple f

<u>Greatest Common Divisor (GCD) Properties</u>

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Extended Euclidean algorithm solves both these problems!

* modulo minor technical caveats; requires a, b odd

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- Extended Euclidean recursion: if $a < b$, swap a, b. Else, update $a \leftarrow a \bmod b$.
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Acta Informatica 1, 139—144 (1971)

Schnelle Berechnung von Kettenbruchentwicklungen

A. Schönhage

Eingegangen am 16. September 1970

Summary. A method, given by D. E. Knuth for the computation of the greatest common divisor of two integers u, v and of the continued fraction for u/v is modified in such a way that only $O(n(\lg n)^2(\lg\lg n))$ elementary steps are used for $u, v < 2^n$.

Zusammenfassung. Ein von D. E. Knuth angegebenes Verfahren, für ganze Zahlen u, v den größten gemeinsamen Teiler und den Kettenbruch für u/v zu berechnen, wird so modifiziert, daß für n-stellige Zahlen nur $O(n(\lg n)^2(\lg\lg n))$ elementare Schritte gebraucht werden.

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A Unified Approach to HGCD Algorithms for polynomials and integers

Klaus Thull and Chee K. Yap*

Freie Universität Berlin
Fachbereich Mathematik
Arnimallee 2-6
D-1000 Berlin 33
West Germany

March, 1990

Abstract

We present a unified framework for the asymptotically fast Half-GCD (HGCD) algorithms, based on properties of the norm. Two other benefits of our approach are (a) a simplified correctness proof of the polynomial HGCD algorithm and (b) the first explicit integer HGCD algorithm. The integer HGCD algorithm turns out to be rather intricate.

Keywords: Integer GCD, Euclidean algorithm, Polynomial GCD, Half GCD algorithm, efficient algorithm.

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ON SCHÖNHAGE'S ALGORITHM AND SUBQUADRATIC INTEGER GCD COMPUTATION

NIELS MÖLLER

Abstract. We describe a new subquadratic left-to-right GCD algorithm, inspired by Schönhage's algorithm for reduction of binary quadratic forms, and compare it to the first subquadratic GCD algorithm discovered by Knuth and Schönhage, and to the binary recursive GCD algorithm of Stehlé and Zimmermann. The new GCD algorithm runs slightly faster than earlier algorithms, and it is much simpler to implement. The key idea is to use a stop condition for HGCD that is based not on the size of the remainders, but on the size of the next difference. This subtle change is sufficient to eliminate the back-up steps that are necessary in all previous subquadratic left-to-right GCD algorithms. The subquadratic GCD algorithms all have the same asymptotic running time, $O(n(\log n)^2 \log\log n)$.

Our Result: Pushing the Space and Depth Down to $\tilde{O}(\log Q)$

   

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- Recall: to solve period finding when the period is T, need to set up a superposition

$$\sum_{a=1}^{\mathrm{poly}(T)} |a\rangle |f(a)\rangle$$

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- **<u>Hope 1:</u>** when factoring P^2Q with Jacobi: the period is just $Q \rightarrow O(\log Q)$ qubits could suffice!

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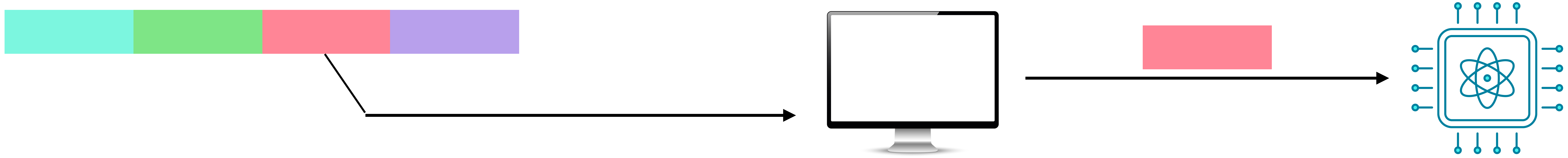
Quantum computer with $\tilde{O}(\log Q)$ qubits



Classical computer sending instructions to the quantum computer

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The "only" bottleneck: computing $|a\rangle \mapsto |a\rangle|N \bmod a\rangle$

Our Result, Distilled

- Theorem (K**R**VV24): for quantum a and classically known N, we can compute

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in $\tilde{O}(\log N)$ gates (near-linear) and $\tilde{O}(\log a)$ qubits (enough qubits to write down a)

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Open question: other applications of these results?

