Factoring with a Quantum Computer: The State of the Art

Seyoon Ragavan (MIT)

Based on joint works with Gregory D. Kahanamoku-Meyer*, Vinod Vaikuntanathan*, and Katherine Van Kirk†

*MIT, †Harvard

Integer Factorisation

Given an n-bit integer $N < 2^n$, find its prime factorisation in $\mathsf{poly}(n)$ time.

Classical Factoring: A Crash Course

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For more: see Pomerance's survey "A Tale of Two Sieves"!

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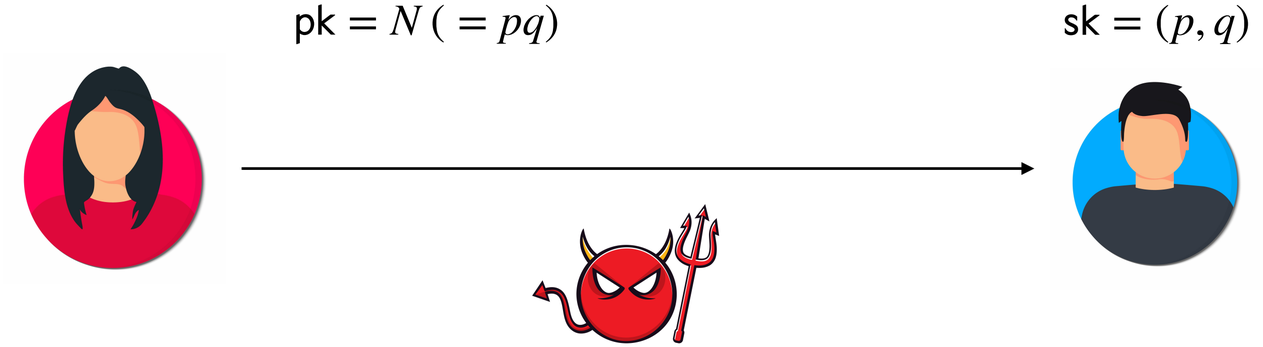
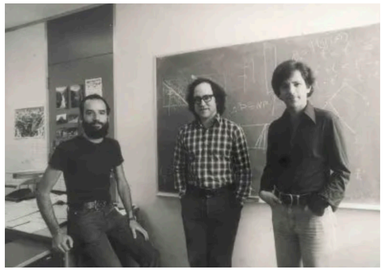
- **Quantum algorithms:** poly(n) **time!** (Shor 1994)

Implication 1: Breaking Cryptography

- RSA public-key cryptography:





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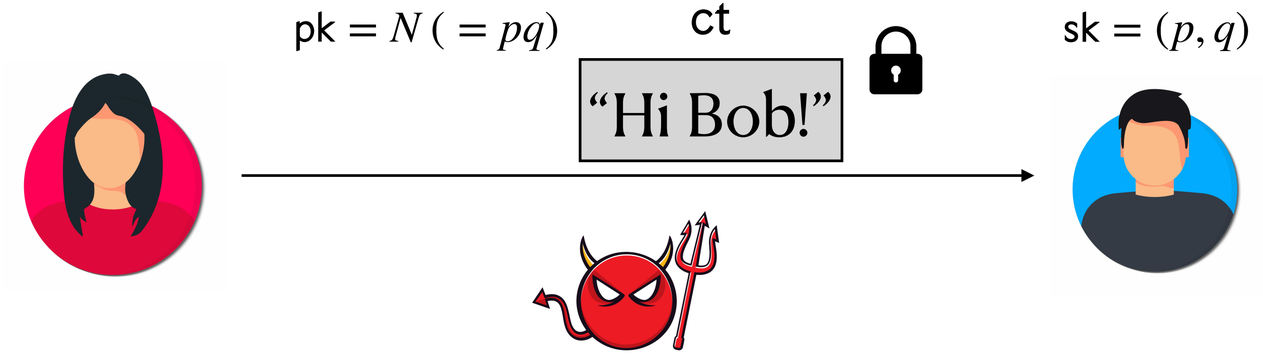

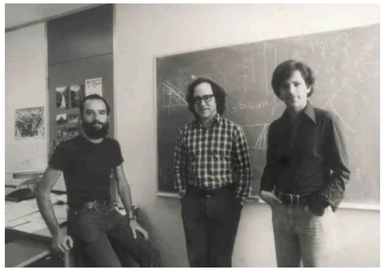

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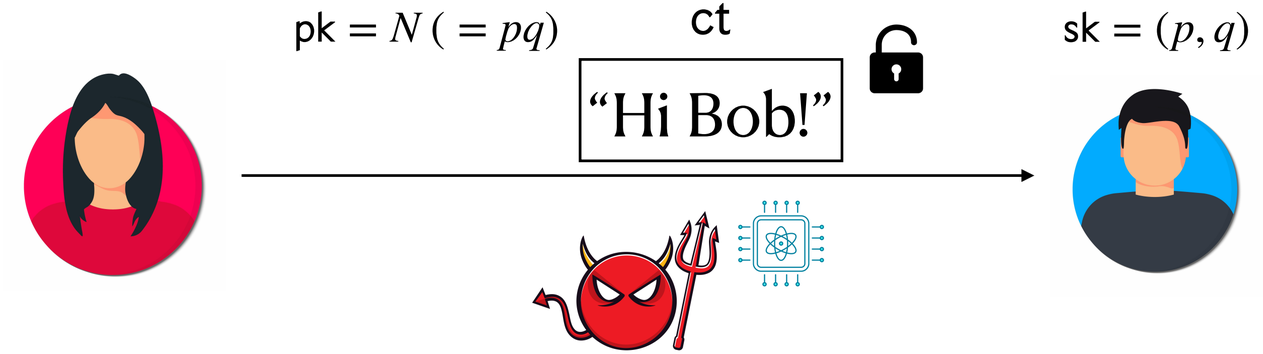

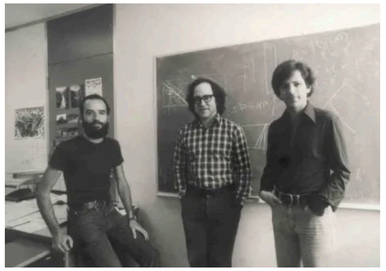

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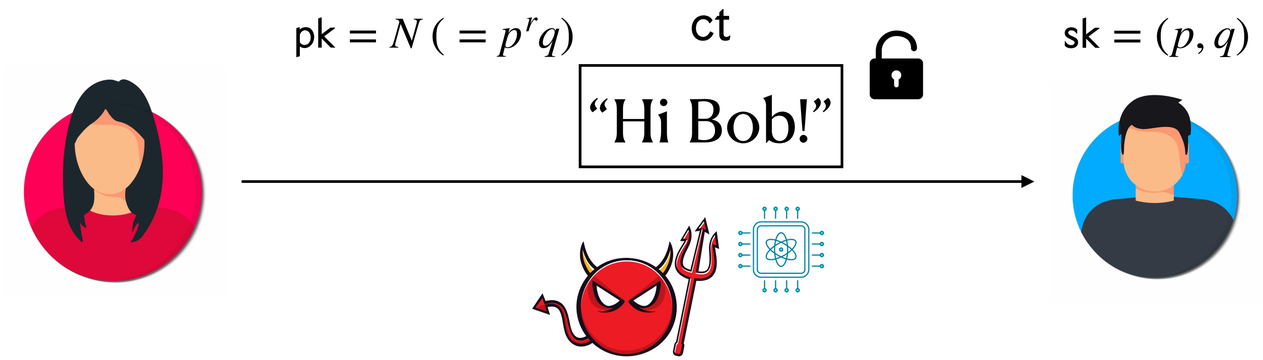
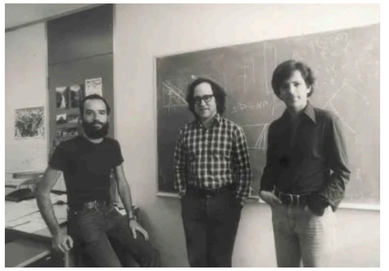
Goal: faster decryption than RSA





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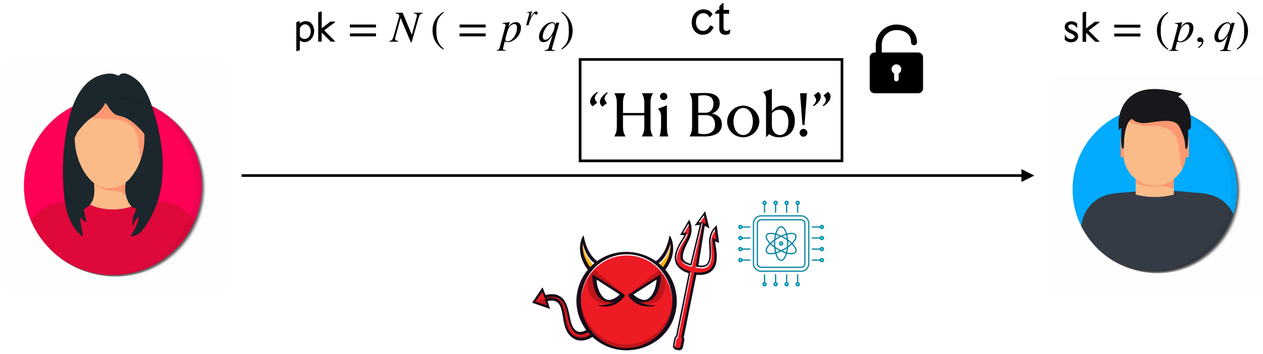

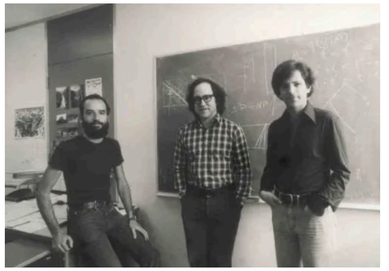

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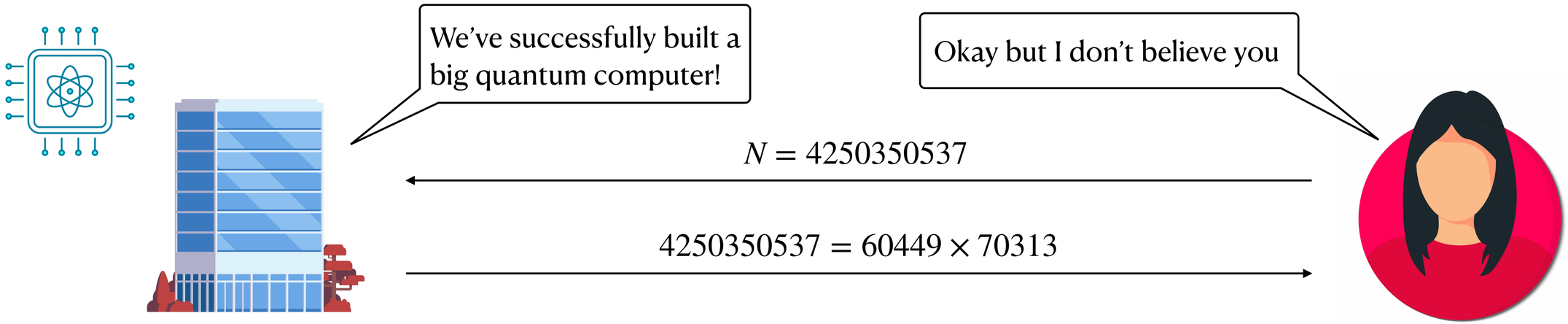
Coming up: even better quantum circuits for factoring $p^r q$ ($r > 1$)

Implication 2: Proofs of Quantumness



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One answer: By factoring a large integer of Alice's choice!