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- **Corollary 2: we can factor $N = P^2Q$ in $\tilde{O}(\log N)$ gates and $\tilde{O}(\log Q)$ qubits**
  - **Just need the above theorem for $a \leq \mathsf{poly}(Q)$**

Computing N mod a with Quantum Streaming

Notation: N has n bits, a has $m = O(\log Q)$ bits

Bits of N, split into chunks of size $O(\log Q)$

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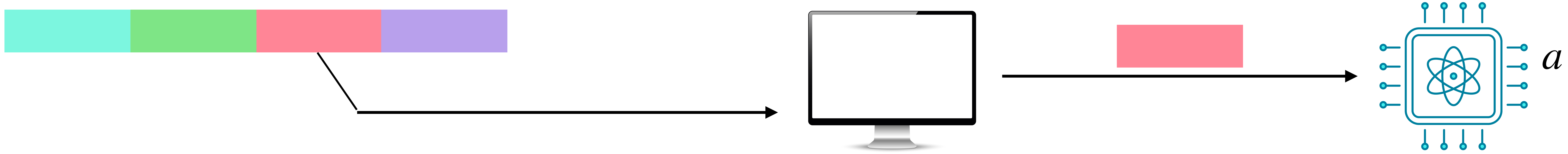

Classical computer sending instructions to the quantum computer

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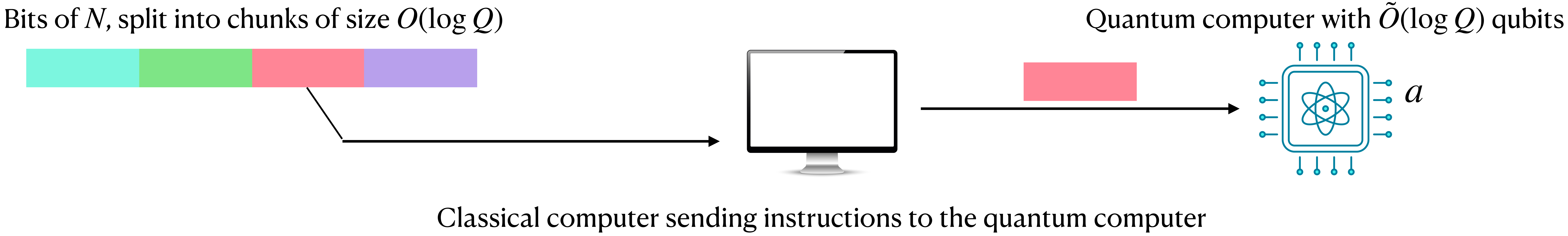

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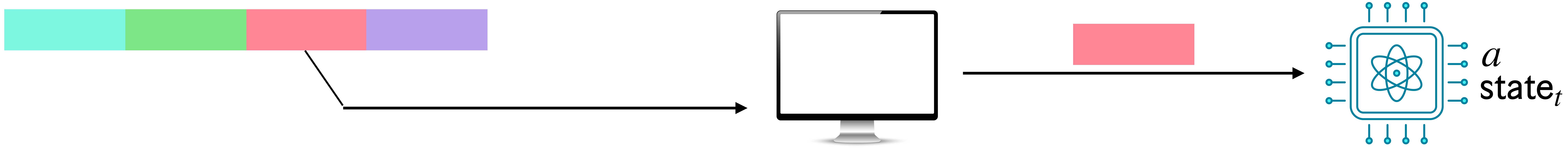

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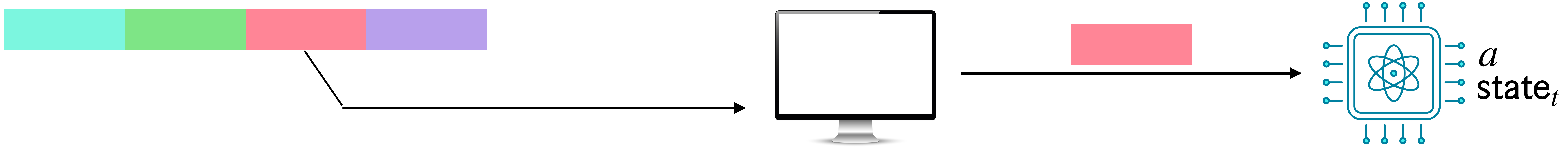

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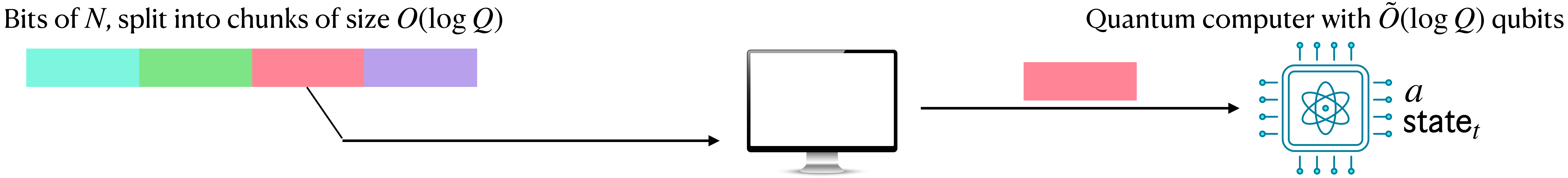

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    - **Reversibility:** state_{t-1} can be reconstructed (and therefore uncomputed) from state_t

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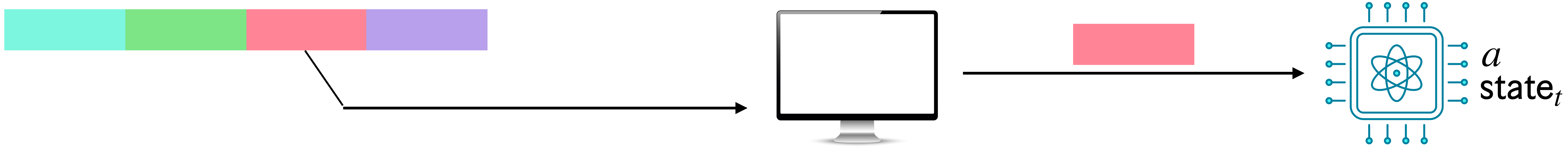

Classical computer sending instructions to the quantum computer

Our Construction, Simplified

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- It turns out that the final state $\mathsf{state}_{n/m}$ suffices to reconstruct N mod a

Our Result: Improving Space Complexity of Regev

Regev's qubit problem

- Performance bottleneck: computing $(z_1, \ldots, z_d) \mapsto a_1^{z_1} \ldots a_d^{z_d} \bmod N$
- Repeated squaring mod N cannot be done in place!