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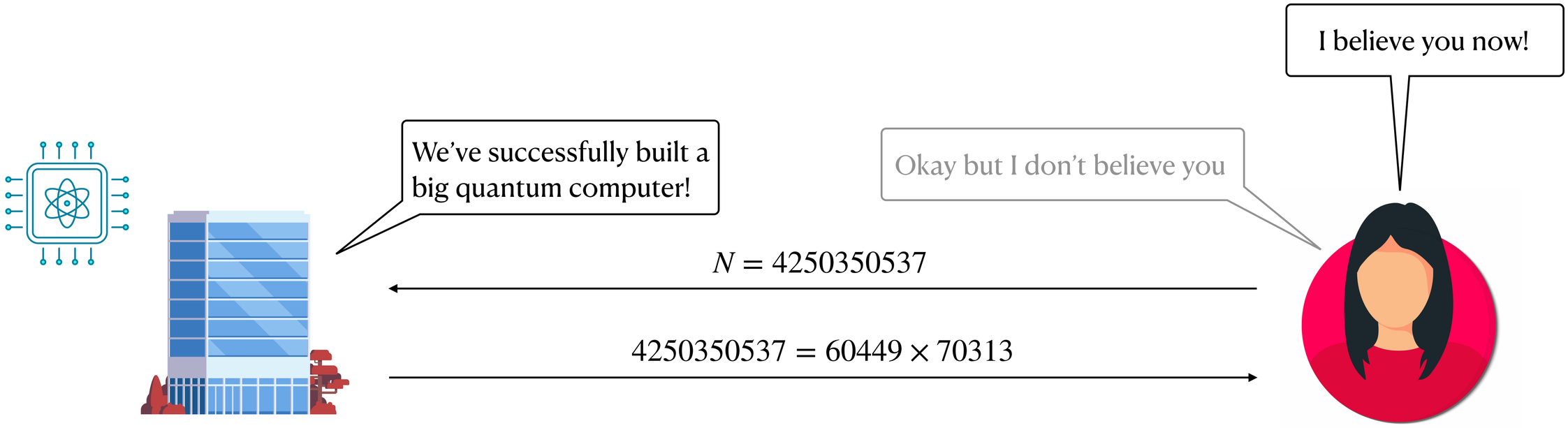

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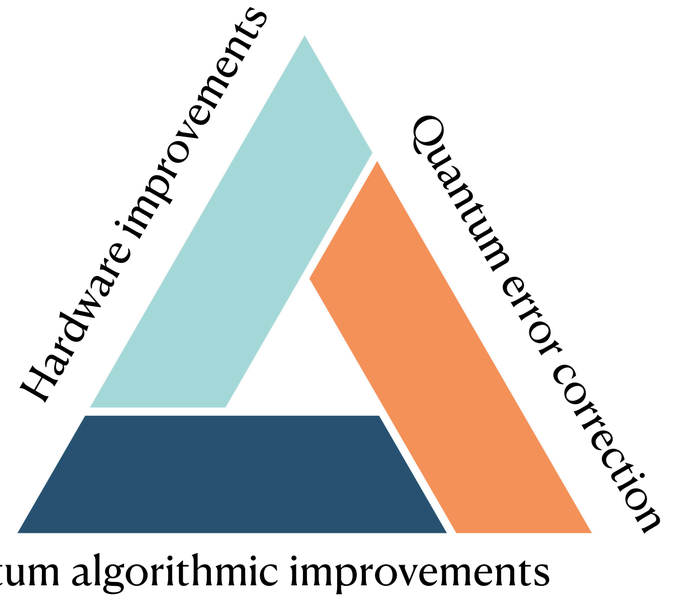
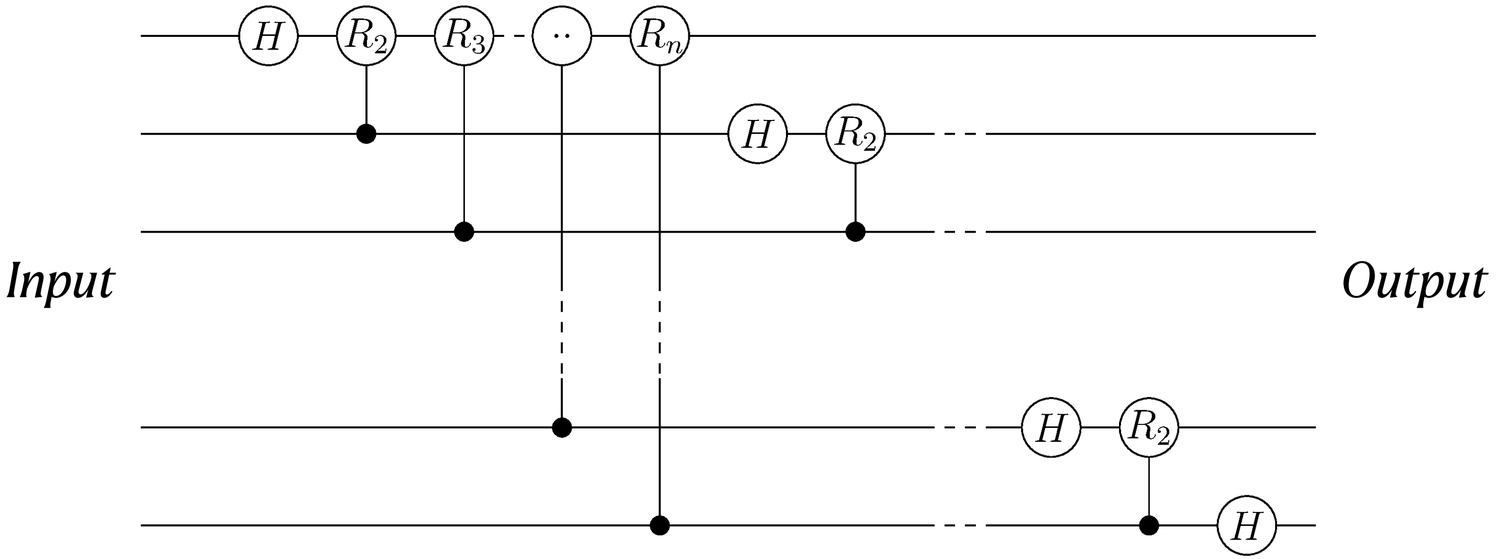
"Our qubits are constantly trying to fall apart… It's like you're trying to write in the sand and the wind is blowing it away."



Efficiency Metrics for Quantum Circuits

- Similar to a classical circuit, but uses reversible logic gates on a fixed number of wires



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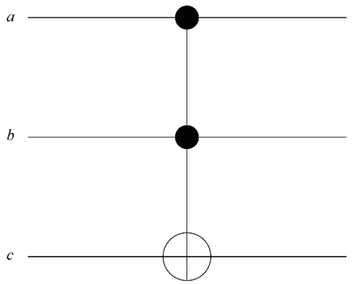

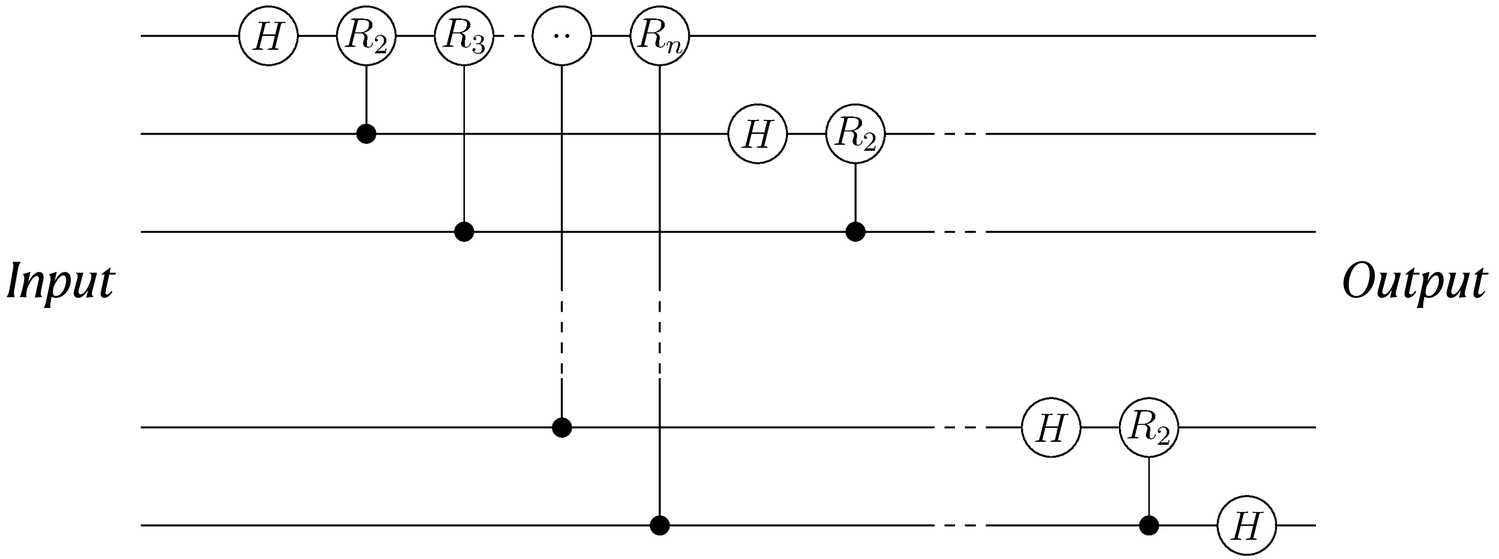

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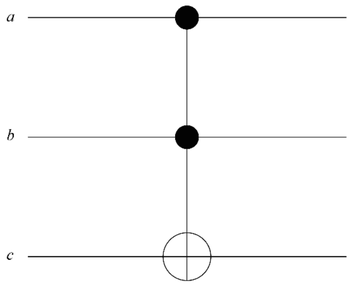

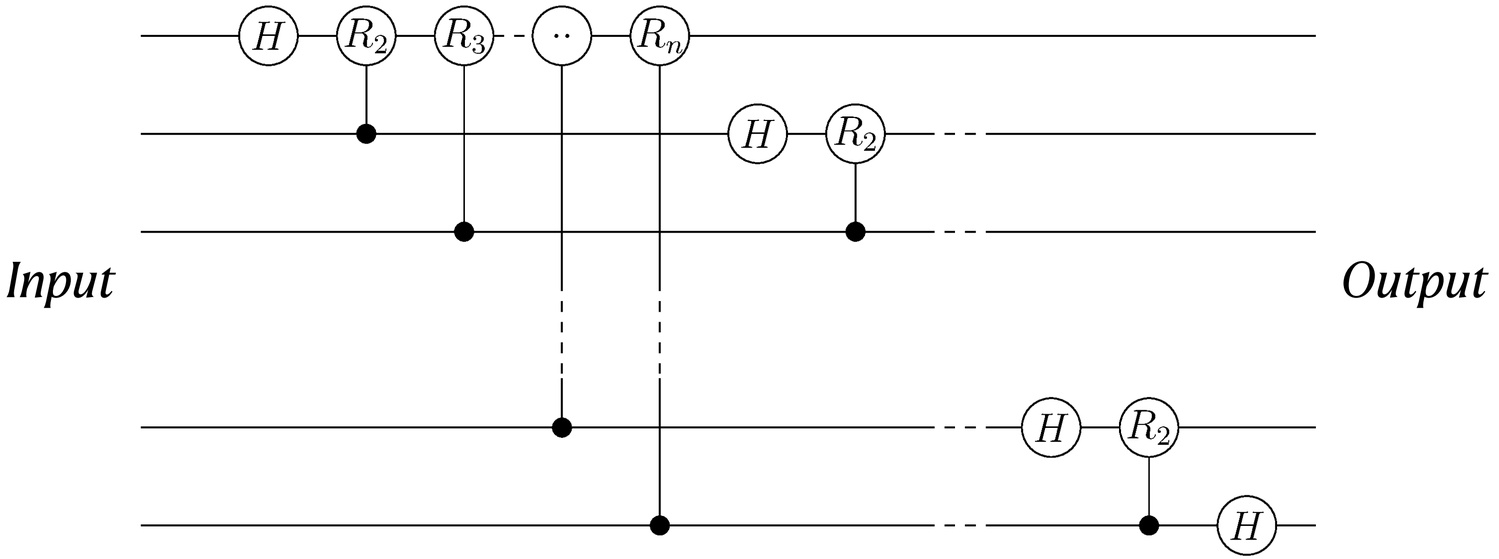

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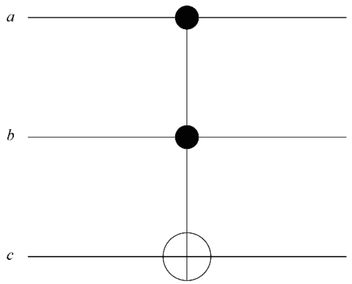

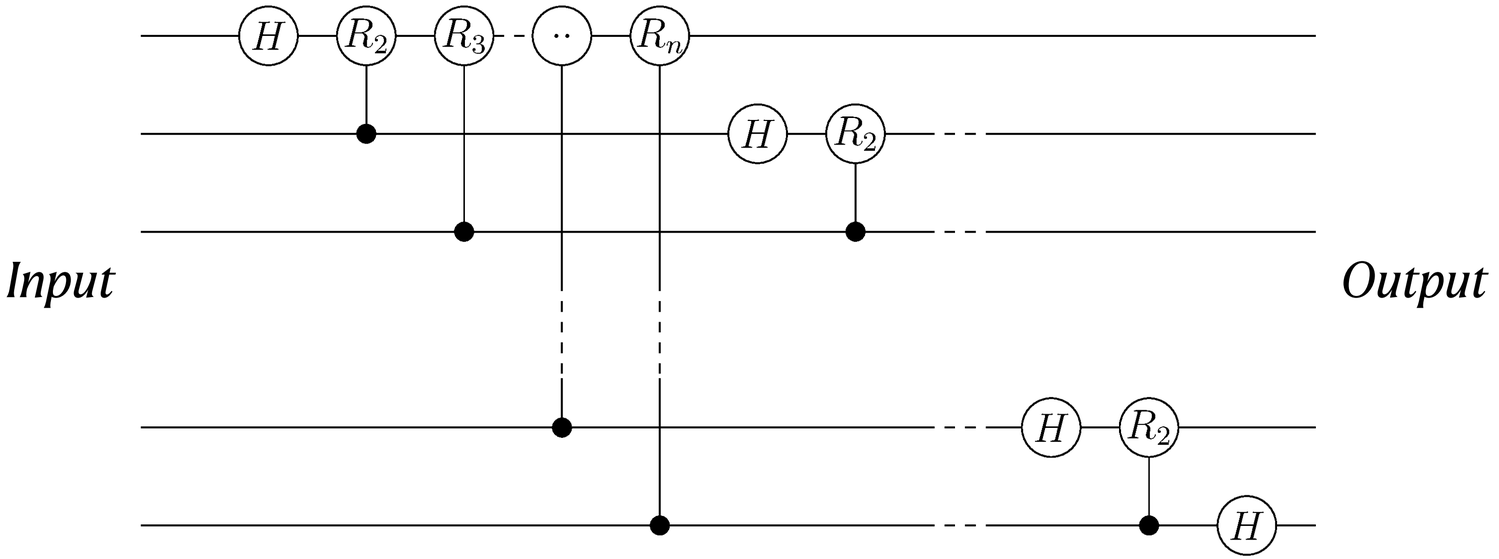

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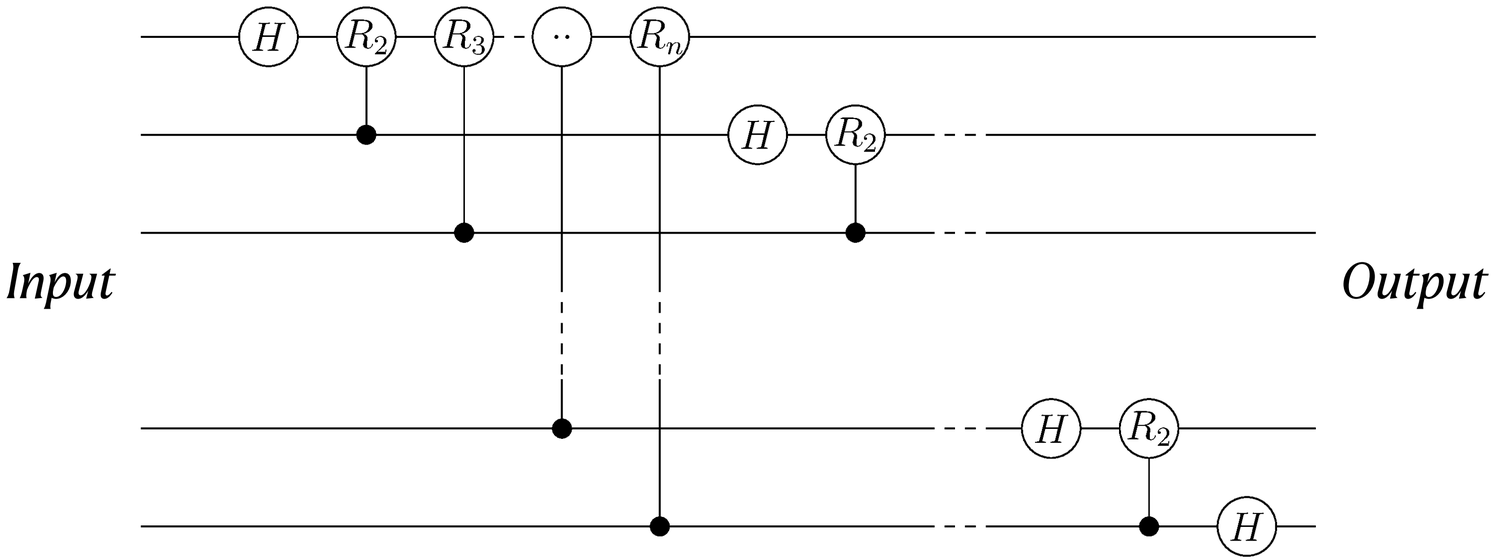
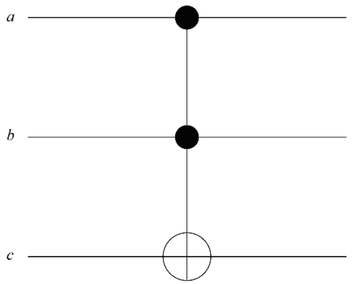
Depth: number of time steps (if gates acting on disjoint wires can be run in parallel)





Proofs of Quantumness from Factoring

Authors	Types of inputs	Gates	Space	Depth
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$

n is the number of bits in the input N

$\tilde{O}(\cdot)$ hides constant and $\text{poly}(\log n)$ factors

All results in this talk are using fast integer multiplication (multiply n-bit integers in $\tilde{O}(n)$ time)

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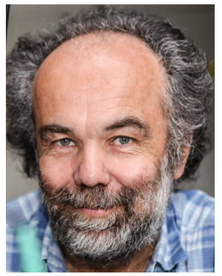

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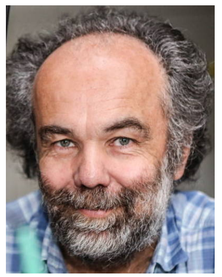
Any input: would break RSA cryptography, and suffice as a proof of quantumness

$N = P^2Q$: only suffices as a proof of quantumness



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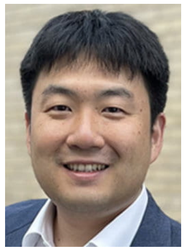

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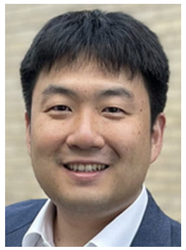

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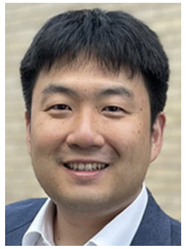

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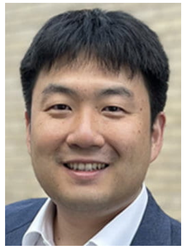
- Con: quantum prover needs to store state between rounds

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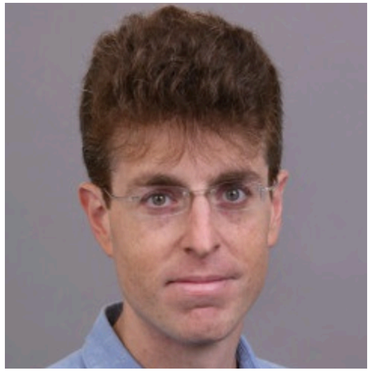

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Result 1: optimising Regev to match the space complexity of Shor

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K**R**VV (2024)	$N = P^2Q$ $(Q < 2^m)$	$\tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m + m)$

Result 2: LPDS with space and depth proportional to $\log Q$ rather than $\log N$




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- Sweet spot: $m = \tilde{O}(n^{2/3}) \rightarrow$ gates $\tilde{O}(n)$, space $\tilde{O}(n^{2/3})$, depth $\tilde{O}(n^{2/3})$

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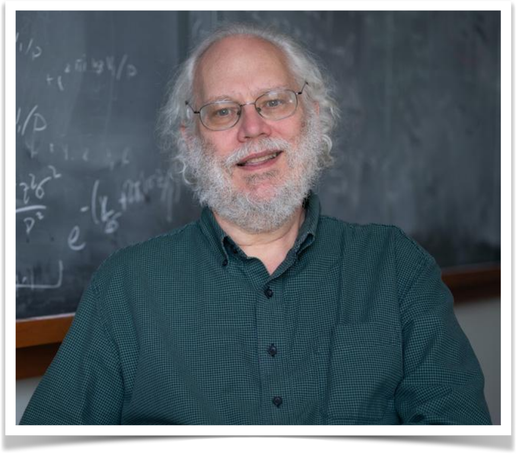
LPDS (2012)	$N = P^2Q$	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(n)$
KCVY (2021)	N/A	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(1)$
Regev (2023)	Any	$\tilde{O}(n^{1.5})$	$O(n^{1.5})$	$\tilde{O}(n^{0.5})$
RV (2024)	Any	$\tilde{O}(n^{1.5})$	$\tilde{O}(n)$	$\tilde{O}(n^{0.5})$
K**R**VV (2024)	$N = P^2Q$ $(Q < 2^{n^{2/3}})$	$\tilde{O}(n)$	$\tilde{O}(n^{2/3})$	$\tilde{O}(n^{2/3})$

Shor's Algorithm: A Sketch



Preliminary: Quantum Period Finding

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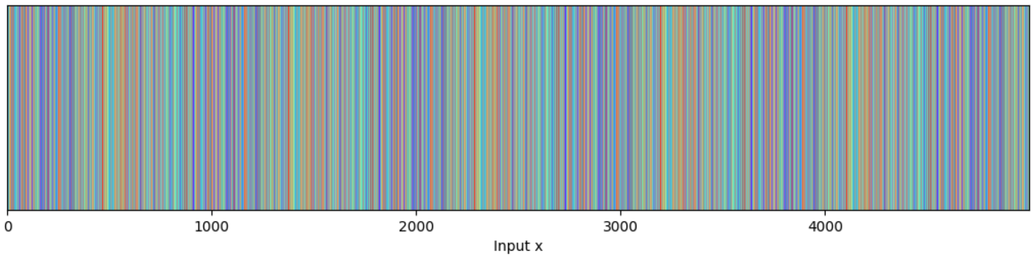
  - $x \equiv y \pmod{T} \Leftrightarrow f(x) = f(y)$



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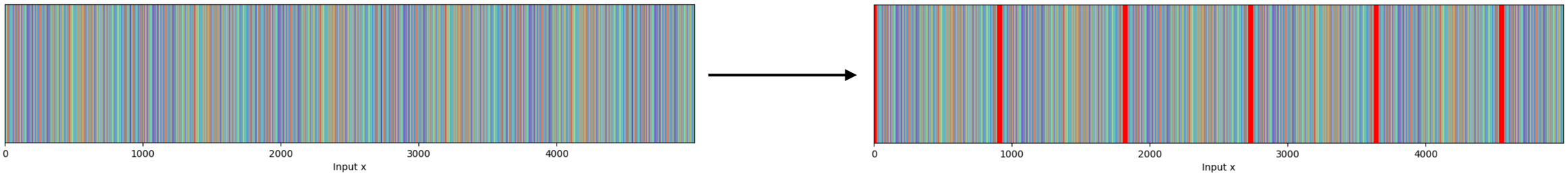