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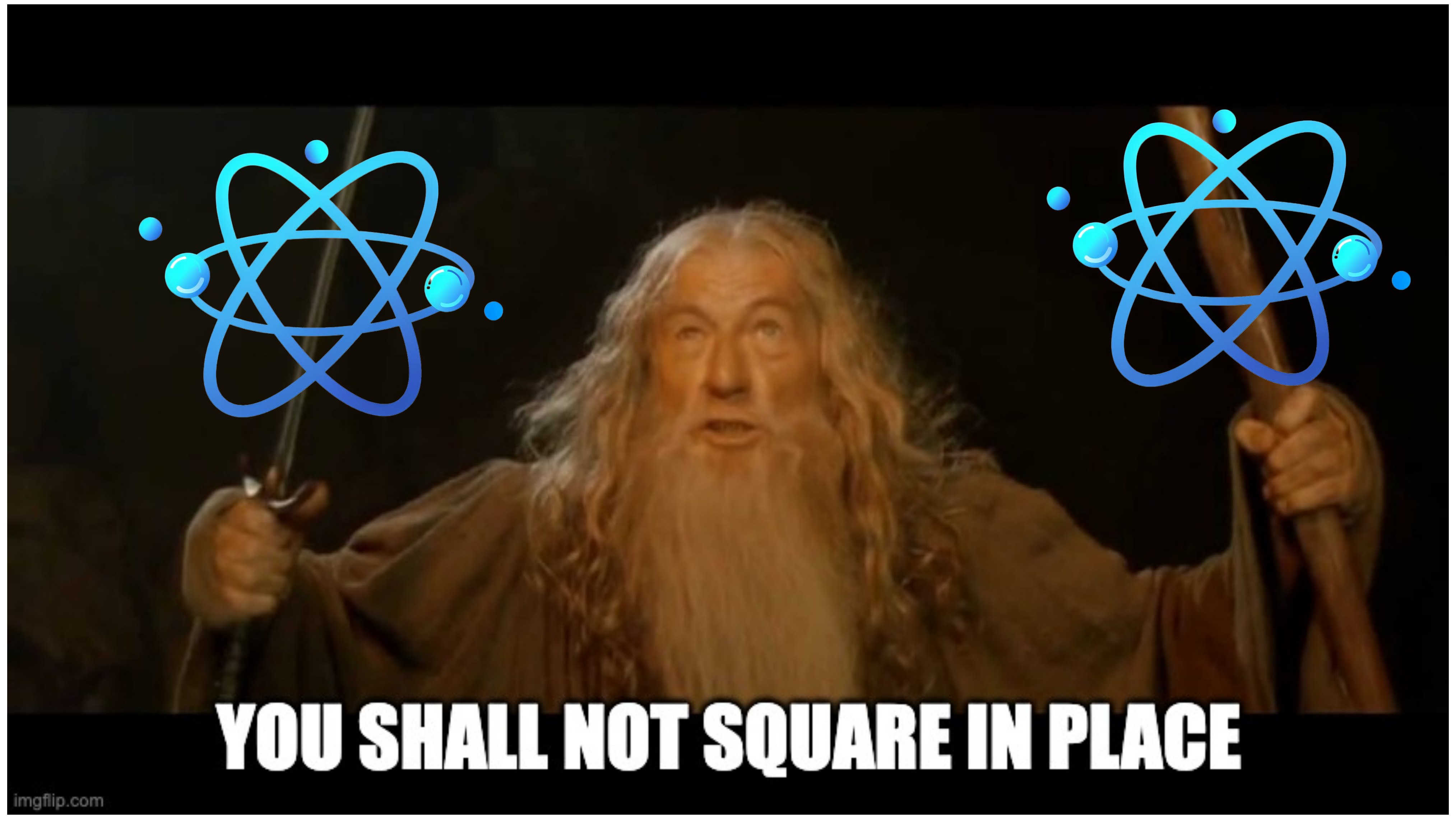
  - Quantum circuits need to be reversible, but squaring mod N is not e.g. $-1, 1$



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Each squaring adds n qubits $\rightarrow O\left(\frac{n}{d} \times n\right) = O(n^{3/2})$ qubits total.

Main Idea: Fibonacci Exponentiation

- What if we used two accumulators?

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We can efficiently compute a^{F_k} for any Fibonacci number F_k!

Regev's Number-Theoretic Assumption

- Regev: relies on $(z_1, \ldots, z_d) \mapsto a_1^{z_1} \ldots a_d^{z_d} \bmod N$ having periods of size $2^{O(n/d)}$
- **But these periods could just yield a trivial square root of 1** $\bmod N$
- Regev relies on a conjecture that *at least one* small period yields a non-trivial square root of 1
- Follow-up work by Pilatte proves* Regev's conjecture

** proves correctness for a variant of Regev's algorithm that is worse by polylog factors and likely impractical*