Shor overview: finding square roots of 1

- Goal: find $z \not\equiv \pm 1 \mod N$ such that $z^2 \equiv 1 \mod N$
  - N divides $z^2 - 1 = (z-1)(z+1)$ but not either factor individually
  - Hence $\gcd(z-1, N)$ is a nontrivial divisor of N

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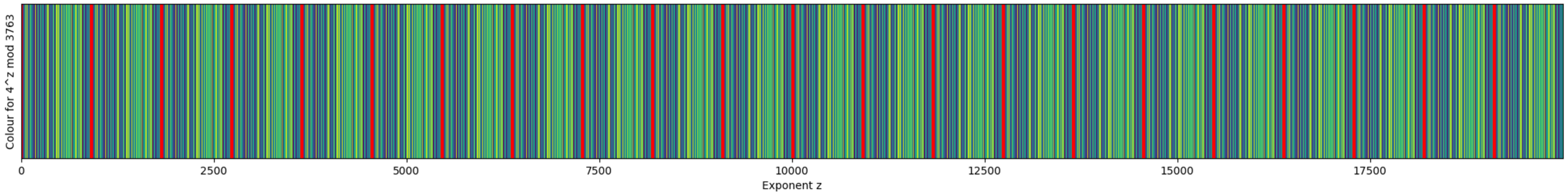

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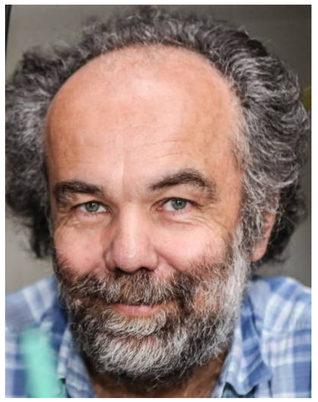
Factoring P^2Q with LPDS12: A Sketch



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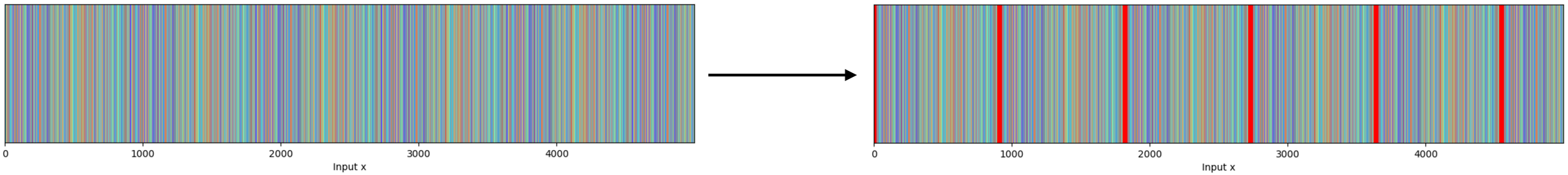

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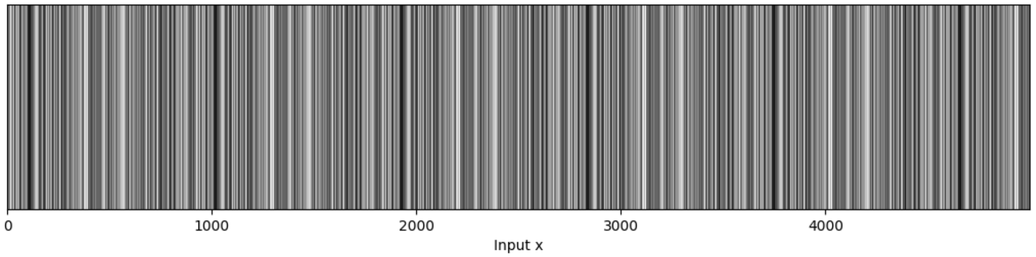

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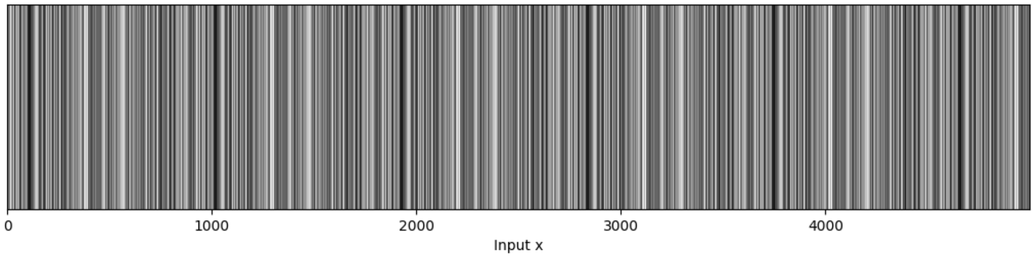

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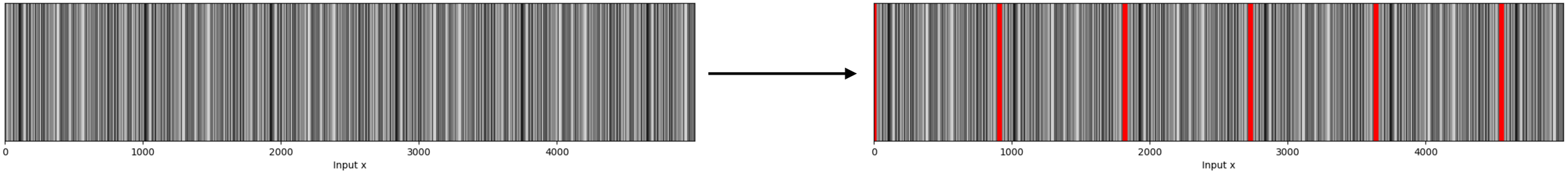

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- Legendre symbol essentially indicates whether this is the case:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a nonzero quadratic residue modulo } p; \text{ and} \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p; \text{ and} \\ 0, \text{ if } a \text{ divisible by } p\,. \end{cases}$$

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- Useful property: $a \equiv b \pmod{N} \Rightarrow \left(\frac{a}{N}\right) = \left(\frac{b}{N}\right)$
- Theorem (from Euclid to Schönhage 1971): can compute $\left(\frac{a}{N}\right)$ efficiently without knowing the factorisation of N — in fact, in time $\tilde{O}(\log N)$

Algorithms Computing the Jacobi Symbol

ft. Euclid, 2000 years ago

Jacobi Properties

- Periodicity: $\left(\frac{a}{b}\right) = \left(\frac{a \bmod b}{b}\right)$
- Reciprocity:* $\left(\frac{a}{b}\right) = (-1)^{f(a,b)}\left(\frac{b}{a}\right)$

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$$f(a,b) = \begin{cases} 0, \text{ if } a \equiv 1 \pmod 4 \text{ or } b \equiv 1 \pmod 4 \\ 1, \text{ if } a \equiv b \equiv 3 \pmod 4 \end{cases}$$

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Acta Informatica 1, 139—144 (1971)

Schnelle Berechnung von Kettenbruchentwicklungen

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Eingegangen am 16. September 1970

Summary. A method, given by D. E. Knuth for the computation of the greatest common divisor of two integers u, v and of the continued fraction for u/v is modified in such a way that only $O(n(\lg n)^2(\lg\lg n))$ elementary steps are used for $u, v < 2^n$.

Zusammenfassung. Ein von D. E. Knuth angegebenes Verfahren, für ganze Zahlen u, v den größten gemeinsamen Teiler und den Kettenbruch für u/v zu berechnen, wird so modifiziert, daß für n-stellige Zahlen nur $O(n(\lg n)^2(\lg\lg n))$ elementare Schritte gebraucht werden.

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A Unified Approach to HGCD Algorithms for polynomials and integers

Klaus Thull and Chee K. Yap*

Freie Universität Berlin
Fachbereich Mathematik
Arnimallee 2-6
D-1000 Berlin 33
West Germany

March, 1990

Abstract

We present a unified framework for the asymptotically fast Half-GCD (HGCD) algorithms, based on properties of the norm. Two other benefits of our approach are (a) a simplified correctness proof of the polynomial HGCD algorithm and (b) the first explicit integer HGCD algorithm. The integer HGCD algorithm turns out to be rather intricate.

Keywords: Integer GCD, Euclidean algorithm, Polynomial GCD, Half GCD algorithm, efficient algorithm.

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ON SCHÖNHAGE'S ALGORITHM AND SUBQUADRATIC INTEGER GCD COMPUTATION

NIELS MÖLLER

Abstract. We describe a new subquadratic left-to-right GCD algorithm, inspired by Schönhage's algorithm for reduction of binary quadratic forms, and compare it to the first subquadratic GCD algorithm discovered by Knuth and Schönhage, and to the binary recursive GCD algorithm of Stehlé and Zimmermann. The new GCD algorithm runs slightly faster than earlier algorithms, and it is much simpler to implement. The key idea is to use a stop condition for HGCD that is based not on the size of the remainders, but on the size of the next difference. This subtle change is sufficient to eliminate the back-up steps that are necessary in all previous subquadratic left-to-right GCD algorithms. The subquadratic GCD algorithms all have the same asymptotic running time, $O(n(\log n)^2 \log\log n)$.

Factoring from Jacobi Symbol Periodicity

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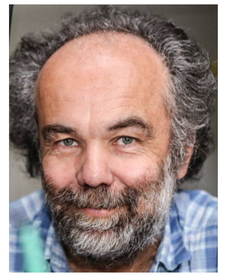

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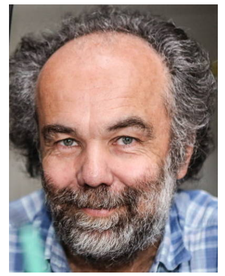

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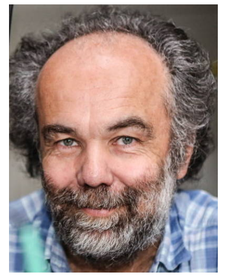
- Space and depth (if naively implemented): also $\tilde{O}(\log N)$



Result 1: Pushing the Space and Depth Down to $\tilde{O}(\log Q)$

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- **<u>Hope 1:</u>** when factoring $P^2 Q$ with Jacobi: the period is just $Q \rightarrow O(\log Q)$ qubits could suffice!

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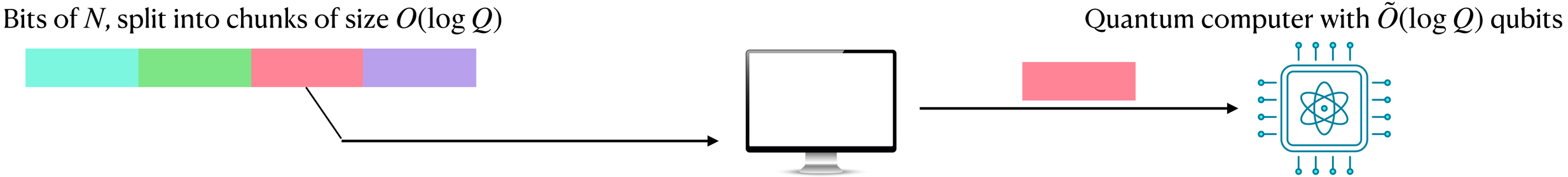

Classical computer sending instructions to the quantum computer

Computing the Jacobi Symbol

It's all about $N \bmod a$

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  2. Now $(N \bmod a) < a < \text{poly}(Q) \rightarrow$ can finish in $\tilde{O}(\log Q)$ gates/space using Schönhage

Computing the Jacobi Symbol

It's all about $N \bmod a$

- Our task: compute $\left(\frac{a}{N}\right)$ for $a \leq \mathsf{poly}(Q)$
- Recall:

$$\left(\frac{a}{N}\right) = (-1)^{f(a,N)}\left(\frac{N}{a}\right) = (-1)^{f(a,N)}\left(\frac{N \bmod a}{a}\right)$$

- Two step procedure:
  1. Compute $N \bmod a$
  2. Now $(N \bmod a) < a < \mathsf{poly}(Q) \rightarrow$ can finish in $\tilde{O}(\log Q)$ gates/space using Schönhage

The "only" bottleneck: computing $|a\rangle \mapsto |a\rangle|N \bmod a\rangle$

Our Result, Distilled

- Theorem (K**R**VV24): for quantum a and classically known N, we can compute

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in $\tilde{O}(\log N)$ gates (near-linear) and $\tilde{O}(\log a)$ qubits (enough qubits to write down a)

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Open question: other applications of these results?

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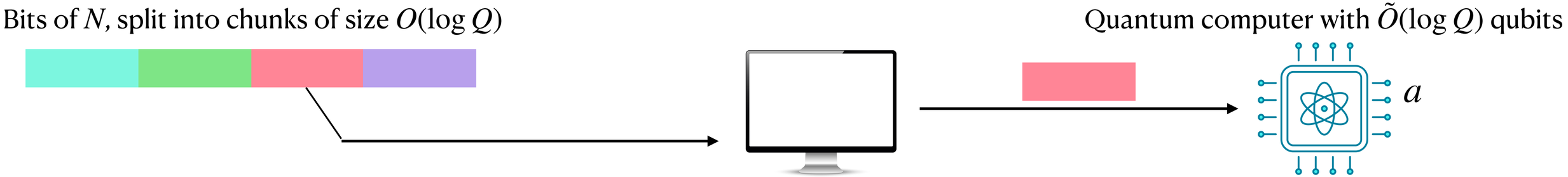
- **Corollary 2: we can factor $N = P^2Q$ in $\tilde{O}(\log N)$ gates and $\tilde{O}(\log Q)$ qubits**
  - **Just need the above theorem for $a \leq \mathrm{poly}(Q)$**

Computing N mod a with Quantum Streaming

Notation: N has n bits, a has $m = O(\log Q)$ bits



Classical computer sending instructions to the quantum computer

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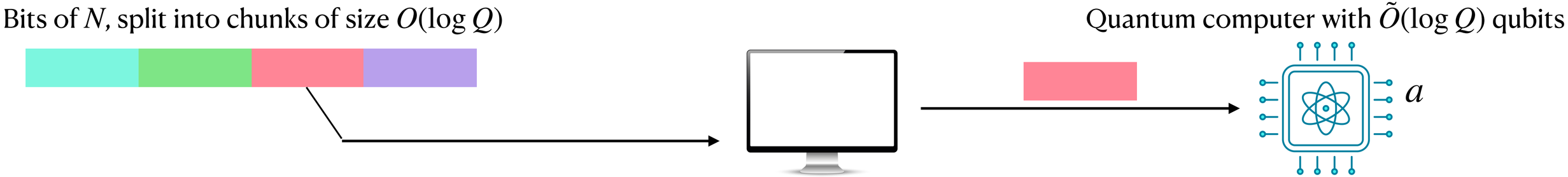

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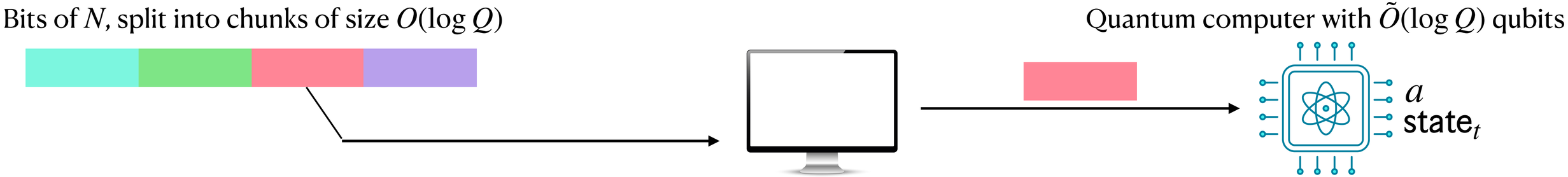

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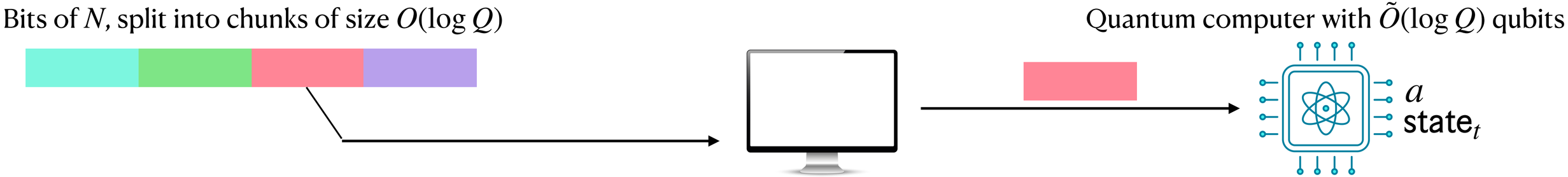

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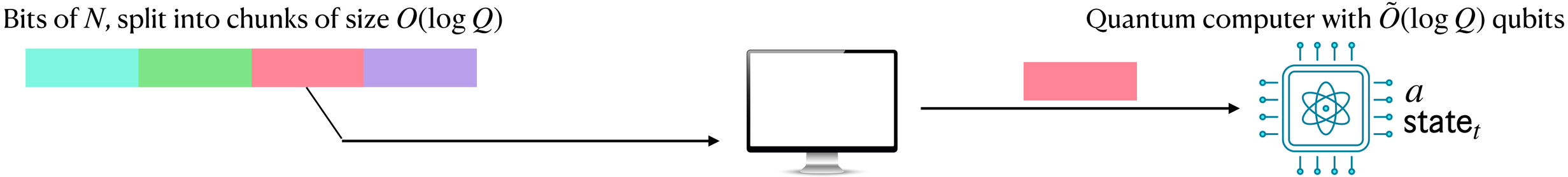

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  - **Reversibility:** state_{t-1} can be reconstructed (and therefore uncomputed) from state_t

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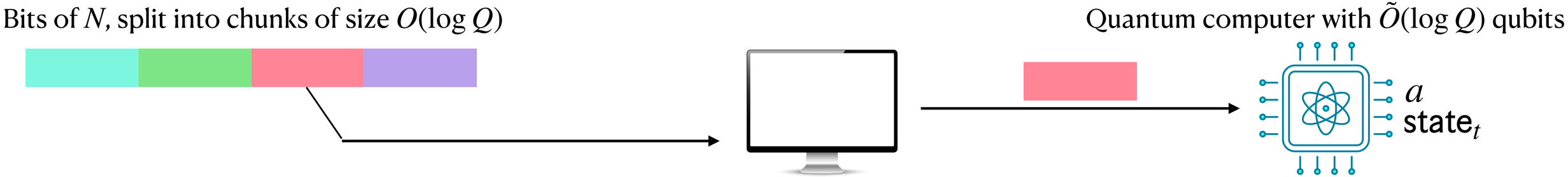

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Efficiency of Our Algorithm for $N \bmod a$

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  - Gates: $O(n/m) \times \tilde{O}(m) = \tilde{O}(n)$
  - Depth: $O(n/m) \times \tilde{O}(1) = \tilde{O}(n/m)$

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   - Contributes an additional depth of $\tilde{O}(m)$ (dominant term in depth if $m \gg \sqrt{n}$)

Summary: Our Factoring Circuit for P^2Q

- Main theorem (K**R**VV24): for $N = P^2Q < 2^n$ such that Q is squarefree and $< 2^m$, we can recover P, Q from N in $\tilde{O}(n)$ gates, $\tilde{O}(m)$ qubits, and $\tilde{O}(n/m + m)$ depth

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Central workhorse: an efficient quantum circuit for computing $N \bmod a$ for classical $N < 2^n$ and quantum $a < 2^m$

Regev's Algorithm: A Sketch

Regev's key idea: add dimensions

Before: exploited periodicity of

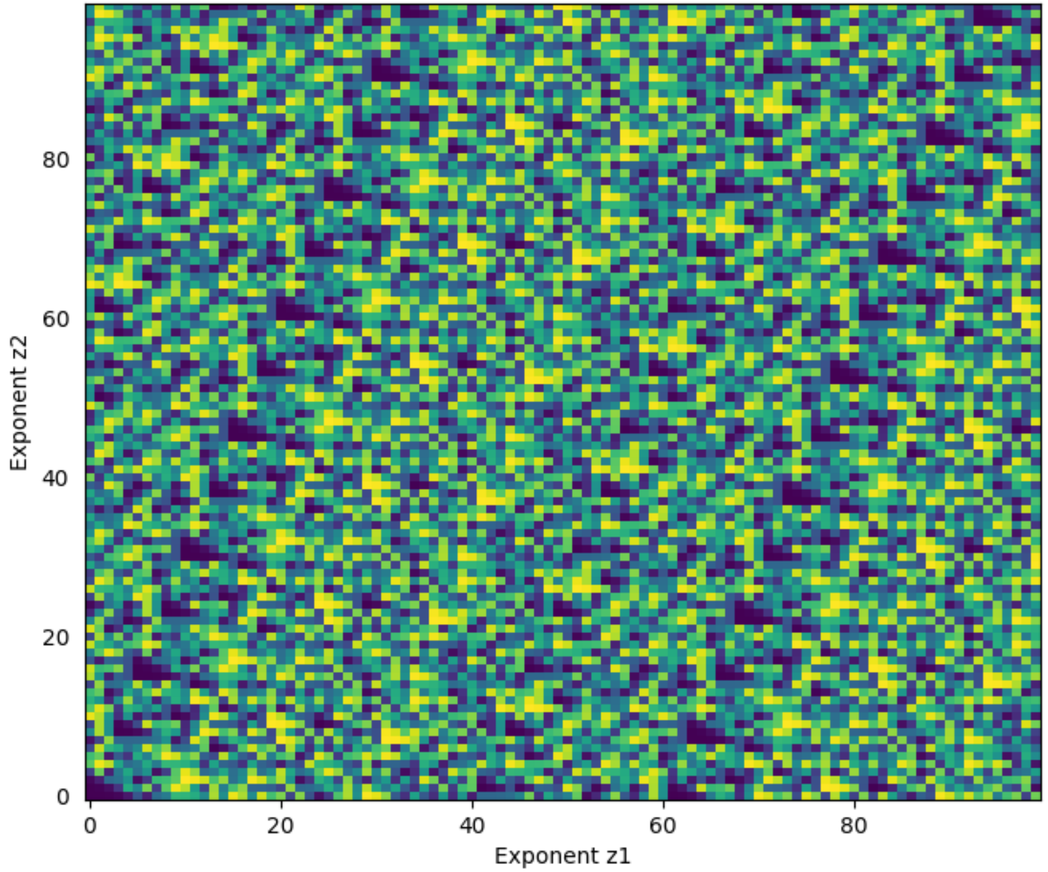
$$z \mapsto 4^z \bmod 3763.$$

Now: let's look at

$$(z_1, z_2) \mapsto 4^{z_1} 9^{z_2} \bmod 3763.$$

Regev's key idea: add dimensions

Q: How does this help us?



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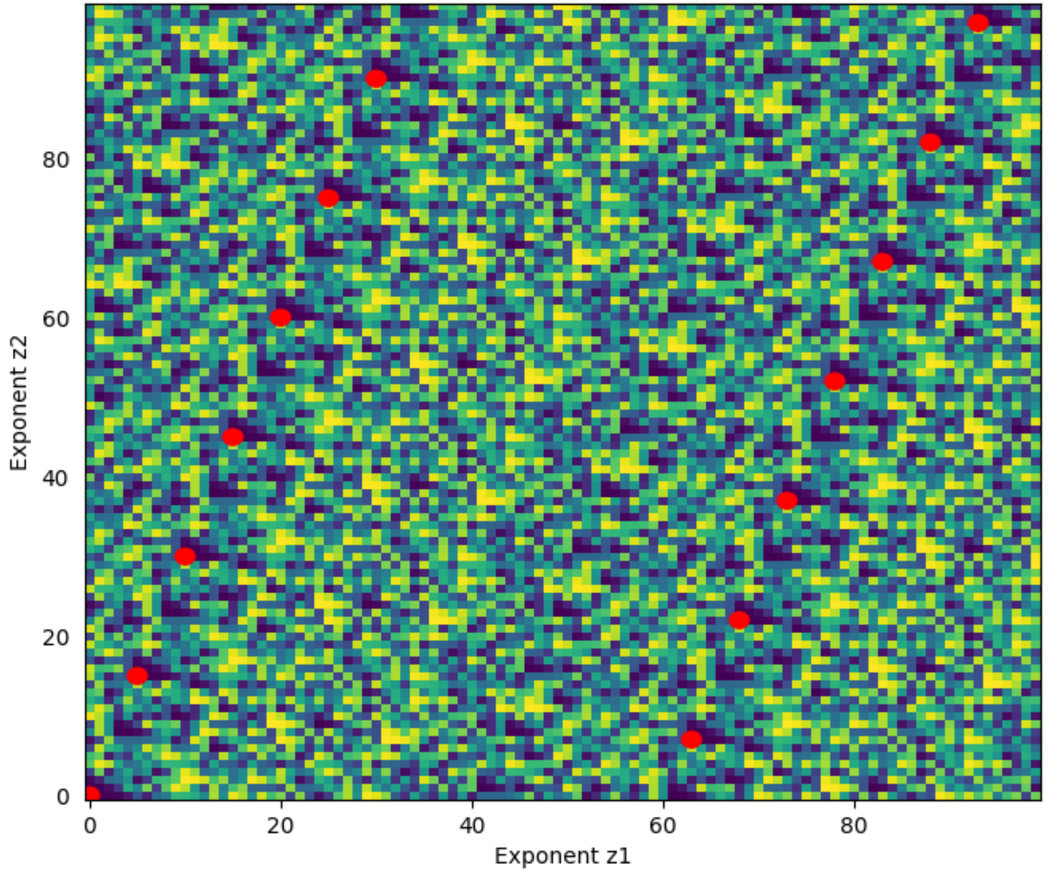
Crucially, these periods are much closer to $(0, 0)$ **than in Shor!**



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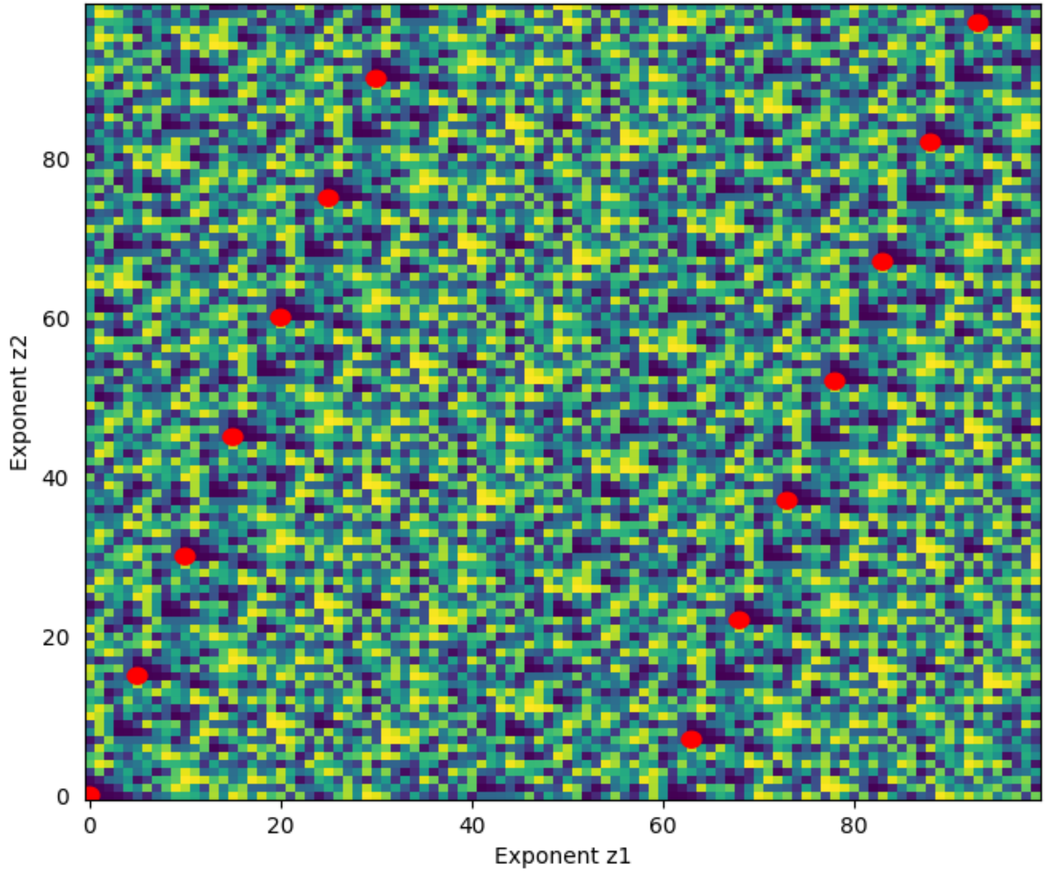
Quantum circuit follows the same blueprint as Shor.



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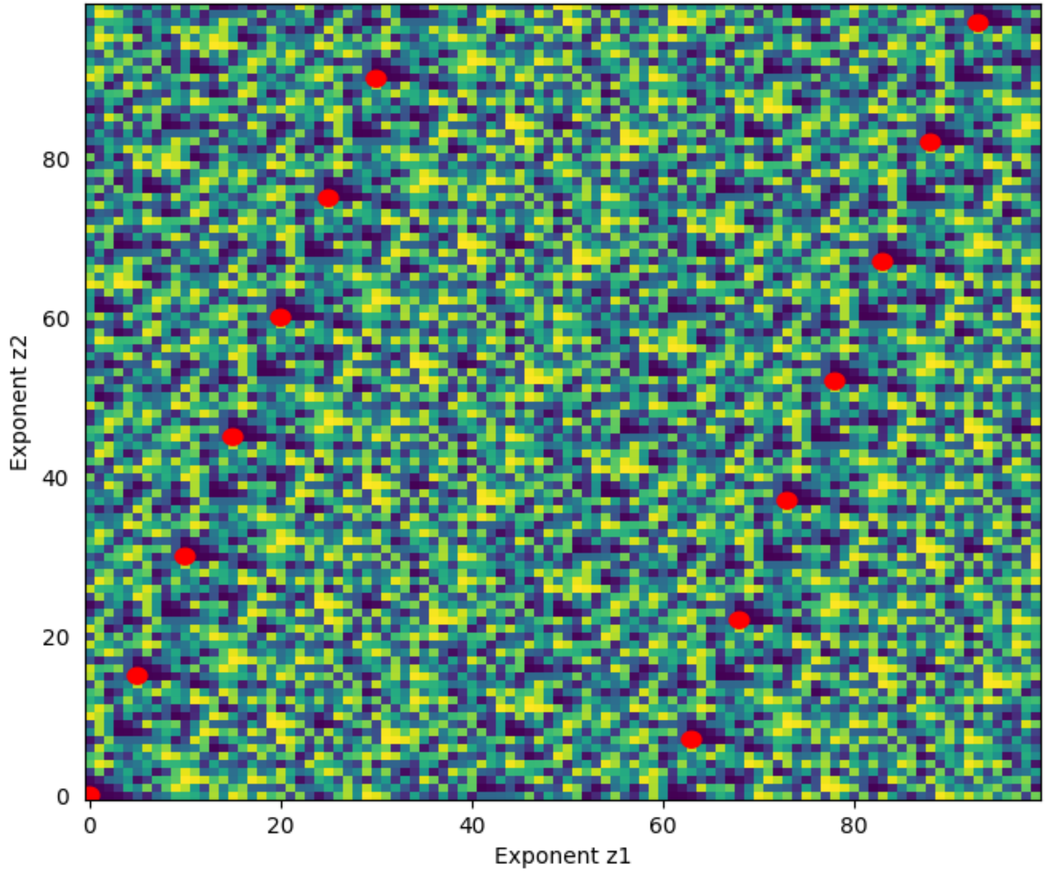

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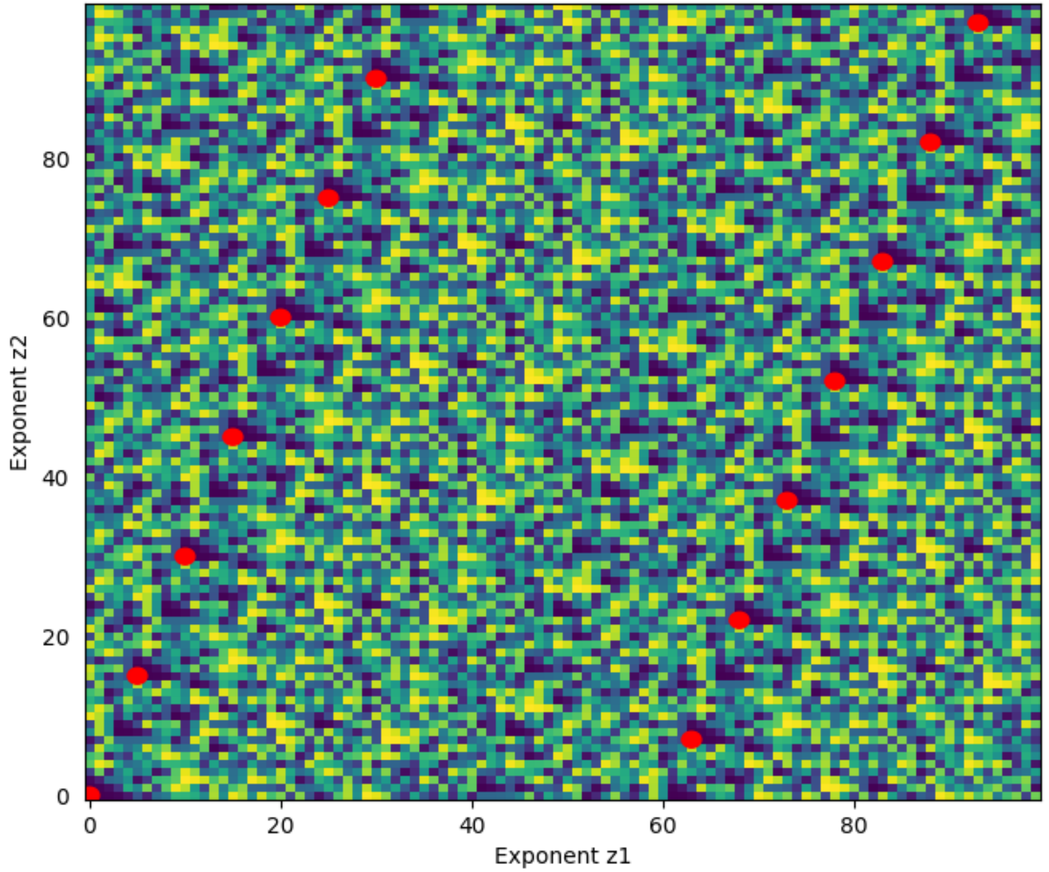

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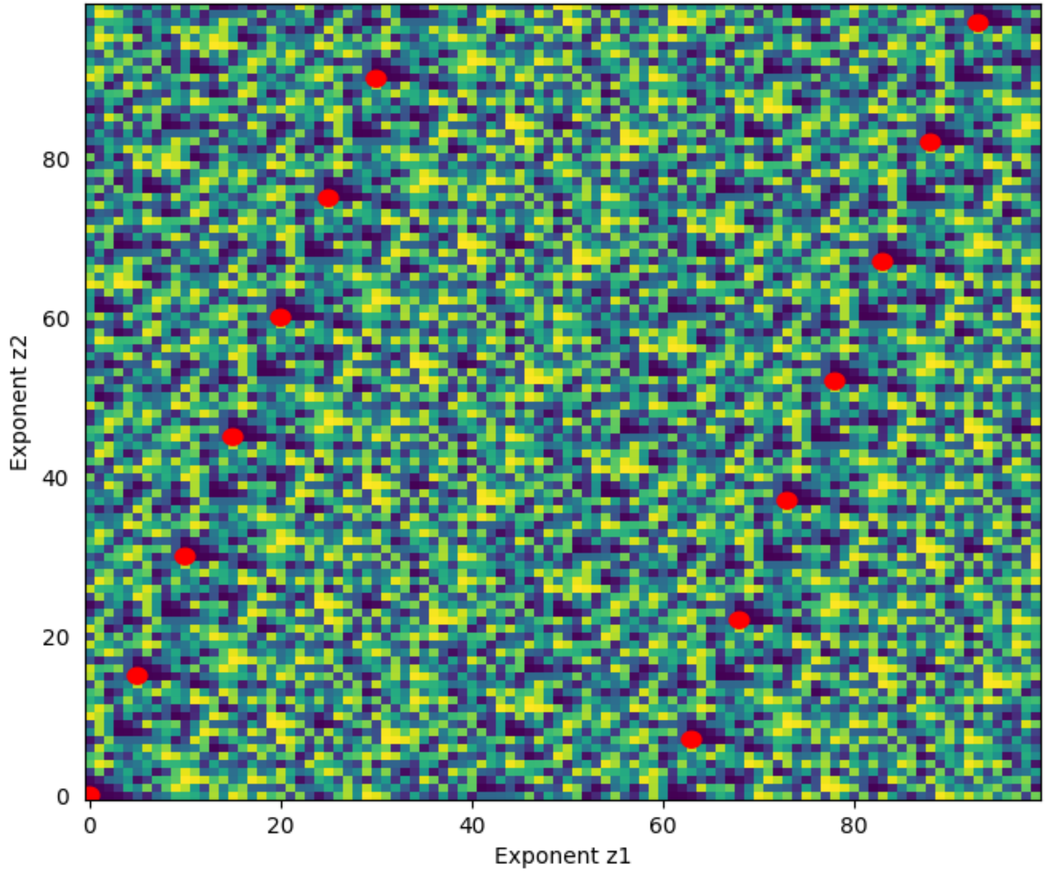

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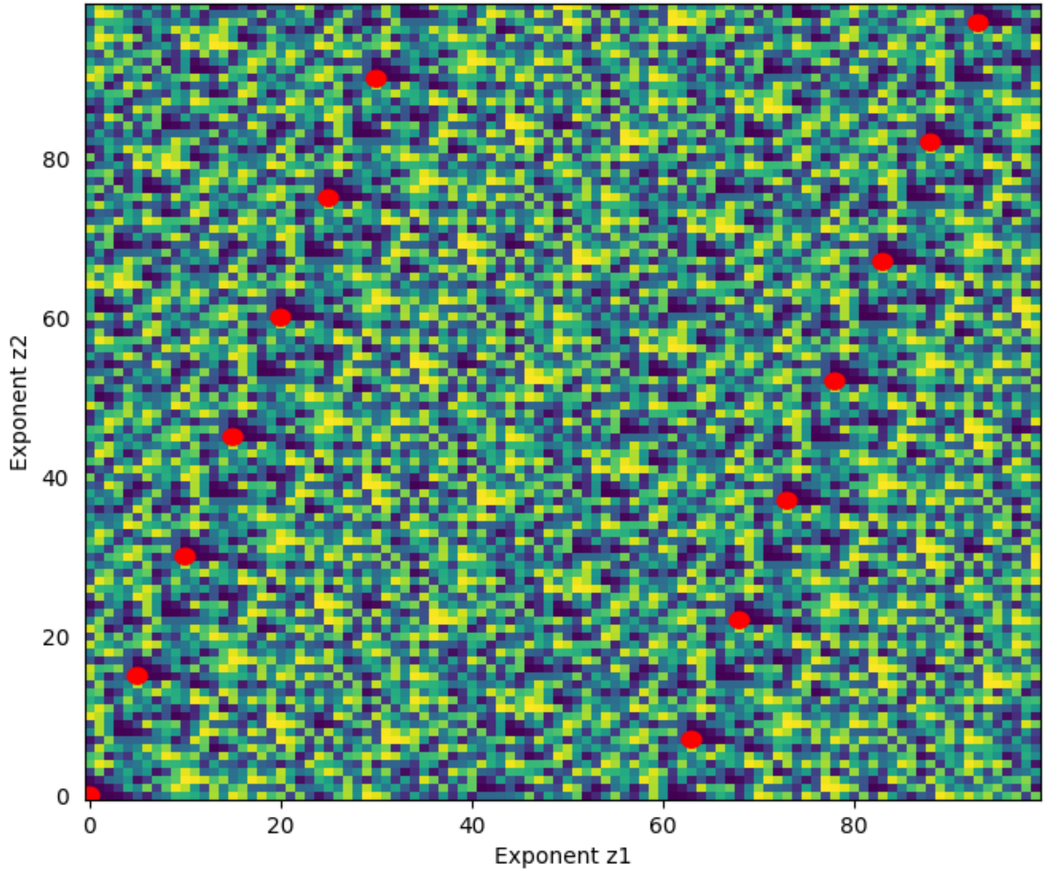

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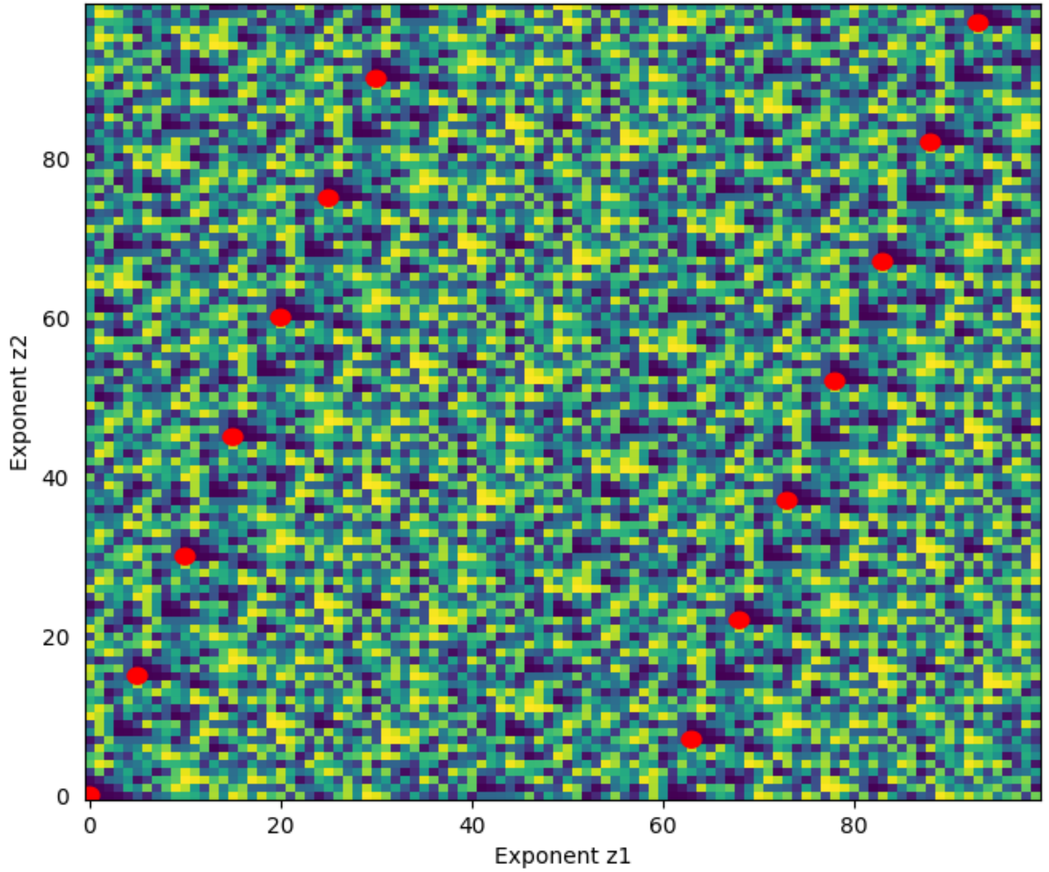
  - In comparison, Shor needs n multiplications → we save a factor of d



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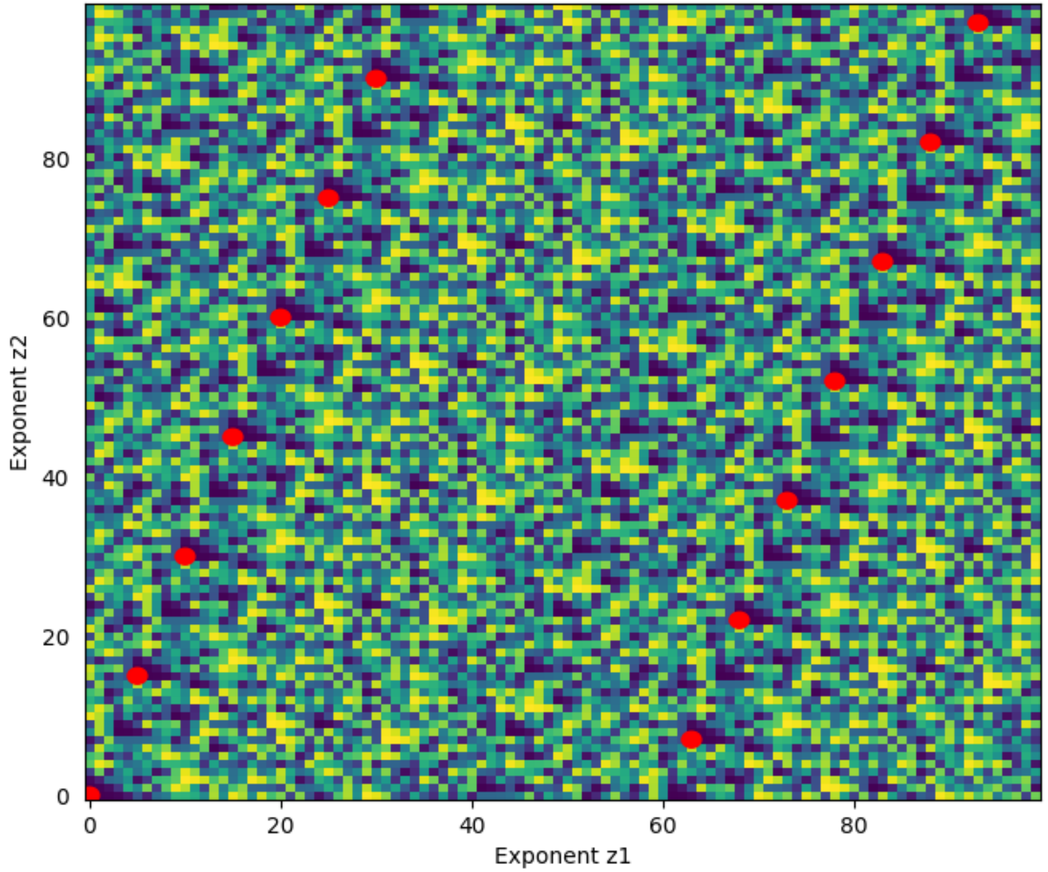

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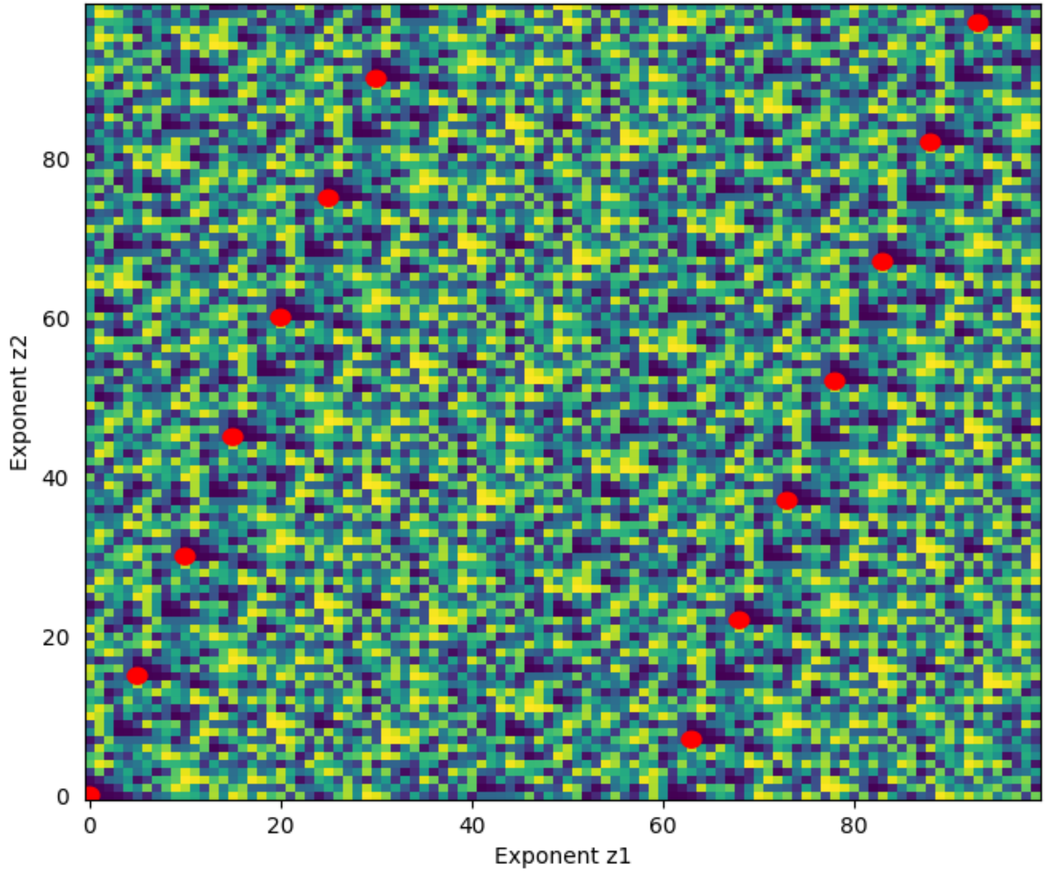

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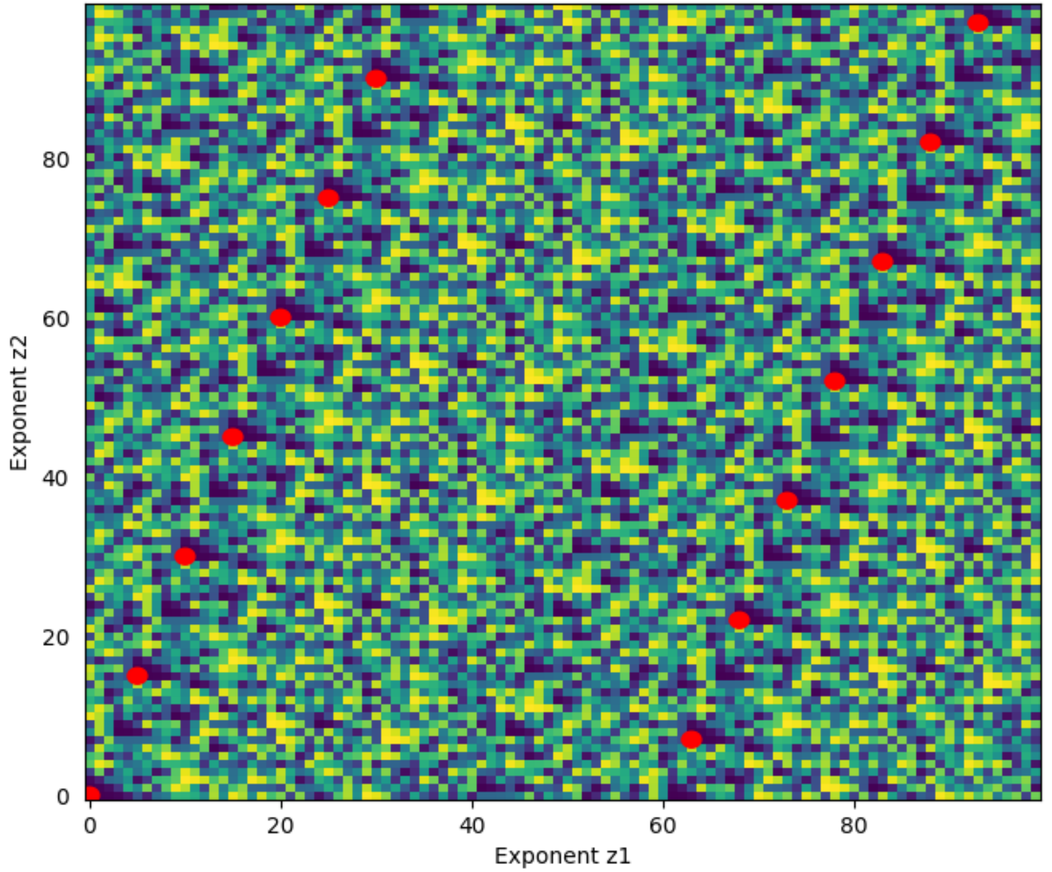
- A: Classical post-processing needs to solve a d-dimensional lattice problem with approximation factor $2^{O(n/d)}$
  - Sweet spot: $d = \sqrt{n}$ (LLL)



Result 2: Improving Space Complexity of Regev

Regev's qubit problem

- Performance bottleneck: computing $(z_1, \ldots, z_d) \mapsto a_1^{z_1} \ldots a_d^{z_d} \bmod N$
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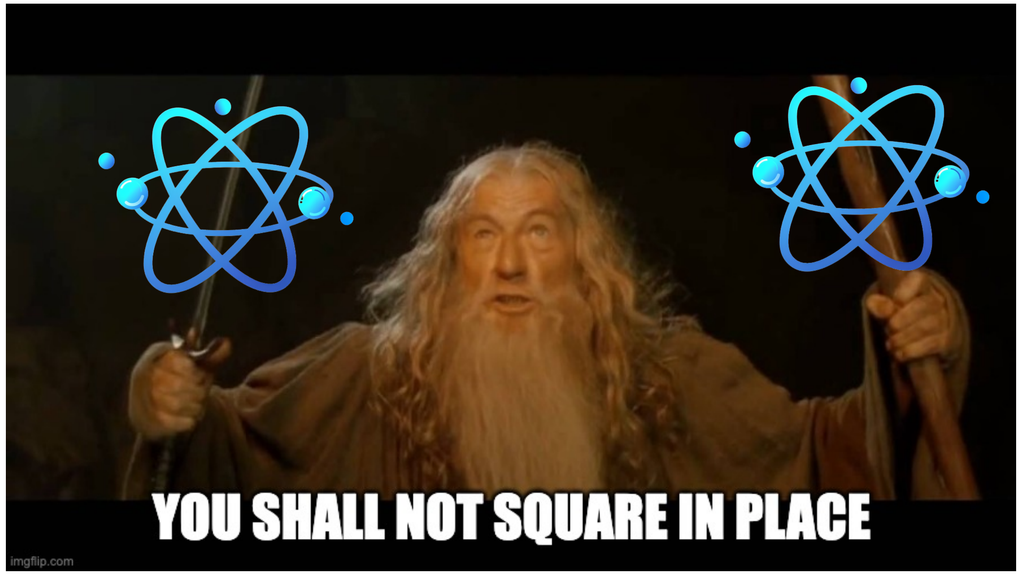
  - Quantum circuits need to be reversible, but squaring mod N is not e.g. $-1,1$



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Each squaring adds n qubits $\rightarrow O\left(\frac{n}{d} \times n\right) = O(n^{3/2})$ qubits total.

Aside: qubit complexity of Shor

- Does this mean Shor also requires $O\left(\frac{n}{d} \times n\right) = O(n^2)$ qubits? **No!**
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    - *Unlike squaring, multiplication can be done reversibly (not obvious... stay tuned)*

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  - Any circuit using these results should require $O(n^2)$ gates

Main Idea: Fibonacci exponentiation

- What if we used two accumulators?

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- Observation by Kaliski 2017:

$$|a, a\rangle \rightsquigarrow |a, a^2\rangle \rightsquigarrow |a^3, a^2\rangle \rightsquigarrow |a^3, a^5\rangle \rightsquigarrow |a^8, a^5\rangle$$

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We can efficiently compute a^{F_k} for any Fibonacci number F_k!

Our Space Improvement

Concrete Results and Summary

- Regev's original circuit: $\approx 3n^{3/2}$ qubits, $\approx 6n^{1/2}$ multiplications mod N
- With our space optimisations: $\approx 10.4n$ qubits, $\approx 45.7n^{1/2}$ multiplications mod N

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Squaring mod N *cannot be implemented in place, while multiplication* mod N *can* $\rightarrow$ *can exponentiate with two registers using Fibonacci exponentiation,* ***without consuming additional space per step.***

Outlook and Future Directions

The True Difficulty of Factoring

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- The smallest quantum circuit for factoring an n-bit integer N has $\tilde{O}(n^{3/2})$ gates (Regev)...
  - ... unless N has a square factor, in which case we just need $\tilde{O}(n)$ gates (Jacobi)

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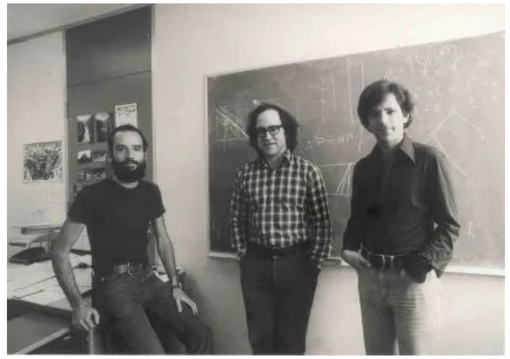
"Idling is just as expensive as doing operations... memory isn't cheap."

Concrete Efficiency of Regev

Q: Does Regev's algorithm (and its follow-ups by us) bring us closer to breaking RSA-2048?



Concrete Efficiency of Regev

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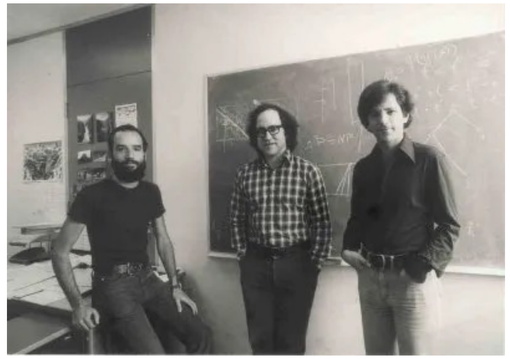
A: It might with further optimisations, but not yet.

Several constant and polylog-factor optimisations to Shor make it very effective for small problem sizes.



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A: This seems hard, but...

Bonus Slides

Our Result, Distilled

- Theorem (K**R**VV24): for $a < 2^m$ (in our application, $m = O(\log Q)$) and classically known $N < 2^n$, we can compute

$$|a\rangle \mapsto |a\rangle |N \bmod a\rangle$$

in $\tilde{O}(n)$ gates, $\tilde{O}(m)$ qubits, and $\tilde{O}(n/m)$ depth

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Open question: other applications of these results?

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simplified presentation based on an observation by Daniel J. Bernstein

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That's it! We computed $N \bmod a$!

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Regev's Number-Theoretic Assumption

- Regev: relies on $(z_1, \ldots, z_d) \mapsto a_1^{z_1} \ldots a_d^{z_d} \bmod N$ having periods of size $2^{O(n/d)}$
- **But these periods could just yield a trivial square root of 1** $\bmod N$
- Regev relies on a conjecture that *at least one* small period yields a non-trivial square root of 1
- Follow-up work by Pilatte proves* Regev's conjecture

** proves correctness for a variant of Regev's algorithm that is worse by polylog factors and likely impractical*